$$0 \in \partial f(x)$$

OPTIMALITY CONDITIONS IN CONVEX OPTIMIZATION

A Finite-Dimensional View

Anulekha Dhara
Joydeep Dutta

CRC Press

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In memory of
Professor M. C. Puri
and
Professor Alex M. Rubinov

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Symbol Description

$\emptyset$	empty set
∞	infinity
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	real line
$\bar{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, +\infty\}$
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}_+$	nonnegative orthant of $\mathbb{R}$
$\mathbb{R}^n_+$	nonnegative orthant of $\mathbb{R}^n$
$[x, y]$	closed line segment joining x and y
(x, y)	open line segment joining x and y
$\mathbb{R}^I$	product space $\prod_I \mathbb{R}$
$\mathbb{R}^{[I]}$	$\{\lambda \in \mathbb{R}^I : \lambda_i \neq 0$ for finitely many $i \in I\}$
$\mathbb{R}^{[I]}_+$	positive cone in $\mathbb{R}^{[I]}$
$supp\ \lambda$	$\{i \in I : \lambda \in \mathbb{R}^{[I]},\ \lambda_i \neq 0\}$
$\mathbb{B}$	open unit ball
$\mathbb{B}_\delta(\bar{x})$	open ball with radius $\delta > 0$ and center at $\bar{x}$
$cl\ F$	closure of F
$co\ F$	convex hull of F
$cl\ co\ F$	closed convex hull of F
$aff\ F$	affine hull of F
$int\ F$	interior of F
$ri\ F$	relative interior of F
$cone\ F$	cone generated by F
F^+	positive polar cone of F
F°	polar cone of F
$x \to \bar{x}$	x converges to $\bar{x}$
$\lim$	limit
$\liminf$	limit infimum
$\limsup$	limit supremum
$\langle .,. \rangle$	inner product
$\\|.\\|$	norm
$\phi(F)$	image space of F under ϕ
$gph\ \Phi$	graph of set-valued map Φ
$dom\ \phi$	effective domain of $\phi : X \to \bar{\mathbb{R}}$
$epi\ \phi$	epigraph of ϕ
$lev_{\leq\alpha}\phi$	α-lower level set of ϕ
δ_F	indicator function to F
d_F	distance function to F
$proj_F(\bar{x})$	projection of $\bar{x}$ to F
$\sigma(\ .\ ; F)$	support function to F
ϕ^*	conjugate function of ϕ
$\phi^+(x)$	$\max\{0, \phi(x)\}$
$\nabla\phi(\bar{x})$	derivative or gradient of ϕ at $\bar{x}$
$\phi^\circ(\bar{x}; d)$	Clarke directional derivative of ϕ at $\bar{x}$ in the direction d
$\dfrac{\partial\phi}{\partial x}$	partial derivative of ϕ with respect to x
$\dfrac{\partial^2\phi}{\partial x_i \partial x_j}$	second-order partial derivative of ϕ with respect to x_i and x_j
$\partial\phi(\bar{x})$	convex subdifferential of ϕ at $\bar{x}$
$\partial_\epsilon\phi(\bar{x})$	ϵ-subdifferential of ϕ at $\bar{x}$
$\partial^\circ\phi(\bar{x})$	Clarke subdifferential or generalized gradient of ϕ at $\bar{x}$
$\nabla^2\phi(\bar{x})$	Hessian of ϕ at $\bar{x}$
$J\phi(\bar{x})$	Jacobian of ϕ at $\bar{x}$
$T_F(\bar{x})$	tangent cone to F at $\bar{x}$
$N_F(\bar{x})$	normal cone to F at $\bar{x}$
$N_{\epsilon,F}(\bar{x})$	ϵ-normal set to F at $\bar{x}$

Foreword

The roots of the mathematical topic of optimization go back to ancient Greece, when Euclid considered the minimal distance of a point to a line; convex sets were investigated by Minkowski about a hundred years ago, and fifty years ago, J.-J. Moreau [87] defined the notion of the subdifferential of a convex function. In 1970, R.T. Rockafellar wrote his monograph [97] on convex analysis. Since then, the field of convex optimization and convex analysis has developed rapidly, a huge number of papers on that topic have been published in scientific journals and a large number of monographs and textbooks have been produced. Now, we have a new book at hand and one can ask why read this book.

A recent topic of research in mathematical optimization is the need to compute global optima of nonconvex problems. To do that, the problem can be convexified using the optimal function value of the resulting convex optimization problem as a bound for the problem investigated. Combining this with an enumeration idea the problem can be solved. The same approach of convexification plus enumeration can serve as a way to solve mixed-integer nonlinear optimization problems which is a second challenging problem of recent and future research. Moreover, many practical situations lead directly to convex programming problems. Hence the need to develop a deep insight into convex optimization.

The theory of convex differentiable optimization is well established. Every student will be introduced in basic courses on mathematical optimization to the Fritz John and Karush–Kuhn–Tucker necessary optimality conditions. For guaranteeing the Karush–Kuhn–Tucker conditions a constraint qualification such as the Slater condition is needed. But, in many applications, this condition is violated. There are a larger number of ways out in such a situation: Abadie constraint qualification can be supposed, sequential optimality conditions can be used or we can try to filter out full information of (enhanced) Fritz John necessary optimality conditions. These nonstandard but essential parts of the theory of convex optimization need to be described in detail and in close relation to each other.

Nonsmooth analysis (see, for example, Mordukhovich [86]) is a quickly developing area in mathematical optimization. The initial point of nonsmooth analysis is convex analysis, but recent developments in nonsmooth analysis are a good influence on convex analysis.

The aim of this book is to develop deep insight into the theory of convex optimization, combining very recent ideas of nonsmooth analysis with standard and nonstandard theoretical results. Lagrange and Fenchel duality use different tools and can be applied successfully in distinct directions. But in the end, both are shown to coincide.

If, at an optimal solution, no constraint qualification is satisfied, algorithms solving the Karush–Kuhn–Tucker conditions cannot be used to compute this point. And, how to characterize such a point? Roughly speaking one idea is the existence of a sequence outside of the feasible set with smaller objective function values converging to that point. These are the enhanced Fritz John necessary optimality conditions. A second idea is to characterize optimality via subgradients of the regular Lagrangian function at perturbed points converging to zero. This is the sequential optimality condition. Both optimality conditions work without constraint qualifications. ε-optimal solutions can be characterized using ε-subgradients.

One special convex optimization problem is also investigated. This is the problem to find a best point within the set of optimal solutions of a convex optimization problem. If the objective function is convex, this is a convex optimization problem called a simple bilevel programming problem. It is easy to see that standard regularity conditions are violated at every feasible point. For this problem, a very general constraint qualification is derived.

A last question is if convexity can successfully be used to investigate nonconvex problems as the maximization of convex functions or the minimization of a function being the difference of convex functions.

Algorithmic approaches for solving convex optimization problems are not described in this book. This results in much more space for theoretical properties. The result is a book illuminating not only the body but also the bounds and corners of the theory of convex optimization. Many of the results presented are usually not contained in books on this topic. But, if more and more (applied) difficult optimization problems need to be solved, we are more likely be faced with instances where usual approaches fail. Then it is necessary to search away from standard tools for applicable approaches. I am sure that this book will be very helpful.

I deeply recommend this book for advanced reading.

Stephan Dempe
Freiberg, Germany

Preface

This is a book on convex optimization. More precisely it is a book on the recent advances in the theory of optimality conditions for convex optimization. The question is why should one need an additional book on the subject? However, possibly the books on convex analysis are much more in number than the ones on convex optimization. In the books dealing with convex analysis, like the classic *Convex Analysis* by Rockafellar [97] or the the more recent *Convex Analysis and Nonlinear Optimization* by Borwein and Lewis [17], one would find convex optimization theory appears as an application to various results of convex analysis. However, from 1970 until now there has been a growing body of research in the area of optimality conditions for a convex optimization. Many of these results address the question as to what happens when the Slater condition fails for a convex optimization problem or are there very general constraint qualification conditions which hold even if the the most popular ones fail? The books on convex analysis usually do not present results of this type and thus these results remain largely scattered in the vast literature on convex optimization. On the other hand, the books dealing with convex optimization largely focus on algorithms or algorithms and theory associated with a certain special class of problems like second-order conic programming or semidedfinite programming. Some recent books like *Introductory Lectures in Convex Optimization* by Nesterov [90] or *Lectures on Modern Convex Optimization* by Ben-Tal and Nemirovskii [8] deal with algorithms and the special problems, respectively.

This book has a completely different focus. It deals with optimality conditions in convex optimization. It attempts to bring in most of the important and recent results in this area that are scattered in the literature. However, we do not ignore the required convex analysis either. We provide a detailed chapter on the main convex analytic tools and also provide some new results that have appeared recently in the literature. These results are usually not found in standard books on convex analysis but they are essential in developing many of the important results in this book. This book actually began as a survey paper but then we realized that it has too much material to be considered as a survey; and then we thought of converting the survey paper into the form of a monograph.

We would look to thank the many people who encouraged us to write the book. Professor Stephan Dempe agreed very kindly to write the foreword. Professor Boris Mordukhovich, Professor Suresh Chandra, Professor Juan Enrique Martinez-Legaz also encouraged us to go ahead and write the book. We are indeed grateful to them. We would also like to thank Aastha Sharma of Taylor & Francis, India, for superb handling of the whole book project and Shashi Kumar from the helpdesk of Taylor & Francis for helping with the formatting. We would also like to extend our deepest gratitude to our families for their support. Joydeep Dutta would like to thank his daughter Naina and his wife Lalty for their understanding and patience during the time this book was written. Anulekha Dhara would like to express her deepest and sincerest regards and gratitude to her parents Dr. Madhu Sudan Dhara and Dolly Dhara for their understanding and support. She would also like to thank the National Board for Higher Mathematics, Mumbai, India, for providing financial support during her tenure at the Indian Institute of Technology Kanpur, India.

The book is intended for research mathematicians in convex optimization and also for graduate students in the area of optimization theory. This could be of interest also to the practitioner who might be interested in the development of the theory. We have tried our best to make the book free of errors. But to err is human, so we take the responsibility for any errors the readers might find in the book. We would also like to request that readers communicate with us by email at the address: jdutta@iitk.ac.in. We sincerely hope that the young researchers in the field of optimization will find this book helpful.

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Chapter 1

What Is Convex Optimization?

1.1 Introduction

Optimization is the heart of applied mathematics. Various problems encountered in the areas of engineering, sciences, management science, and economics are based on the fundamental idea of mathematical formulation. Optimization is an essential tool for the formulation of many such problems expressed in the form of minimization of a function under certain constraints like inequalities, equalities, and/or abstract constraints. It is thus rightly considered a science of selecting the best of the many possible decisions in a complex real-life environment.

Even though optimization problems have existed since very early times, the optimization theory has settled as a solid and autonomous field only in recent decades. The origin of analytic optimization lies in the classical calculus of variations and is interrelated with the development of calculus. The very concept of derivative introduced by Fermat in the mid-seventeenth century via the tangent slope to the graph of a function was motivated by solving an optimization problem, leading to the Fermat stationary principle. Around 1684, Leibniz developed a method to distinguish between minima and maxima via second-order derivatives. The *calculus of variations* was introduced by Euler while solving the Brachistochrone problem, which was posed by Bernoulli in 1696. The problem is stated as *"Given two points x and y in the vertical plane. A particle is allowed to move under its own gravity from x to y. What should be the curve along which the particle should move so as to reach y from x in the shortest time?"* In 1759, Lagrange gave a completely different approach to solve the problems in calculus of variations, today known as the *Lagrange multiplier rule*. The Lagrange multipliers are viewed as the auxiliary variables that are primarily used to derive the optimality conditions for constrained optimization problems. These optimality conditions are the building blocks of optimization theory.

During the second world war, Dantzig developed the *simplex method* to solve *linear programming problems*. The first attempt to develop the Lagrange multiplier rules for nonlinear optimization problem was made by Fritz John [71] in 1948. In 1951, Kuhn and Tucker [73] gave the Lagrange multiplier rule for convex and other nonlinear optimization problems involving differentiable functions. It was later found that Karush in 1939 had independently established the optimality conditions similar to those of Kuhn and Tucker. These optimality conditions are today famous as the *Karush–Kuhn–Tucker (KKT) optimality conditions*. All the initial theories were developed with the differentiability assumptions of the functions involved.

Meanwhile, efforts were made to shed the differentiability hypothesis, thereby leading to the development of *nonsmooth convex analysis* as a subject in itself. This added a new chapter to optimization theory. The key contributors in the development of convexity theory are Fenchel [45], Moreau [88], and Rockafellar [97]. An important milestone in this direction was the publication of *Convex Analysis* by Rockafellar [97], where the theory of nonsmooth convex analysis was presented in detail for the first time. No wonder this text is by far a must for all optimization researchers. In the early 1970s, his student Clarke coined the term *nonsmooth optimization* to categorize the theory involving nondifferentiable optimization problems. He extended the calculus rules and applied them to optimization problems involving locally Lipschitz functions. This was just the beginning. The subsequent decade witnessed a large development in the field of nonsmooth nonconvex optimization. For details on nonsmooth analysis, one may refer to Borwein and Lewis [17]; Borwein and Zhu [18]; Clarke [27]; Clarke, Ledyaev, Stern and Wolenshi [28]; Mordukhovich [86]; and Rockafellar and Wets [101].

However, such developments have not overshadowed the importance of convex optimization, which still is and will remain a pivotal area of research. It has paved a path not only for theoretical improvements, but also algorithmic designing aspects. In this book we focus mainly on convex analysis and its application to the development of convex optimization theory.

1.2 Basic Concepts

By *convex optimization* we simply mean the problem of minimizing a convex function over a convex set. More precisely, we are concerned with the following problem

$$\min\ f(x) \quad \text{subject to} \quad x \in C, \tag{CP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $C \subset \mathbb{R}^n$ is a convex set. Of course in most cases the the set C is described by a system of convex inequalities and affine equalities. Thus we can write

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m \ \text{ and } \\ h_j(x) = 0,\ j = 1, 2, \ldots, l\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$ are convex functions and $h_j : \mathbb{R}^n \to \mathbb{R}$,

$j = 1, 2, \ldots, l$ are affine functions. When C is expressed explicitly as above, (CP) is called the *convex programming problem.*

A set $C \subset \mathbb{R}^n$ is a *convex set* if for any $x, y \in \mathbb{R}^n$, the line segment joining them, that is

$$[x, y] = \{z \in \mathbb{R}^n : z = (1-\lambda)x + \lambda y,\ 0 \leq \lambda \leq 1\},$$

is also in C. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is a *convex function* if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y),$$

while it is an *affine function* if it is a translate of a linear function; that is, ϕ is affine if

$$\phi(x) = \langle a, x \rangle + b,$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

It is important to note at the very outset that in optimization theory it is worthwhile to consider extended-valued functions, that is, functions that take values in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The need to do so arises when we seek to convert a constrained optimization problem into an unconstrained one. Consider for example the problem (CP), which can be restated as

$$\min\ f_0(x) \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

where

$$f_0(x) = \begin{cases} f(x), & x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

All the modern books on convex analysis beginning with the classic *Convex Analysis* by Rockafellar [97] follow this framework. However, when we include infinities, we need to know how to deal with them. Most rules with infinity are intuitively clear except possibly $0 \times (+\infty)$ and $\infty - \infty$. Because we will be dealing mainly with minimization problems, we will follow the convention $0 \times (+\infty) = (+\infty) \times 0 = 0$ and $\infty - \infty = \infty$. This convention was adopted in Rockafellar and Wets [101] and we shall follow it. However, we would like to ascertain that we really need not get worried about $\infty - \infty$ as the functions considered in this book are real-valued or proper functions. An extended-valued function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be a *proper function* if $\phi(x) > -\infty$ for every $x \in \mathbb{R}^n$ and *dom* ϕ is nonempty where *dom* $\phi = \{x \in \mathbb{R}^n : \phi(x) < +\infty\}$ is the *domain* of ϕ.

It is worthwhile to note that the definition of a convex function given above can be extended to the case when ϕ is an extended-valued function. An extended-valued function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a *convex function* if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y),$$

with the convention that $\infty - \infty = +\infty$. A better way to handle the convexity of an extended-valued convex function is to use its associated geometry. In this direction we describe the *epigraph* of a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$, which is given as

$$epi\ \phi = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \phi(x) \leq \alpha\}.$$

A function is said to be convex if the epigraph is convex. We leave it as a simple exercise for the reader to show that if the epigraph of a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is convex in $\mathbb{R}^n \times \mathbb{R}$, then ϕ is a convex function over $\mathbb{R}^n$. For more details see Chapter 2.

In case of extended-valued functions, one can work with the semicontinuity of the functions rather than the continuity. Before we define those notions, we present certain notations that will be used throughout.

For any two sets $F_1, F_2 \subset \mathbb{R}^n$, define

$$F_1 + F_2 = \{x_1 + x_2 \in \mathbb{R}^n : x_1 \in F_1,\ x_2 \in F_2\}.$$

For any set $F \subset \mathbb{R}^n$ and any scalar $\lambda \in \mathbb{R}$,

$$\lambda F = \{\lambda x \in \mathbb{R}^n : x \in F\}.$$

The *closure* of a set F is denoted by $cl\ F$ while the *interior* is given by $int\ F$.

The *open unit ball*, or simply *unit ball*, is denoted by $\mathbb{B}$. By $\mathbb{B}_\delta(\bar{x})$ we mean an open ball of radius $\delta > 0$ with center at $\bar{x}$. Explicitly,

$$\mathbb{B}_\delta(\bar{x}) = \bar{x} + \delta\mathbb{B}.$$

For vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, the *inner product* of x and y is denoted by $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ while the *norm* of x is given by $\|x\| = \sqrt{\langle x, x\rangle}$. We state a standard result on the norm.

Proposition 1.1 *(Cauchy–Schwarz Inequality) For any two vectors* $x, y \in \mathbb{R}^n$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

The above inequality holds as equality if and only if $x = \alpha y$ *for some scalar* $\alpha \in \mathbb{R}$.

To discuss the concept of continuities of a function, we shall consider the notions of limit infimum and limit supremum of a function. But first we discuss the convergence of sequences in $\mathbb{R}^n$.

Definition 1.2 A sequence $\{x_k \in \mathbb{R} : k = 1, 2, \ldots\}$ or simply $\{x_k\} \subset \mathbb{R}$ is said to *converge* to $\bar{x} \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists k_ε such that

$$|x_k - \bar{x}| < \varepsilon,\ \forall\ k \geq k_\varepsilon.$$

A sequence $\{x_k\} \subset \mathbb{R}^n$ *converges* to $\bar{x} \in \mathbb{R}^n$ if the i-th component of x_k

converges to the i-th component of $\bar{x}$. The vector $\bar{x}$ is called the *limit* of $\{x_k\}$. Symbolically it is expressed as

$$x_k \to \bar{x} \qquad \text{or} \qquad \lim_{k\to\infty} x_k = \bar{x}.$$

The sequence $\{x_k\} \subset \mathbb{R}^n$ is *bounded* if each of its components is bounded. Equivalently, $\{x_k\}$ is bounded if and only if there exists $M \in \mathbb{R}$ such that $\|x_k\| \leq M$ for every $k \in \mathbb{N}$. A *subsequence* of $\{x_k\} \subset \mathbb{R}^n$ is a sequence $\{x_{k_j}\}$, $j = 1, 2, \ldots$, where each x_{k_j} is a member of the original sequence and the order of the elements as in the original sequence is maintained. A vector $\bar{x} \in \mathbb{R}^n$ is a *limit point* of $\{x_k\} \subset \mathbb{R}^n$ if there exists a subsequence of $\{x_k\}$ converging to $\bar{x}$. If the limit point is unique, it is the limit of $\{x_k\}$. Next we state the classical result on the bounded sequences.

Proposition 1.3 *(Bolzano–Weierstrass Theorem) A bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence.*

For a sequence $\{x_k\} \subset \mathbb{R}$, define

$$z_r = \inf\{x_k : k \geq r\} \quad \text{and} \quad y_r = \sup\{x_k : k \geq r\}.$$

It is obvious that the sequences $\{z_r\}$ and $\{y_r\}$ are nondecreasing and nonincreasing, respectively. If $\{x_k\}$ is bounded below or bounded above, the sequences $\{z_r\}$ or $\{y_r\}$, respectively, have a limit. The limit of $\{z_r\}$ is called the *limit infimum* or *lower limit* of $\{x_k\}$ and denoted by $\liminf_{k\to\infty} x_k$, while that of $\{y_r\}$ is called the *limit supremum* or *upper limit* of $\{x_k\}$ and denoted by $\limsup_{k\to\infty} x_k$. Equivalently,

$$\liminf_{k\to\infty} x_k = \lim_{k\to\infty} \{\inf_{r\geq k} x_r\} \qquad \text{and} \qquad \limsup_{k\to\infty} x_k = \lim_{k\to\infty} \{\sup_{r\geq k} x_r\}.$$

For a sequence $\{x_k\}$, $\liminf_{k\to\infty} x_k = -\infty$ if the sequence is unbounded below while $\limsup_{k\to\infty} x_k = +\infty$ if the sequence is unbounded above. Therefore, $\{x_k\}$ converges to $\bar{x}$ if and only if

$$-\infty < \liminf_{k\to\infty} x_k = \bar{x} = \limsup_{k\to\infty} x_k < +\infty.$$

Now we move on to define the semicontinuities of a function that involve the limit infimum and limit supremum of the function.

Definition 1.4 A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be *lower semicontinuous* (*lsc*) at $\bar{x} \in \mathbb{R}^n$ if for every sequence $\{x_k\} \subset \mathbb{R}^n$ converging to $\bar{x}$,

$$\phi(\bar{x}) \leq \liminf_{k\to\infty} \phi(x_k).$$

Equivalently,

$$\phi(\bar{x}) \leq \liminf_{x\to\bar{x}} \phi(x),$$

where the term on the right-hand side of the inequality denotes the *limit infimum* or the *lower limit* of the function ϕ defined as

$$\liminf_{x \to \bar{x}} \phi(x) = \lim_{\delta \downarrow 0} \inf_{x \in \mathbb{B}_\delta(\bar{x})} \phi(x).$$

The function ϕ is lsc over a set $F \subset \mathbb{R}^n$ if ϕ is lsc at every $\bar{x} \in F$.

For a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$,

$$\inf_{x \in \mathbb{B}_\delta(\bar{x})} \phi(x) \leq \phi(\bar{x}).$$

Taking the limit as $\delta \downarrow 0$ in the above inequality leads to

$$\liminf_{x \to \bar{x}} \phi(x) \leq \phi(\bar{x}).$$

Thus, the inequality in the above definition of lsc can be replaced by an equality, that is, $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is lsc at $\bar{x}$ if

$$\phi(\bar{x}) = \liminf_{x \to \bar{x}} \phi(x).$$

Similar to the concept of lower semicontinuity and limit infimum, we next define the upper semicontinuity and the limit supremum of a function.

Definition 1.5 A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be *upper semicontinuous* (*usc*) at $\bar{x} \in \mathbb{R}^n$ if for every sequence $\{x_k\} \subset \mathbb{R}^n$ converging to $\bar{x}$,

$$\phi(\bar{x}) \geq \limsup_{k \to \infty} \phi(x_k).$$

Equivalently,

$$\phi(\bar{x}) \geq \limsup_{x \to \bar{x}} \phi(x),$$

where the term on the right-hand side of the inequality denotes the *limit supremum* or the *upper limit* of the function ϕ defined as

$$\limsup_{x \to \bar{x}} \phi(x) = \lim_{\delta \downarrow 0} \sup_{x \in \mathbb{B}_\delta(\bar{x})} \phi(x).$$

The function ϕ is usc over a set $F \subset \mathbb{R}^n$ if ϕ is usc at every $\bar{x} \in F$.

Definition 1.6 A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be *continuous* at $\bar{x}$ if it is lsc as well as usc at $\bar{x}$, that is,

$$\lim_{x \to \bar{x}} \phi(x) = \phi(\bar{x}).$$

Alternatively, ϕ is continuous at $\bar{x}$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon, \bar{x}) > 0$ such that

$$|\phi(x) - \phi(\bar{x})| \leq \varepsilon \quad \text{whenever} \quad \|x - \bar{x}\| < \delta(\varepsilon, \bar{x}).$$

The function ϕ is continuous over a set $F \subset \mathbb{R}^n$ if ϕ is continuous at every $\bar{x} \in F$.

Because we will be considering minimization problems, the continuity of a function will be replaced by lower semicontinuity. Before moving on, we state a result on the infimum and supremum operations from Rockafellar and Wets [101].

Proposition 1.7 *(i) Consider an extended-valued function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and sets $F_i \subset \mathbb{R}^n,\ i = 1, 2$ such that $F_1 \subset F_2$. Then*

$$\inf_{x_1 \in F_1} \phi(x_1) \geq \inf_{x_2 \in F_2} \phi(x_2) \quad \text{and} \quad \sup_{x_1 \in F_1} \phi(x_1) \leq \sup_{x_2 \in F_2} \phi(x_2).$$

(ii) Consider the functions $\phi_1,\ \phi_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$ and a set $F \subset \mathbb{R}^n$. Then

$$\begin{aligned} \inf_{x \in F} \phi_1(x) + \inf_{x \in F} \phi_2(x) \ &\leq \ \inf_{x \in F} (\phi_1 + \phi_2)(x) \\ &\leq \ \sup_{x \in F} (\phi_1 + \phi_2)(x) \ \leq \ \sup_{x \in F} \phi_1(x) + \sup_{x \in F} \phi_2(x). \end{aligned}$$

Also, for functions $\phi_i : \mathbb{R}^{n_i} \to \bar{\mathbb{R}}$ and sets $F_i \subset \mathbb{R}^{n_i},\ i = 1, 2,$

$$\begin{aligned} \inf_{x_1 \in F_1} \phi_1(x_1) + \inf_{x_2 \in F_2} \phi_2(x_2) &= \inf_{(x_1, x_2) \in F_1 \times F_2} (\phi_1(x_1) + \phi_2(x_2)), \\ \sup_{x_1 \in F_1} \phi_1(x_1) + \sup_{x_2 \in F_2} \phi_2(x_2) &= \sup_{(x_1, x_2) \in F_1 \times F_2} (\phi_1(x_1) + \phi_2(x_2)). \end{aligned}$$

(iii) Consider an extended-valued function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and a set $F \subset \mathbb{R}^n$. Then for $\lambda \geq 0$,

$$\inf_{x \in F} (\lambda \phi)(x) = \lambda \inf_{x \in F} \phi(x) \quad \text{and} \quad \sup_{x \in F} (\lambda \phi)(x) = \lambda \sup_{x \in F} \phi(x),$$

provided $0 \times (+\infty) = 0 = 0 \times (-\infty)$.

The next result from Rockafellar and Wets [101] gives a characterization of limit infimum of an arbitrary extended-valued function.

Lemma 1.8 *For an extended-valued function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$,*

$$\liminf_{x \to \bar{x}} \phi(x) = \min\{\alpha \in \bar{\mathbb{R}} : \text{there exists} \ \ x_k \to \bar{x} \ \ \text{satisfying} \ \ \phi(x_k) \to \alpha\}.$$

Proof. Suppose that $\liminf_{x \to \bar{x}} \phi(x) = \bar{\alpha}$. We claim that for $x_k \to \bar{x}$ with $\phi(x_k) \to \alpha$, $\alpha \geq \bar{\alpha}$. As $x_k \to \bar{x}$, for any $\delta > 0$, there exists $k_\delta \in \mathbb{N}$ such that $x_k \in \mathbb{B}_\delta(\bar{x})$ for every $k \geq k_\delta$. Therefore,

$$\phi(x_k) \geq \inf_{x \in \mathbb{B}_\delta(\bar{x})} \phi(x).$$

Taking the limit as $k \to +\infty$ in the above inequality,

$$\alpha \geq \inf_{x \in \mathbb{B}_\delta(\bar{x})} \phi(x), \ \forall\ \delta > 0.$$

Because δ is arbitrarily chosen, so taking the limit $\delta \downarrow 0$ along with the definition of the limit infimum of ϕ leads to

$$\alpha \geq \liminf_{x \to \bar{x}} \phi(x),$$

that is, $\alpha \geq \bar{\alpha}$.

To prove the result, we shall show that there exists a sequence $x_k \to \bar{x}$ such that $\phi(x_k) \to \bar{\alpha}$. For a nonnegative sequence $\{\delta_k\}$, define

$$\bar{\alpha}_k = \inf_{x \in \mathbb{B}_{\delta_k}(\bar{x})} \phi(x).$$

As $\delta_k \to 0$, by Definition 1.4 of limit infimum, $\bar{\alpha}_k \to \bar{\alpha}$. Now for every $k \in \mathbb{N}$, by the definition of infimum it is possible to find $x_k \in \mathbb{B}_{\delta_k}(\bar{x})$ for which $\phi(x_k)$ is very close to $\bar{\alpha}_k$, that is, in an interval $[\bar{\alpha}_k, \alpha_k]$ where $\bar{\alpha}_k < \alpha_k$ and $\alpha_k \to \bar{\alpha}$. Therefore, as $k \to +\infty$, $x_k \to \bar{x}$, and $\phi(x_k) \to \bar{\alpha}$, thereby establishing the result. □

After the characterization of limit infimum of a function, the result below gives an equivalent characterization of lower semicontinuity of the function in terms of the epigraph and lower level set.

Theorem 1.9 *Consider a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then the following conditions are equivalent:*

(i) ϕ is lsc over $\mathbb{R}^n$.

(ii) The epigraph of ϕ, epi ϕ, is a closed set in $\mathbb{R}^n \times \mathbb{R}$.

(iii) The lower-level set $lev_{\leq \alpha}\phi = \{x \in \mathbb{R}^n : \phi(x) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

Proof. If $\phi \equiv \infty$, the result holds trivially. So assume that *dom* ϕ is nonempty and thus *epi* ϕ and $lev_{\leq \alpha}\phi$ are nonempty.

We will first show that (i) implies (ii). Consider a sequence $\{(x_k, \alpha_k)\} \subset$ *epi* ϕ such that $(x_k, \alpha_k) \to (\bar{x}, \bar{\alpha})$. Therefore, $\phi(x_k) \leq \alpha_k$, which implies that

$$\liminf_{x \to \bar{x}} \phi(x) \leq \liminf_{k \to \infty} \phi(x_k) \leq \bar{\alpha}.$$

By the lower semicontinuity of ϕ,

$$\phi(\bar{x}) = \liminf_{x \to \bar{x}} \phi(x),$$

which reduces the preceding condition to $\phi(\bar{x}) \leq \bar{\alpha}$, thereby proving that *epi* ϕ is a closed set in $\mathbb{R}^n \times \mathbb{R}$.

Next we show that (ii) implies (iii). For a fixed $\alpha \in \mathbb{R}$, suppose that $\{x_k\} \subset lev_{\leq \alpha}\phi$ such that $x_k \to \bar{x}$. Therefore, $\phi(x_k) \leq \alpha$, that is, $(x_k, \alpha) \in$ *epi* ϕ. By (ii), *epi* ϕ is closed, which implies $(\bar{x}, \alpha) \in$ *epi* ϕ, that is, $\phi(\bar{x}) \leq \alpha$. Thus, $\bar{x} \in lev_{\leq \alpha}\phi$, thereby yielding condition (iii).

Finally, to obtain the equivalence, we will establish that (iii) implies (i). To show that ϕ is lsc, we need to show that for every $\bar{x} \in \mathbb{R}^n$,

$$\phi(\bar{x}) \leq \liminf_{k\to\infty} \phi(x_k) \quad \text{whenever} \quad x_k \to \bar{x}.$$

On the contrary, assume that for some $\bar{x} \in \mathbb{R}^n$ and some sequence $x_k \to \bar{x}$,

$$\phi(\bar{x}) > \liminf_{k\to\infty} \phi(x_k),$$

which implies there exists $\alpha \in \mathbb{R}$ such that

$$\phi(\bar{x}) > \alpha > \liminf_{k\to\infty} \phi(x_k). \tag{1.1}$$

Thus, there exists a subsequence, without relabeling, say $\{x_k\}$ such that $\phi(x_k) \leq \alpha$ for every $k \in \mathbb{N}$, which implies $x_k \in lev_{\leq\alpha}\phi$. By (iii), the lower level set $lev_{\leq\alpha}\phi$ is closed and hence $\bar{x} \in lev_{\leq\alpha}\phi$, that is, $\phi(\bar{x}) \leq \alpha$, which contradicts (1.1). Therefore, ϕ is lsc over $\mathbb{R}^n$. □

The proof of the last implication, that is, (iii) implies (i) of Theorem 1.9 by contradiction was from Bertsekas [12]. We present an alternative proof for the same from Rockafellar and Wets [101].

It is obvious that for any $\bar{x} \in \mathbb{R}^n$,

$$\bar{\alpha} = \liminf_{x\to\bar{x}} \phi(x) \leq \phi(\bar{x}).$$

Therefore, to establish the lower semicontinuity of ϕ at $\bar{x}$, we need to prove that $\phi(\bar{x}) \leq \bar{\alpha}$. By Lemma 1.8, there exists a sequence $\{x_k\} \subset \mathbb{R}^n$ with $x_k \to \bar{x}$ such that $\phi(x_k) \to \bar{\alpha}$. Thus, for every $\alpha > \bar{\alpha}$, $\phi(x_k) \leq \alpha$, which implies $x_k \in lev_{\leq\alpha}\phi$. Now if condition (iii) of the above theorem holds, that is, $lev_{\leq\alpha}\phi$ is closed in $\mathbb{R}^n$,

$$\bar{x} \in lev_{\leq\alpha}\phi, \ \forall\ \alpha > \bar{\alpha}.$$

Thus, $\phi(\bar{x}) \leq \alpha$, which leads to $\phi(\bar{x}) \leq \bar{\alpha}$. Because $\bar{x} \in \mathbb{R}^n$ was arbitrarily chosen, ϕ is lsc over $\mathbb{R}^n$.

Theorem 1.9 gives equivalent characterization of lower semicontinuity of a function. But if the function is not lsc, its epigraph is not closed. The result below gives an equivalent characterization of the closure of the epigraph of any arbitrary function.

Proposition 1.10 *For any arbitrary extended-valued function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$, $(\bar{x}, \bar{\alpha}) \in cl\ epi\ \phi$ if and only if*

$$\liminf_{x\to\bar{x}} \phi(x) \leq \bar{\alpha}.$$

Proof. Suppose that $(\bar{x}, \bar{\alpha}) \in cl\ epi\ \phi$, which implies that there exists $\{(x_k, \alpha_k)\} \subset epi\ \phi$ such that $(x_k, \alpha_k) \to (\bar{x}, \bar{\alpha})$. Thus, taking the limit as $k \to +\infty$, the condition

$$\liminf_{x \to \bar{x}} \phi(x) \leq \liminf_{x_k \to \bar{x}} \phi(x_k)$$

yields

$$\liminf_{x \to \bar{x}} \phi(x) \leq \bar{\alpha},$$

as desired.

Conversely, assume that $\liminf_{x \to \bar{x}} \phi(x) \leq \bar{\alpha}$ but $(\bar{x}, \bar{\alpha}) \notin cl\ epi\ \phi$. We claim that, $\liminf_{x \to \bar{x}} \phi(x) = \bar{\alpha}$. On the contrary, suppose that $\liminf_{x \to \bar{x}} \phi(x) < \bar{\alpha}$. As $(\bar{x}, \bar{\alpha}) \notin cl\ epi\ \phi$, there exists $\bar{\delta} > 0$ such that for every $\delta \in (0, \bar{\delta})$,

$$\mathbb{B}_\delta((\bar{x}, \bar{\alpha})) \cap cl\ epi\ \phi = \emptyset,$$

which implies for every $(x, \alpha) \in \mathbb{B}_\delta((\bar{x}, \bar{\alpha}))$, $\phi(x) > \alpha$. In particular for $(x, \bar{\alpha}) \in \mathbb{B}_\delta((\bar{x}, \bar{\alpha}))$, $\phi(x) > \bar{\alpha}$, that is,

$$\phi(x) > \bar{\alpha}, \ \forall\ x \in \mathbb{B}_\delta(\bar{x}).$$

Therefore, taking the limit as $\delta \to 0$ along with the definition of limit infimum of a function yields

$$\liminf_{x \to \bar{x}} \phi(x) \geq \bar{\alpha},$$

which is a contradiction. Therefore, $\liminf_{x \to \bar{x}} \phi(x) = \bar{\alpha}$. By Lemma 1.8, there exists a sequence $x_k \to \bar{x}$ such that $\phi(x_k) \to \bar{\alpha}$. Because $(x_k, \phi(x_k)) \in epi\ \phi$, $(\bar{x}, \bar{\alpha}) \in cl\ epi\ \phi$, thereby reaching a contradiction and hence the result. $\square$

Now the question is whether it is possible to construct a function that is the closure of the epigraph of another function. This leads to the concept of closure of a function.

Definition 1.11 For any function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$, an lsc function that is constructed in such a way that its epigraph is the closure of the epigraph of ϕ is called the *lower semicontinuous hull* or the *closure* of the function ϕ and is denoted by $cl\ \phi$. Therefore,

$$epi(cl\ \phi) = cl\ epi\ \phi.$$

Equivalently, the closure of ϕ is defined as

$$cl\ \phi(\bar{x}) = \liminf_{x \to \bar{x}} \phi(x), \ \forall\ \bar{x} \in \mathbb{R}^n.$$

By Proposition 1.10, it is obvious that $(\bar{x}, \bar{\alpha}) \in cl\ epi\ \phi$ if and only if $(\bar{x}, \bar{\alpha}) \in epi\ cl\ \phi$. The function ϕ is said to be *closed* if $cl\ \phi = \phi$.

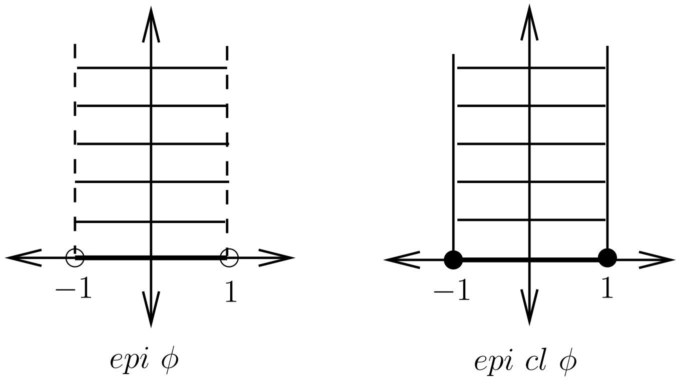

FIGURE 1.1: Lower semicontinuous hull.

If ϕ is lsc, then it is closed as well. Also $cl\ \phi$ is lsc and the greatest of all lsc functions ψ such that $\psi(x) \leq \phi(x)$ for every $x \in \mathbb{R}^n$. From Theorem 1.9, one has that *closedness is the same as lower semicontinuity over* $\mathbb{R}^n$. In this discussion, the function ϕ was defined over $\mathbb{R}^n$. But what if ϕ is defined over some subset of $\mathbb{R}^n$. Then one cannot talk about the lower semicontinuity of the function over $\mathbb{R}^n$. In such a case, how is the closedness of a function related to lower semicontinuity? This issue was addressed by Bertsekas [12]. Consider a set $F \subset \mathbb{R}^n$ and a function $\phi : F \to \bar{\mathbb{R}}$. Observe that here we define ϕ over the set F and not $\mathbb{R}^n$. The function ϕ can be extended over $\mathbb{R}^n$ by defining a function $\bar{\phi} : \mathbb{R}^n \to \bar{\mathbb{R}}$ as

$$\bar{\phi}(x) = \begin{cases} \phi(x), & x \in F, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that both the extended-valued functions ϕ and $\bar{\phi}$ have the same epigraph. Thus from the above discussion, one has ϕ *is closed if and only if* $\bar{\phi}$ *is lsc over* $\mathbb{R}^n$. Also observe that the lower semicontinuity of ϕ over *dom* ϕ is not sufficient for ϕ to be closed. In addition, one has to assume the closedness of *dom* ϕ. To emphasize this fact, let us consider a simple example. Consider $\phi : \mathbb{R} \to \bar{\mathbb{R}}$ defined as

$$\phi(x) = \begin{cases} 0, & x \in (-1, 1), \\ +\infty, & \text{otherwise.} \end{cases}$$

Here, *dom* $\phi = (-1, 1)$ over which the function is lsc but *epi* ϕ is not closed and hence, ϕ is not closed. The closure of ϕ is given by

$$cl\ \phi(x) = \begin{cases} 0, & x \in [-1, 1], \\ +\infty, & \text{otherwise.} \end{cases}$$

Observe that in Figure 1.1, *epi* ϕ is not closed while *epi cl* ϕ is closed. Therefore, we have the following result from Bertsekas [12].

Proposition 1.12 *Consider $F \subset \mathbb{R}^n$ and a function $\phi : F \to \bar{\mathbb{R}}$. If dom ϕ is closed and ϕ is lsc over dom ϕ, then ϕ is closed.*

Because we are interested in studying the minimization problem, it is important to know whether a minimizer exists or not. In this respect, we have the classical Weierstrass theorem, according to which "*A continuous function attains its minimum over a compact set.*" For a more general version of this theorem from Bertsekas [12], we require the notion of coercivity.

Definition 1.13 A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be *coercive* over a set $F \subset \mathbb{R}^n$ if for every sequence $\{x_k\} \subset F$

$$\lim_{k\to\infty} \phi(x_k) = +\infty \quad \text{whenever} \quad \|x_k\| \to +\infty.$$

For $F = \mathbb{R}^n$, ϕ is simply called *coercive.*

Observe that for a coercive function, every nonempty lower level set is bounded. Below we prove the Weierstrass Theorem.

Theorem 1.14 *(Weierstrass Theorem) Consider a proper lsc function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and assume that one of the following holds:*

(i) dom ϕ is bounded.

(ii) there exists $\alpha \in \mathbb{R}$ such that the lower level set $lev_{\leq\alpha}\phi$ is nonempty and bounded.

(iii) ϕ is coercive.

Then the set of minimizers of ϕ over $\mathbb{R}^n$ is nonempty and compact.

Proof. Suppose that condition (i) holds, that is, *dom* ϕ is bounded. Because ϕ is proper, $\phi(x) > -\infty$ for every $x \in \mathbb{R}^n$ and *dom* ϕ is nonempty. Denote $\phi_{inf} = \inf_{x\in\mathbb{R}^n} \phi(x)$, which implies $\phi_{inf} = \inf_{x\in dom\ \phi} \phi(x)$. Therefore, there exists a sequence $\{x_k\} \subset dom\ \phi$ such that $\phi(x_k) \to \phi_{inf}$. Because *dom* ϕ is bounded, $\{x_k\}$ is a bounded sequence, which by Bolzano–Weierstrass Theorem, Proposition 1.3, has a convergent subsequence. Without loss of generality, assume that $x_k \to \bar{x}$. By the lower semicontinuity of ϕ,

$$\phi(\bar{x}) \leq \liminf_{k\to\infty} \phi(x_k) = \lim_{k\to\infty} \phi(x_k) = \phi_{inf},$$

which implies that $\bar{x}$ is a point of minimizer of ϕ over $\mathbb{R}^n$. Denote the set of minimizers by $\mathcal{S}$. Therefore, $\bar{x} \in \mathcal{S}$ and hence $\mathcal{S}$ is nonempty. Because $\mathcal{S} \subset dom\ \phi$ which is bounded, $\mathcal{S}$ is a bounded set. Also, $\mathcal{S}$ is the intersection of the lower level sets $lev_{\leq\alpha}\phi$, where $\alpha > m$. For an lsc function ϕ, $lev_{\leq\alpha}\phi$ is closed by Theorem 1.9 and thus $\mathcal{S}$ is closed. Hence $\mathcal{S}$ is compact.

Assume that condition (ii) holds; that is, for some $\alpha \in \mathbb{R}$, the lower level set $lev_{\leq\alpha}\phi$ is nonempty and bounded. Consider a proper function $\bar{\phi} : \mathbb{R}^n \to \bar{\mathbb{R}}$ defined as

$$\bar{\phi}(x) = \begin{cases} \phi(x), & \phi(x) \leq \alpha, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore, *dom* $\bar{\phi} = lev_{\leq\alpha}\phi$ which is nonempty and bounded by condition (ii). Since ϕ is lsc which by Theorem 1.9 implies that *dom* $\bar{\phi}$ is closed. Also by the lower semicontinuity of ϕ along with Proposition 1.12, $\bar{\phi}$ is closed and hence lsc. Moreover, the set of minimizers of $\bar{\phi}$ is the same as that of ϕ. The result follows by applying condition (i) to $\bar{\phi}$.

Suppose that condition (iii) is satisfied, that is, ϕ is coercive. Because ϕ is proper, *dom* ϕ is nonempty and thus has a nonempty lower level set. By the coercivity of ϕ, it is obvious that the nonempty lower level sets of ϕ are bounded, thereby satisfying condition (ii), and therefore leading to the desired result. □

As we all know, the next concept that comes to mind after limit and continuity is the derivative of a function. Below we define this very notion.

Definition 1.15 For a scalar-valued function $\phi : \mathbb{R}^n \to \mathbb{R}$, the *derivative* of ϕ at $\bar{x}$ is denoted by $\nabla\phi(\bar{x}) \in \mathbb{R}^n$ and is defined as

$$\lim_{\|h\|\to 0} \frac{\phi(\bar{x}+h) - \phi(\bar{x}) - \langle \nabla\phi(\bar{x}), h\rangle}{\|h\|} = 0.$$

Equivalently, the derivative can also be expressed as

$$\phi(x) = \phi(\bar{x}) + \langle \nabla\phi(\bar{x}), x - \bar{x}\rangle + o(\|x - \bar{x}\|),$$

where $\lim_{x\to\bar{x}} \dfrac{o(\|x-\bar{x})\|}{\|x-\bar{x}\|} = 0$. A function ϕ is differentiable if it is differentiable at every $x \in \mathbb{R}^n$. The derivative, $\nabla\phi(\bar{x})$, of ϕ at $\bar{x}$ is also called the *gradient* of ϕ at $\bar{x}$, which can be expressed as

$$\nabla\phi(\bar{x}) = \left(\frac{\partial\phi}{\partial x_1}(\bar{x}), \frac{\partial\phi}{\partial x_2}(\bar{x}), \ldots, \frac{\partial\phi}{\partial x_n}(\bar{x})\right),$$

where $\dfrac{\partial\phi}{\partial x_i}$, $i = 1, 2, \ldots, n$ denotes the i-th partial derivative of ϕ. If ϕ is continuously differentiable, that is, the map $x \mapsto \nabla\phi(x)$ is continuous over $\mathbb{R}^n$, then ϕ is called a *smooth function*. If ϕ is not smooth, it is called a *nonsmooth function*.

Similar to the first-order differentiability, we have the second-order differentiability notion as follows.

Definition 1.16 For a scalar-valued function $\phi : \mathbb{R}^n \to \mathbb{R}$, the *second-order derivative* of ϕ at $\bar{x}$ is denoted by $\nabla^2\phi(\bar{x}) \in \mathbb{R}^{n\times n}$ and is defined as

$$\lim_{\|h\|\to 0} \frac{\phi(\bar{x}+h) - \phi(\bar{x}) - \langle \nabla\phi(\bar{x}), h\rangle - \dfrac{1}{2}\langle \nabla^2\phi(\bar{x})h, h\rangle}{\|h\|^2} = 0,$$

which is equivalent to

$$\phi(x) = \phi(\bar{x}) + \langle \nabla\phi(\bar{x}), x-\bar{x}\rangle + \langle \nabla^2\phi(\bar{x})(x-\bar{x}), x-\bar{x}\rangle + o(\|x-\bar{x}\|^2).$$

The matrix $\nabla^2\phi(\bar{x})$ is also referred to as the *Hessian* with the ij-th entry of the matrix being the second-order partial derivative $\dfrac{\partial^2\phi}{\partial x_i \partial x_j}(\bar{x})$. If ϕ is twice continuously differentiable, then the matrix $\nabla^2\phi(\bar{x})$ is a symmetric matrix.

In the above definitions we considered the function ϕ to be a scalar-valued function. Next we define the notion of differentiability for a vector-valued function Φ.

Definition 1.17 For a vector-valued function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, the *derivative* of Φ at $\bar{x}$ is denoted by $J\Phi(\bar{x}) \in \mathbb{R}^{m\times n}$ and is defined as

$$\lim_{\|h\|\to 0} \frac{\|\Phi(\bar{x}+h) - \Phi(\bar{x}) - \langle J\Phi(\bar{x}), h\rangle\|}{\|h\|} = 0.$$

The matrix $J\Phi(\bar{x})$ is also called the *Jacobian* of Φ at $\bar{x}$. If $\Phi = (\phi_1, \phi_2, \ldots, \phi_m)$, Φ is differentiable if each $\phi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ is differentiable. The Jacobian of Φ at $\bar{x}$ can be expressed as

$$J\Phi(\bar{x}) = \begin{bmatrix} \nabla\phi_1(\bar{x}) \\ \nabla\phi_2(\bar{x}) \\ \vdots \\ \nabla\phi_m(\bar{x}) \end{bmatrix}$$

with the ij-th entry of the matrix being the partial derivative $\dfrac{\partial\phi_i}{\partial x_j}(\bar{x})$. In the above expression of $J\Phi(\bar{x})$, the vectors $\nabla\phi_1(\bar{x})$, $\nabla\phi_2(\bar{x}), \ldots, \nabla\phi_m(\bar{x})$ are written as row vectors.

Observe that the derivative is a local concept and it is defined at a point x if $x \in$ *int dom* ϕ. Below we state the Mean Value Theorem, which plays a pivotal role in the study of optimality conditions.

Theorem 1.18 *(Mean Value Theorem) Consider a continuously differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$. Then for every $x, y \in \mathbb{R}^n$, there exists $z \in [x, y]$ such that*

$$\phi(y) - \phi(x) = \langle \nabla\phi(z), y - x\rangle.$$

With all these basic concepts we now move on to the study of convexity. The importance of convexity in optimization stems from the fact that whenever we minimize a convex function over a convex set, every local minimum is a global minimum. Many other issues in optimization depend on convexity. However, convex functions suffer from the drawback that they need not be differentiable at every point of their domain of definition and the nondifferentiability may be precisely at the point where the minimum is achieved. For instance, consider the minimization of the absolute value function, $|x|$, over $\mathbb{R}$. At the point of minima, $\bar{x} = 0$, the function is nondifferentiable. How this major difficulty was overcome by the development of a completely different type of analysis is possibly one of the most thrilling developments in optimization theory. This analysis depends on set-valued maps, which we briefly present below.

Definition 1.19 A *set-valued map* Φ from $\mathbb{R}^n$ to $\mathbb{R}^m$ associates every $x \in \mathbb{R}^n$ to a set in $\mathbb{R}^m$; that is, for every $x \in \mathbb{R}^n$, $\Phi(x) \subset \mathbb{R}^m$. Symbolically it is expressed as $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. A set-valued map is associated with its *graph* defined as

$$gph\ \Phi = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in \Phi(x)\}.$$

Φ is said to be a *proper map* if there exists $x \in \mathbb{R}^n$ such that $\Phi(x) \neq \emptyset$. Φ is said to be *closed-valued* or *convex-valued* or *bounded-valued* if for every $x \in \mathbb{R}^n$, the sets $\Phi(x)$ are closed or convex or bounded, respectively. Φ is *locally bounded* at $\bar{x} \in \mathbb{R}^n$ if there exists $\delta > 0$ and a bounded set $\mathcal{F} \subset \mathbb{R}^n$ such that

$$\Phi(x) \subset V, \ \forall\ x \in \mathbb{B}_\delta(\bar{x}).$$

The set-valued map Φ is said to be *closed* if it has a closed graph; that is, for any sequence $\{x_k\} \subset \mathbb{R}^n$ with $x_k \to \bar{x}$ and $y_k \in \Phi(x_k)$ with $y_k \to \bar{y}$, $\bar{y} \in \Phi(\bar{x})$. A set-valued map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *upper semicontinuous* (*usc*) at $\bar{x} \in \mathbb{R}^n$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Phi(x) \subset \Phi(\bar{x}) + \varepsilon\mathbb{B}, \ \forall\ x \in \mathbb{B}_\delta(\bar{x}),$$

where the balls are in the respective spaces. If Φ is locally bounded and has a closed graph, then it is usc. If Φ is single-valued, that is, $\Phi(x)$ is singleton for every x, the upper semicontinuity of Φ coincides with continuity.

For more on set-valued maps, the readers may refer to Berge [10]. A detailed analysis of convex function appears in Chapter 2.

1.3 Smooth Convex Optimization

Recall the convex optimization problem (CP) stated in Section 1.1, that is,

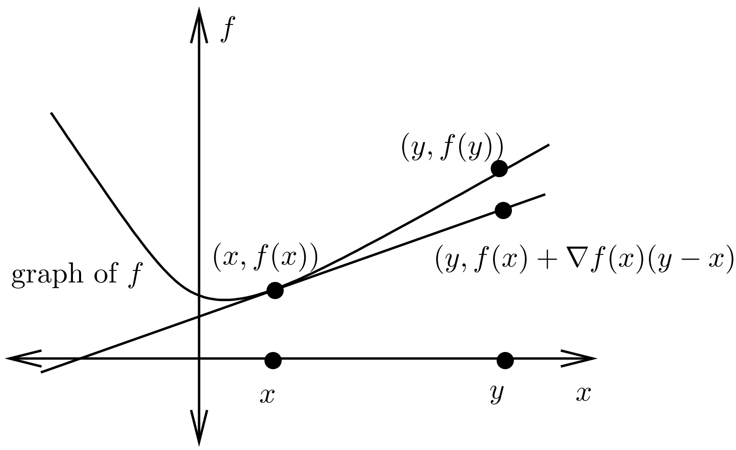

FIGURE 1.2: Graph of a real-valued differentiable convex function.

$$\min \; f(x) \quad \text{subject to } x \in C, \tag{CP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and C is a closed convex set in $\mathbb{R}^n$. Let us additionally assume that f is differentiable. It is mentioned in Chapter 2 that if f is differentiable, then for any $x \in \mathbb{R}^n$,

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle, \; \forall \; y \in \mathbb{R}^n.$$

Conversely, if the above relation holds for a function, then the function is convex. This fact appears as Theorem 2.81 in the next chapter. It is mentioned there as a consequence of more general facts. However, we provide a direct proof here.

Observe that if f is convex, then for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$,

$$(1 - \lambda) f(x) + \lambda f(y) \geq f(x + \lambda(y - x)).$$

Hence, for any $\lambda \in (0, 1)$,

$$f(y) - f(x) \geq \frac{f(x + \lambda(y - x)) - f(x)}{\lambda}.$$

Taking the limit as $\lambda \downarrow 0$, the above inequality yields

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle. \tag{1.2}$$

For the converse, suppose that (1.2) holds for any $x, y \in \mathbb{R}^n$. Setting $z = x + \lambda(y - x)$ with $\lambda \in (0, 1)$, then

$$f(y) - f(z) \geq \langle \nabla f(z), y - z \rangle \tag{1.3}$$

$$f(x) - f(z) \geq \langle \nabla f(z), x - z \rangle \tag{1.4}$$

The result is obtained by simply multiplying (1.3) with λ and (1.4) with $(1-\lambda)$ and then adding them up. This description geometrically means that the tangent plane should always lie below the graph of the function. For a convex function $f : \mathbb{R} \to \mathbb{R}$, it looks something like Figure 1.2. This important characterization of a convex function leads to the following result.

Theorem 1.20 *Consider the convex optimization problem (CP) where f is a differentiable convex function and C is a closed convex set in $\mathbb{R}^n$. Then $\bar{x}$ is a point of minimizer of (CP) if and only if*

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in C. \tag{1.5}$$

Proof. It is simple to see that as C is a convex set, for $x \in C$,

$$\bar{x} + \lambda(x - \bar{x}) \in C, \ \forall\, \lambda \in [0, 1].$$

Therefore, if $\bar{x}$ is a point of minimum,

$$f(\bar{x} + \lambda(x - \bar{x})) \geq f(\bar{x}),$$

that is,

$$f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x}) \geq 0.$$

Dividing both sides by $\lambda > 0$ and taking the limit as $\lambda \downarrow 0$ leads to

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0.$$

Because $x \in C$ was arbitrarily chosen,

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in C.$$

Also as f is convex, by the condition (1.2), for any $x \in C$,

$$f(x) - f(\bar{x}) \geq \langle \nabla f(\bar{x}), x - \bar{x} \rangle.$$

Now if (1.5) is satisfied, then the above inequality reduces to

$$f(x) \geq f(\bar{x}), \ \forall\, x \in C,$$

thereby proving the requisite result. $\square$

Remark 1.21 Expressing the optimality condition in the form of (1.5) leads to what is called a *variational inequality*. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a given function and C be a closed convex set in $\mathbb{R}^n$. Then the variational inequality $VI(F, C)$ is the problem of finding $\bar{x} \in C$ such that

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in C.$$

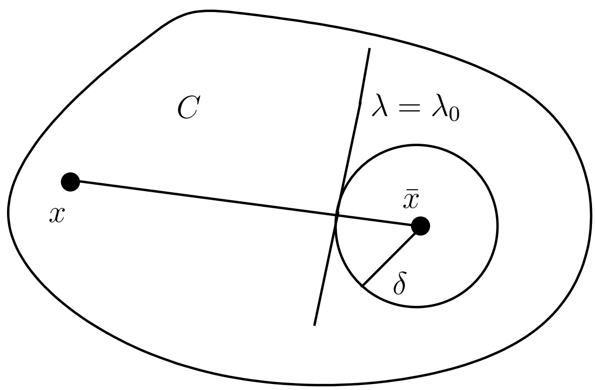

FIGURE 1.3: Local minimizer is global minimizer.

When f is a differentiable convex function, for $F = \nabla f$, $VI(\nabla f, C)$ is nothing but the condition (1.5). In order to solve $VI(F, C)$ efficiently, one needs an additional property on F which is monotonicity. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *monotone* if for any $x, y \in \mathbb{R}^n$,

$$\langle F(y) - F(x), y - x \rangle \geq 0.$$

However, when f is a convex function, one has the following pleasant property.

Theorem 1.22 *A differentiable function f is convex if and only if ∇f is monotone.*

For proof, see Rockafellar [97]. However, the reader should try to prove it on his/her own. We have shown that when (CP) has a smooth f, one can write down a necessary and sufficient condition for a point $\bar{x} \in C$ to be a global minimizer of (CP). In fact, as already mentioned, the importance of studying convexity in optimization stems from the following fact. For the problem (CP), every local minimizer is a global minimizer irrespective of the fact whether f is smooth or not. This can be proved in a simple way as follows. If $\bar{x}$ is a local minimizer of (CP), then there exists $\delta > 0$ such that

$$f(x) \geq f(\bar{x}), \ \forall \, x \in C \cap \mathbb{B}_\delta.$$

Now consider any $x \in C$. Then it is easy to observe from Figure 1.3 that there exists $\lambda_0 \in (0, 1)$ such that for every $\lambda \in (0, \lambda_0)$,

$$\lambda x + (1 - \lambda)\bar{x} \in C \cap \mathbb{B}_\delta.$$

Hence

$$f(\lambda x + (1 - \lambda)\bar{x}) \geq f(\bar{x}).$$

The convexity of f shows that

$$\lambda(f(x) - f(\bar{x})) \geq 0.$$

Because $\lambda > 0$, $f(x) \geq f(\bar{x})$. As $x \in C$ was arbitrary, our claim is established. The result can also be obtained using the approach of contradiction as done in Theorem 2.90.

Now consider the following function

$$\theta(x) = \sup_{y \in C} \langle \nabla f(x), x - y \rangle.$$

The interesting feature of the function is that

$$\theta(x) \geq 0, \ \forall \ x \in C$$

and if $\theta(x) = 0$ for $x \in C$, then x solves the problem (CP). Furthermore, if x solves the problem (CP), we have $\theta(x) = 0$. The function θ is usually called the *gap function* or the *merit function* associated with (CP). For the variational inequality problem, such a function was first introduced by Auslender [5]. The next question is how useful is the function θ to the problem (CP). What we will now show is that for certain classes of the problem (CP), the function θ can provide an *error bound* for the problem (CP). By an error bound we mean an upper estimate of the distance of a point in C to the solution set of (CP). The class of convex optimization problems where such a thing can be achieved is the class of *strongly convex optimization problems.* A function $f : \mathbb{R}^n \to \mathbb{R}$ is *strongly convex* with modulus of strong convexity $\rho > 0$ if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$(1 - \lambda) f(x) + \lambda f(y) \geq f(x + \lambda(y - x)) + \rho\lambda(1 - \lambda)\|y - x\|^2.$$

If f is differentiable, then f is strongly convex if and only if for any $x, y \in \mathbb{R}^n$,

$$f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle + \rho\|y - x\|^2.$$

Observe that $f(x) = \dfrac{1}{2}\langle x, Ax \rangle$, where $x \in \mathbb{R}^n$ and A is a positive definite $n \times n$ matrix, is strongly convex with $\rho = \lambda_{min}(A)$, the minimum eigenvalue of A while $f(x) = x$ with $x \in \mathbb{R}^n$ is not strongly convex.

If f is a twice continuously differentiable strongly convex function, then $\nabla^2 f(x)$ is always positive definite for each x. Now if f is strongly convex with modulus of convexity $\rho > 0$, then for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} f(y) - f(x) &\geq \langle \nabla f(x), y - x \rangle + \rho\|y - x\|^2, \\ f(x) - f(y) &\geq \langle \nabla f(y), x - y \rangle + \rho\|y - x\|^2. \end{aligned}$$

Adding the above inequalities leads to

$$0 \geq \langle \nabla f(x), y - x \rangle + \langle \nabla f(y), x - y \rangle + 2\rho\|y - x\|^2,$$

that is,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 2\rho \|y - x\|^2. \tag{1.6}$$

The property of ∇f given by (1.6) is called *strong monotonicity* with 2ρ as the modulus of monotonicity. It is in fact interesting to observe that if $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function for which there exists $\rho > 0$ such that for every $x, y \in \mathbb{R}^n$,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 2\rho \|y - x\|^2,$$

which implies that

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \rho \|y - x\|^2, \ \forall \ x, y \in \mathbb{R}^n.$$

Now we request the reader to show that f is strongly convex with modulus $\rho > 0$. In fact, if f is strongly convex with $\rho > 0$ one can also show that ∇f is strongly monotone with $\rho > 0$. Thus we conclude that f is strongly convex with modulus of strong convexity $\rho > 0$ if and only if ∇f is strongly monotone with modulus of monotonicity $\rho > 0$.

It is important to note that one cannot guarantee θ to be finite unless C has some additional conditions, for example, C is compact. Assume that C is compact and let $\bar{x}$ be a solution of the problem (CP), where f is strongly convex. (Think why a solution should exist.) Now as f is strongly convex, it is simple enough to see that $\bar{x}$ is the unique solution of (CP). Thus from the definition of θ, for any $x \in C$ and $y = \bar{x}$,

$$\theta(x) \geq \langle \nabla f(x), x - \bar{x} \rangle.$$

By strong convexity of f with $\rho > 0$ as the modulus of strong convexity, ∇f is strongly monotone with modulus 2ρ. Thus,

$$\langle \nabla f(x), x - \bar{x} \rangle \geq \langle \nabla f(\bar{x}), x - \bar{x} \rangle + 2\rho \|x - \bar{x}\|^2,$$

thereby yielding

$$\theta(x) \geq \langle \nabla f(\bar{x}), x - \bar{x} \rangle + 2\rho \|x - \bar{x}\|^2. \tag{1.7}$$

But by the optimality condition in Theorem 1.20,

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall \ x \in C.$$

Therefore, the inequality (1.7) reduces to

$$\theta(x) \geq 2\rho \|x - \bar{x}\|^2,$$

which leads to

$$\|x - \bar{x}\| \leq \sqrt{\frac{\theta(x)}{2\rho}}.$$

This provides an error bound for (CP), where f is strongly convex and C is compact. In this derivation if ∇f was strongly monotone with modulus $\rho > 0$, then the error bound will have the expression

$$\|x - \bar{x}\| \leq \sqrt{\frac{\theta(x)}{\rho}}.$$

Observe that as $\rho > 0$,

$$\sqrt{\frac{\theta(x)}{2\rho}} \leq \sqrt{\frac{\theta(x)}{\rho}}$$

and hence the error bound provided by considering that f is strongly monotone with modulus 2ρ gives a sharper error bound.

Now the question is can we design a merit function for (CP) that can be used to develop an error bound even when C is noncompact. Such a merit function was first developed by Fukushima [48] for general variational inequalities. In our context, the function given by

$$\hat{\theta}_\alpha(x) = \sup_{y \in C} \left(\langle \nabla f(x), x - y \rangle - \frac{\alpha}{2}\|y - x\|^2 \right), \quad \alpha > 0.$$

It will be an interesting exercise for the reader to show that

$$\hat{\theta}_\alpha(x) \geq 0, \ \forall\ x \in C$$

and $\hat{\theta}_\alpha(x) = 0$ for $x \in C$ if and only if x is a solution of (CP). Observe that

$$\hat{\theta}_\alpha(x) = -\inf_{y \in C} \left(\langle \nabla f(x), y - x \rangle + \frac{\alpha}{2}\|y - x\|^2 \right), \quad \alpha > 0.$$

For a fixed x, observe that the function

$$\phi_x^\alpha(y) = \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2}\|y - x\|^2$$

is a strongly convex function and is coercive (Definition 1.13). Hence ϕ_x^α attains a lower bound on C. The point of minimum is unique as ϕ_x^α is strongly convex. Hence for each x, the function ϕ_x^α has a finite minimum value. Thus $\hat{\theta}_\alpha(x)$ is always finite, thereby leading to the following error bound.

Theorem 1.23 *Consider the convex optimization problem (CP) where f is a differentiable strongly convex function with modulus $\rho > 0$ and C is a closed convex set in $\mathbb{R}^n$. Let $\bar{x} \in C$ be the unique solution of (CP). Furthermore, if $\rho > \frac{\alpha}{2}$, then for any $x \in C$,*

$$\|x - \bar{x}\| \leq \sqrt{\frac{2\hat{\theta}_\alpha(x)}{2\rho - \alpha}}.$$

Proof. For any $x \in C$ and $y = \bar{x}$ in particular,

$$\hat{\theta}_\alpha(x) \geq \langle \nabla f(x), x - \bar{x} \rangle - \frac{\alpha}{2} \|x - \bar{x}\|^2.$$

By the fact that ∇f is strongly monotone with modulus $\rho > 0$,

$$\hat{\theta}_\alpha(x) \geq \langle \nabla f(\bar{x}), x - \bar{x} \rangle + \rho \|x - \bar{x}\|^2 - \frac{\alpha}{2} \|x - \bar{x}\|^2. \tag{1.8}$$

Because $\bar{x}$ is the unique point of minimizer of (CP), by Theorem 1.20,

$$\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0,$$

thereby reducing the inequality (1.8) to

$$\hat{\theta}_\alpha(x) \geq (\rho - \frac{\alpha}{2}) \|x - \bar{x}\|^2.$$

Therefore,

$$\|x - \bar{x}\| \leq \sqrt{\frac{2\hat{\theta}_\alpha(x)}{2\rho - \alpha}},$$

as desired. □

The reader is urged to show that under the hypothesis of the above theorem, one can prove a more tighter error bound of the form

$$\|x - \bar{x}\| \leq \sqrt{\frac{2\hat{\theta}_\alpha(x)}{4\rho - \alpha}}.$$

The study of optimality conditions with C explicitly given by functional constraints will begin in Chapter 3.

Chapter 2

Tools for Convex Optimization

2.1 Introduction

With the basic concepts discussed in the previous chapter, we devote this chapter to the study of concepts related to the convex analysis. *Convex analysis* is the branch of mathematics that studies convex objects, namely, convex sets, convex functions, and convex optimization theory. These concepts will be used in the subsequent chapters to discuss the details of convex optimization theory and in the development of the book.

2.2 Convex Sets

Recall that for any $x, y \in \mathbb{R}^n$, the set

$$[x, y] = \{z \in \mathbb{R}^n : z = (1-\lambda)x + \lambda y,\ 0 \leq \lambda \leq 1\}$$

denotes the *line segment* joining the points x and y. The *open line segment* joining x and y is given by

$$(x, y) = \{z \in \mathbb{R}^n : z = (1-\lambda)x + \lambda y,\ 0 < \lambda < 1\}.$$

Definition 2.1 A set $F \subset \mathbb{R}^n$ is a *convex set* if

$$\lambda x + (1-\lambda)y \in F,\ \forall\ x, y \in F,\ \forall\ \lambda \in [0, 1].$$

Equivalently, for any $x, y \in F$, the line segment $[x, y]$ is contained in F. Figure 2.1 present convex and nonconvex sets.

Consider the *hyperplane* defined as

$$H(a, b) = \{x \in \mathbb{R}^n : \langle a, x\rangle = b\},$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Observe that it is a convex set. Similarly, the *closed half spaces* given by

$$H_{\leq}(a, b) = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\} \quad \text{and} \quad H_{\geq}(a, b) = \{x \in \mathbb{R}^n : \langle a, x\rangle \geq b\},$$

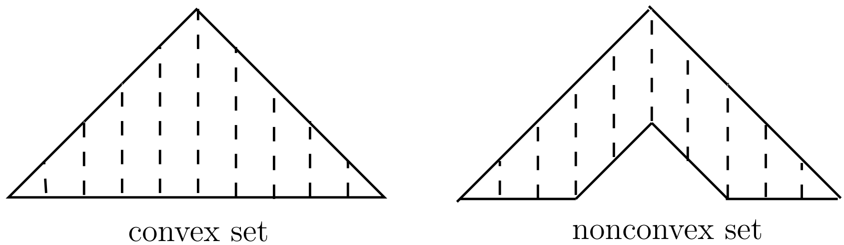

FIGURE 2.1: Convex and nonconvex sets.

and the *open half spaces* given by

$$H_<(a,b) = \{x \in \mathbb{R}^n : \langle a, x\rangle < b\} \quad \text{and} \quad H_>(a,b) = \{x \in \mathbb{R}^n : \langle a, x\rangle > b\}$$

are also convex. Another class of sets that are also convex is the affine sets.

Definition 2.2 A set $M \subset \mathbb{R}^n$ is said to be an *affine set* if

$$(1-\lambda)x + \lambda y \in M, \ \forall\ x, y \in M, \ \forall\ \lambda \in \mathbb{R},$$

where the set $\{z \in \mathbb{R}^n : z = (1-\lambda)x + \lambda y, \ \lambda \in \mathbb{R}\}$ denotes the line passing through x and y. Equivalently, $M \in \mathbb{R}^n$ is affine if for any $x, y \in M$, the line passing through them is contained in M.

Note that a hyperplane is an example of an affine set. The empty set $\emptyset$ and the whole space $\mathbb{R}^n$ are the trivial examples of affine sets. Even though affine sets are convex, the converse need not be true, as is obvious from the example of half spaces.

Next we state some basic properties of convex sets.

Proposition 2.3 *(i) The intersection of an arbitrary collection of convex sets is convex.*

(ii) For two convex sets $F_1, F_2 \subset \mathbb{R}^n$, $F_1 + F_2$ is convex.

(iii) For a convex set $F \subset \mathbb{R}^n$ and scalar $\lambda \in \mathbb{R}$, λF is convex.

(iv) For a convex set $F \subset \mathbb{R}^n$ and scalars $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$,

$$(\lambda_1 + \lambda_2)F = \lambda_1 F + \lambda_2 F$$

which is convex.

Proof. The properties (i)-(iii) can be established by simply using Definition 2.1. The readers are urged to prove (i)-(iii) on their own. Here we will prove only (iv). Consider $z \in (\lambda_1 + \lambda_2)F$. Thus, there exists $x \in F$ such that

$$z = (\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x \in \lambda_1 F + \lambda_2 F.$$

Because $z \in (\lambda_1 + \lambda_2)F$ was arbitrary,

$$(\lambda_1 + \lambda_2)F \subset \lambda_1 F + \lambda_2 F. \tag{2.1}$$

Conversely, let $z \in \lambda_1 F + \lambda_2 F$, which implies that there exist $x_1, x_2 \in F$ such that

$$z = \lambda_1 x_1 + \lambda_2 x_2 = (\lambda_1 + \lambda_2)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2\right). \tag{2.2}$$

Because $\dfrac{\lambda_i}{\lambda_1 + \lambda_2} \in [0, 1],\ i = 1, 2$, which along with the convexity of F implies that

$$x = \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2 \in F. \tag{2.3}$$

Combining the conditions (2.2) and (2.3) lead to

$$z = (\lambda_1 + \lambda_2)x \in (\lambda_1 + \lambda_2)F.$$

As $z \in \lambda_1 F + \lambda_2 F$ was arbitrarily chosen,

$$(\lambda_1 + \lambda_2)F \supset \lambda_1 F + \lambda_2 F,$$

which along with the inclusion (2.1) yields the desired equality. Observe that (ii) and (iii) lead to the convexity of $(\lambda_1 + \lambda_2)F = \lambda_1 F + \lambda_2 F$. $\square$

From Proposition 2.3, it is obvious that intersection of finitely many closed half spaces is again a convex set. Such sets that can be expressed in this form are called *polyhedral sets*. These sets play an important role in linear programming problems. We will deal with polyhedral sets later in this chapter.

However, unlike the intersection and sum of convex sets, the union as well as the complement of convex sets need not be convex. For instance, consider the sets

$$F_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \qquad \text{and} \qquad F_2 = \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}.$$

Observe from Figure 2.2 that both F_1 and F_2 along with their intersection are convex sets but neither their complements nor the union of these two sets is convex.

To overcome such situations where nonconvex sets come into the picture in convex analysis, one has to convexify the nonconvex sets. This leads to the notion of convex combination and convex hull.

Definition 2.4 A point $x \in \mathbb{R}^n$ is said to be a *convex combination* of the points $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$ if

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m$$

with $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i = 1$.

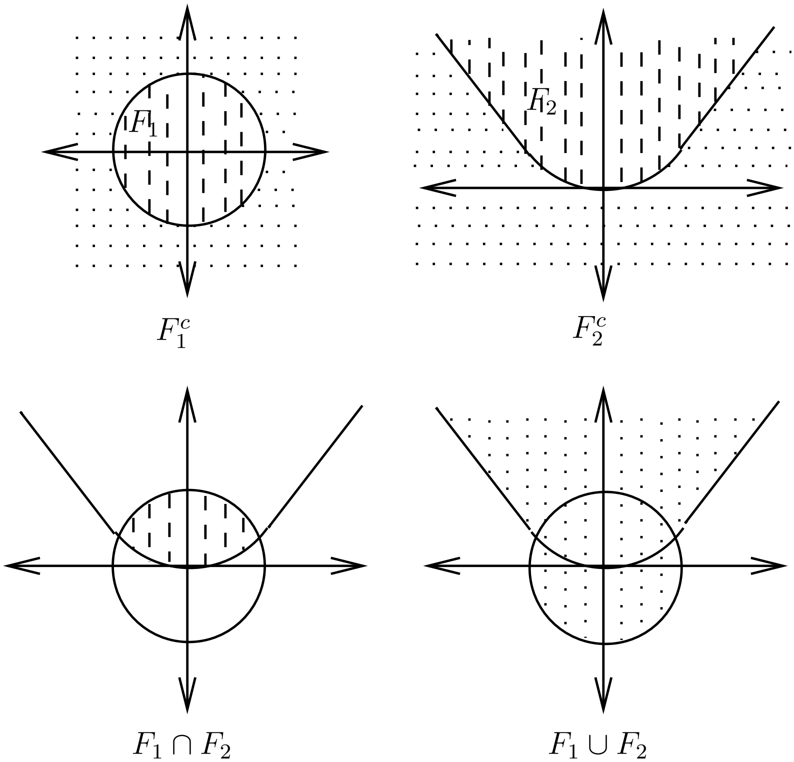

FIGURE 2.2: F_1, F_2, and $F_1 \cap F_2$ are convex while F_1^c, F_2^c, and $F_1 \cup F_2$ are nonconvex.

The next result expresses the concept of convex set in terms of the convex combination of its elements.

Theorem 2.5 *A set $F \subset \mathbb{R}^n$ is convex if and only if it contains all the convex combinations of its elements.*

Proof. From Definition 2.1 of convex set, $F \subset \mathbb{R}^n$ is convex if and only if

$$(1-\lambda)x_1 + \lambda x_2 \in F, \ \forall \ x_1, x_2 \in F, \ \lambda \in [0,1],$$

that is, the convex combination for $m = 2$ belongs to F.

To establish the result, we will use the induction approach. Suppose that the convex combination of $m = l - 1$ elements of F belong to F. Consider $m = l$. The convex combination of l elements is

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_l x_l,$$

where $x_i \in F$ and $\lambda_i \geq 0$, $i = 1, 2, \ldots, l$ with $\sum_{i=1}^{l} \lambda_i = 1$. Because $\sum_{i=1}^{l} \lambda_i = 1$, there exists at least one $\lambda_j > 0$ for some $j \in \{1, 2, \ldots, l\}$. Denote

$$\tilde{x} = \tilde{\lambda}_1 x_1 + \ldots + \tilde{\lambda}_{j-1} x_{j-1} + \tilde{\lambda}_{j+1} x_{j+1} + \ldots + \tilde{\lambda}_l x_l,$$

where $\tilde{\lambda}_i = \dfrac{\lambda_i}{1 - \lambda_j} \geq 0$, $i = 1, \ldots, j-1, j+1, \ldots, l$. Observe that $\sum_{i=1,\, i \neq j}^{l} \tilde{\lambda}_i = 1$ and thus $\tilde{x} \in F$ because it is a convex combination of $l-1$ elements of F. The element x can now be expressed in terms of x_j and $\tilde{x}$ as

$$x = \lambda_j x_j + (1 - \lambda_j)\tilde{x}.$$

Therefore, x is a convex combination of two elements from F, which implies that $x \in F$. Thus, the convex set F can be equivalently expressed as a convex combination of its elements, as desired. $\square$

Similar to the concept of convex combination of points, next we introduce the notion of the convex hull of a set.

Definition 2.6 The *convex hull* of a set $F \subset \mathbb{R}^n$ is the smallest convex set containing F and is denoted by *co* F. It is basically nothing but the intersection of all the convex sets containing F.

Further, the convex hull of F can be expressed in terms of the convex combination of the elements of F as presented in the theorem below.

Theorem 2.7 *For any set $F \subset \mathbb{R}^n$, the convex hull of F, co F, consists of all the convex combinations of the elements of F, that is,*

$$co\ F = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \lambda_i x_i,\ x_i \in F,\ \lambda_i \geq 0,\ i = 1, 2, \ldots, m,$$
$$\sum_{i=1}^{m} \lambda_i = 1,\ m \geq 0\}.$$

Proof. Denote the set of convex combination of the elements of F by $\mathcal{F}$, that is,

$$\mathcal{F} = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \lambda_i x_i,\ x_i \in F,\ \lambda_i \geq 0,\ i = 1, 2, \ldots, m,$$
$$\sum_{i=1}^{m} \lambda_i = 1,\ m \geq 0\}.$$

From Definition 2.6, *co* F is the smallest convex set containing F. Therefore, $F \subset co\ F$. By Theorem 2.5, the convex combination of the elements of F also belong to the convex set *co* F, that is,

$$co\ F \supset \mathcal{F}. \tag{2.4}$$

To establish the result, we will show that $\mathcal{F}$ is also convex. Suppose that $x, \tilde{x} \in \mathcal{F}$, which implies there exist $x_i \in F,\ \lambda_i \geq 0,\ i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $\tilde{x}_i \in F,\ \tilde{\lambda}_i \geq 0,\ i = 1, 2, \ldots, l$ with $\sum_{i=1}^{l} \tilde{\lambda}_i = 1$ such that

$$\begin{aligned} x &= \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m, \\ \tilde{x} &= \tilde{\lambda}_1 \tilde{x}_1 + \tilde{\lambda}_2 \tilde{x}_2 + \ldots + \tilde{\lambda}_l \tilde{x}_l. \end{aligned}$$

For any $\lambda \in [0, 1]$,

$$(1-\lambda)x + \lambda\tilde{x} = (1-\lambda)\lambda_1 x_1 + \ldots + (1-\lambda)\lambda_m x_m + \lambda\tilde{\lambda}_1\tilde{x}_1 + \ldots + \lambda\tilde{\lambda}_l\tilde{x}_l.$$

Observe that $(1-\lambda)\lambda_i \geq 0,\ i = 1, 2, \ldots, m$ and $\lambda\tilde{\lambda}_i \geq 0,\ i = 1, 2, \ldots, l$ satisfying

$$(1-\lambda)\sum_{i=1}^{m} \lambda_i + \lambda \sum_{i=1}^{l} \tilde{\lambda}_i = 1.$$

Thus, for any $\lambda \in [0, 1]$,

$$(1-\lambda)x + \lambda\tilde{x} \in \mathcal{F}.$$

As $x, \tilde{x} \in \mathcal{F}$ were arbitrary, the above relation holds for any $x, \tilde{x} \in \mathcal{F}$, thereby implying the convexity of $\mathcal{F}$. Also, $F \subset \mathcal{F}$. Therefore, by Definition 2.6 of convex hull, *co* $F \subset \mathcal{F}$, which along with the inclusion (2.4) leads to the desired result. $\square$

It follows from the above discussion that a set $F \subset \mathbb{R}^n$ is convex if *co* $F = F$ and thus equivalent to the fact that a convex set $F \subset \mathbb{R}^n$ contains all the convex combinations of the elements of F. From the above theorem we observe that *co* F is expressed as a convex combination of m elements of F, where $m \geq 0$ is arbitrary. But the obvious question is how large this m has to be chosen in the result. This is answered in the famous Carathéodory Theorem, which we present next. Though one finds various approaches to prove the result [12, 97, 101], we present a simple proof from Mangasarian [82].

Theorem 2.8 *(Carathéodory Theorem) Consider a nonempty set $F \subset \mathbb{R}^n$. Then any point of the convex hull of F is representable as a convex combination of at most $n + 1$ points of F.*

Proof. From Theorem 2.7, any element in *co* F can be expressed as a convex combination of m elements of F. We have to show that $m \leq (n+1)$. Suppose that $x \in$ *co* F, which implies that there exist $x_i \in F,\ \lambda_i \geq 0,\ i = 1, 2, \ldots, m$, with $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m.$$

Assume that $m > (n + 1)$. We will prove that x can be expressed as a convex combination of $(m - 1)$ elements. The result can be established by applying the reduction process until $m = n+1$. In case for some $i \in \{1,2,\ldots,m\}$, $\lambda_i = 0$, then x is a convex combination of $(m-1)$ elements of F.

Suppose that $\lambda_i > 0,\ i = 1,2,\ldots,m$. It is known that for $l > n$, any l elements in $\mathbb{R}^n$ are linearly dependent. As $m-1 > n$, there exist $\alpha_i \in \mathbb{R}$, $i = 1,2,\ldots,m-1$, not all zeroes, such that

$$\alpha_1(x_1 - x_m) + \alpha_2(x_2 - x_m) + \ldots + \alpha_{m-1}(x_{m-1} - x_m) = 0.$$

Define $\alpha_m = -(\alpha_1 + \alpha_2 + \ldots + \alpha_{m-1})$. Observe that

$$\sum_{i=1}^{m} \alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i x_i = 0. \tag{2.5}$$

Define $\tilde{\lambda}_i = \lambda_i - \gamma\alpha_i$, $i = 1,2,\ldots,m$, where $\gamma > 0$ is chosen such that $\tilde{\lambda}_i \geq 0$, $i \in \{1,2,\ldots,m\}$ and for some $j \in \{1,2,\ldots,m\}$, $\tilde{\lambda}_j = 0$. This is possible by taking

$$\frac{1}{\gamma} = \max_{i=1,\ldots,m} \left\{\frac{\alpha_i}{\lambda_i}\right\} = \frac{\alpha_j}{\lambda_j}.$$

By choice, $\tilde{\lambda}_j = 0$ and $\tilde{\lambda}_i \geq 0,\ i = 1,\ldots,j-1,j+1,\ldots,m$, which by the condition (2.5) yields

$$\sum_{i=1,\ i\neq j}^{m} \tilde{\lambda}_i = \sum_{i=1}^{m} \tilde{\lambda}_i = \sum_{i=1}^{m} \lambda_i - \gamma \sum_{i=1}^{m} \alpha_i = 1$$

and

$$x = \sum_{i=1}^{m} \lambda_i x_i = \sum_{i=1}^{m} \tilde{\lambda}_i x_i + \gamma \sum_{i=1}^{m} \alpha_i x_i = \sum_{i=1,\ i\neq j}^{m} \tilde{\lambda}_i x_i,$$

which implies that x is now expressed as a convex combination of $(m-1)$ elements of F. This reduction can be carried out until $m = (n+1)$ and thus any element in the convex hull of F is representable as a convex combination of at most $(n+1)$ elements of F, as desired. $\square$

Using the above theorem, we have the following important result for a compact set from Bertsekas [12] and Rockafellar [97].

Theorem 2.9 *For a compact set $F \subset \mathbb{R}^n$, its convex hull co F is also a compact set.*

Proof. We claim that *co* F is closed. Consider a sequence $\{z_k\} \subset co\ F$. By the Carathéodory Theorem, Theorem 2.8, there exist sequences $\{\lambda_i^k\} \subset \mathbb{R}_+$, $i = 1,2,\ldots,n+1$, satisfying $\sum_{i=1}^{n+1} \lambda_i^k = 1$ and $\{x_i^k\} \subset F$, $i = 1,2,\ldots,n+1$, such that

$$z_k = \sum_{i=1}^{m} \lambda_i^k x_i^k,\ \forall\ k \in \mathbb{N}. \tag{2.6}$$

As for any $k \in \mathbb{N}$, $\lambda_i^k \geq 0$, $i = 1, 2, \ldots, n+1$ with $\sum_{i=1}^{m} \lambda_i^k = 1$, $\{\lambda_i^k\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, $\{\lambda_i^k\}$, $i = 1, 2, \ldots, n+1$, has a convergent subsequence. Without loss of generality, assume that $\lambda_i^k \to \lambda_i$, $i = 1, 2, \ldots, n+1$, such that $\lambda_i \geq 0$, $i = 1, 2, \ldots, n+1$ with $\sum_{i=1}^{n+1} \lambda_i = 1$. By the compactness of F, the sequence $\{x_i^k\}$, $i = 1, 2, \ldots, n+1$, is bounded. Again by the Bolzano–Weierstrass Theorem, $\{x_i^k\}$ has a convergent subsequence. Without loss of generality, let $x_i^k \to x_i$, $i = 1, 2, \ldots, n+1$. By the compactness of F, F is a closed set and thus $x_i \in F$, $i = 1, 2, \ldots, n+1$. Taking the limit as $k \to +\infty$, (2.6) yields that

$$z_k \to z = \sum_{i=1}^{n+1} \lambda_i x_i \in co\ F.$$

Because $\{z_k\} \subset co\ F$ was arbitrary sequence, $co\ F$ is a closed set.

To prove that $co\ F$ is compact, we will establish that $co\ F$ is bounded. As F is compact, it is a bounded set, which implies that there exists $M > 0$ such that $\|x\| \leq M$ for every $x \in F$. Now consider $z \in co\ F$, which by the Carathéodory Theorem implies that there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, n+1$, satisfying $\sum_{i=1}^{n+1} \lambda_i = 1$ and $x_i \in F$, $i = 1, 2, \ldots, n+1$, such that

$$z = \sum_{i=1}^{n+1} \lambda_i x_i.$$

Therefore, by the boundedness of F along with the fact that $\lambda_i \in [0, 1]$, $i = 1, 2, \ldots, n+1$ yields that

$$\|z\| = \|\sum_{i=1}^{n+1} \lambda_i x_i\| \leq \sum_{i=1}^{n+1} \lambda_i \|x_i\| \leq (n+1)M.$$

Because $z \in co\ F$ was arbitrary, every element in $co\ F$ is bounded above by $(n+1)M$ and thus $co\ F$ is bounded. Hence $co\ F$ is a compact set. $\square$

However, the above result does not hold true if one replaces the compactness of F by simply the closedness of the set. To verify this fact, we present an example from Bertsekas [12]. Consider the closed set F defined as

$$F = \{(0, 0)\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\},$$

while the convex hull of F is

$$co\ F = \{(0, 0)\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\},$$

which is not a closed set.

Now, similar to the concepts of convex combination and convex hull, we present the notions of affine combination and affine hulls.

Definition 2.10 A point $x \in \mathbb{R}^n$ is said to be a *affine combination* of the points $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$ if

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m$$

with $\lambda_i \in \mathbb{R},\ i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i = 1$.

Definition 2.11 The *affine hull* of a set $F \subset \mathbb{R}^n$ is the smallest affine set containing F and is denoted by $aff\ F$. It consists of all affine combinations of the elements of F.

Now we move on to the properties of closure, interior, and relative interior of convex sets.

Definition 2.12 The *closure* of a set $F \subset \mathbb{R}^n$, $cl\ F$, is expressed as

$$cl\ F = \bigcap_{\varepsilon > 0} (F + \varepsilon \mathbb{B}) = \bigcap_{\varepsilon > 0} \{x + \varepsilon \mathbb{B} : x \in F\},$$

while the *interior* of the set F, $int\ F$, is defined as

$$int\ F = \{x \in \mathbb{R}^n : \text{there exists } \varepsilon > 0 \text{ such that } (x + \varepsilon \mathbb{B}) \subset F\}.$$

It is well known that the arbitrary intersection of closed sets is closed but not for the union. However, for the case of union, the following relation holds. For any arbitrary family of set $\{F_\lambda\}$, $\lambda \in \Lambda$, where the index set Λ may possibly be infinite,

$$\bigcup_{\lambda \in \Lambda} cl\ F_\lambda \subset cl \bigcup_{\lambda \in \Lambda} F_\lambda.$$

The notion of interior suffers from a drawback that even for a nonempty convex set, it may turn out to be empty. For example, consider a line in $\mathbb{R}^2$. From the above definition, it is obvious that the interior is empty. But the set of interior points relative to the affine hull of the set is nonempty. This motivates us to introduce the notion of relative interior.

Definition 2.13 The *relative interior* of a convex set $F \subset \mathbb{R}^n$, $ri\ F$, is the interior of F relative to the affine hull of F, that is,

$$ri\ F = \{x \in \mathbb{R}^n : \text{there exists } \varepsilon > 0 \text{ such that } (x + \varepsilon \mathbb{B}) \cap aff\ F \subset F\}.$$

For an n-dimensional convex set $F \subset \mathbb{R}^n$, $aff\ F = \mathbb{R}^n$ and thus $ri\ F = int\ F$. Though the notion of relative interior helps in overcoming the emptiness of the interior of a convex set, it also suffers from a drawback. For nonempty convex sets $F_1, F_2 \subset \mathbb{R}^n$,

$$F_1 \subset F_2 \quad \Longrightarrow \quad cl\ F_1 \subset cl\ F_2 \quad \text{and} \quad int\ F_1 \subset int\ F_2,$$

but $ri\ F_1 \subset ri\ F_2$ need not hold. For instance, consider $F_1 = \{(0,0)\}$ and $F_2 = \{(0,y) \in \mathbb{R}^2 : y \geq 0\}$. Here $F_1 \subset F_2$ with $ri\ F_1 = \{(0,0)\}$ and $ri\ F_2 = \{(0,y) \in \mathbb{R}^2 : y > 0\}$. Here the relative interiors are nonempty and disjoint.

Next we present some properties of closure and relative interior of convex sets. The proofs are from Bertsekas [11, 12] and Rockafellar [97].

Proposition 2.14 *Consider a nonempty convex set $F \subset \mathbb{R}^n$. Then the following hold:*

(i) $ri\ F$ is nonempty.

(ii) (Line Segment Principle) Let $x \in ri\ F$ and $y \in cl\ F$. Then for $\lambda \in [0,1)$,

$$(1-\lambda)x + \lambda y \in ri\ F.$$

(iii) (Prolongation Principle) $x \in ri\ F$ if and only if every line segment in F having x as one end point can be prolonged beyond x without leaving F, that is, for every $y \in F$ there exists $\gamma > 1$ such that

$$x + (\gamma - 1)(x - y) \in F.$$

(iv) $ri\ F$ and $cl\ F$ are convex sets with the same affine hulls as that of F.

Proof. (i) Without loss of generality assume that $0 \in F$. Then the affine hull of F, $aff\ F$, is a subspace containing F. Denote the dimension of $aff\ F$ by m. If $m = 0$, then F as well as $aff\ F$ consist of a single point and hence $ri\ F$ is the point itself, thus proving the result.

Suppose that $m > 0$. Then one can always find linearly independent elements $x_1, x_2, \ldots, x_m$ from F such that $aff\ F = span\{x_1, x_2, \ldots, x_m\}$, that is, $x_1, x_2, \ldots, x_m$ form a basis of the subspace $aff\ F$. If this was not possible, then there exist linearly independent elements $y_1, y_2, \ldots, y_l$ with $l < m$ from F such that $F \subset span\{y_1, y_2, \ldots, y_l\}$, thereby contradicting the fact that the dimension of $aff\ F$ is m. Observe that $co\ \{0, x_1, x_2, \ldots, x_m\} \subset F$ has a nonempty interior with respect to $aff\ F$, which implies $co\ \{0, x_1, x_2, \ldots, x_m\} \subset ri\ F$, thereby yielding that $ri\ F$ is nonempty.

(ii) Suppose that $y \in cl\ F$, which implies there exists $\{y_k\} \subset F$ such that $y_k \to y$. As $x \in ri\ F$, there exists $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon(x) \cap aff\ F \subset F$. For $\lambda \in [0,1)$, define $y_\lambda = (1-\lambda)x + \lambda y$ and $y_{k,\lambda} = (1-\lambda)x + \lambda y_k$. Therefore, from Figure 2.3, it is obvious that each point of $\mathbb{B}_{(1-\lambda)\varepsilon}(y_{k,\lambda}) \cap aff\ F$ is a convex combination of y_k and some point from $\mathbb{B}_\varepsilon(x) \cap aff\ F$. By the convexity of F,

$$\mathbb{B}_{(1-\lambda)\varepsilon}(y_{k,\lambda}) \cap aff\ F \subset F, \ \forall\ k \in \mathbb{N}.$$

Because $y_k \to y$, $y_{k,\lambda} \to y_\lambda$. Thus, for sufficiently large k,

$$\mathbb{B}_{(1-\lambda)\varepsilon/2}(y_\lambda) \subset \mathbb{B}_{(1-\lambda)\varepsilon}(y_{k,\lambda}),$$

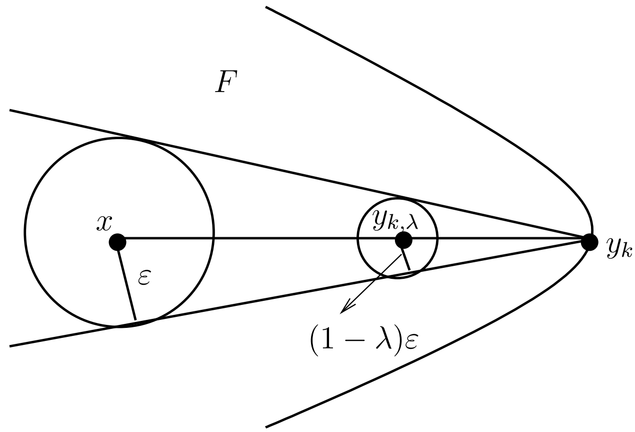

FIGURE 2.3: Line segment principle.

which implies

$$\mathbb{B}_{(1-\lambda)\varepsilon/2}(y_\lambda) \cap aff\ F \subset \mathbb{B}_{(1-\lambda)\varepsilon}(y_{k,\lambda}) \cap aff\ F \subset F.$$

Hence, $y_\lambda = (1-\lambda)x + \lambda y \in ri\ F$ for $\lambda \in [0,1)$.

Aliter. The above approach was direct and somewhat cumbersome. In the proof to follow, we use the fact that relative interiors are preserved under one-to-one affine transformation of $\mathbb{R}^n$ to itself and hence these transformations preserve the affine hulls. This property simplifies the proofs. If F is an m-dimensional set in $\mathbb{R}^n$, there exists a one-to-one affine transformation of $\mathbb{R}^n$ to itself that carries $aff\ F$ to the subspace

$$\mathcal{S} = \{(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \in \mathbb{R}^n : x_{m+1} = 0, \ldots, x_n = 0\}.$$

Thus, $\mathcal{S}$ can be considered a copy of $\mathbb{R}^m$. From this view, one can simply consider the case when $F \subset \mathbb{R}^n$ is an n-dimensional set, which implies that $ri\ F = int\ F$. We will now establish the result for $int\ F$ instead of $ri\ F$. Because $y \in cl\ F$,

$$y \in F + \varepsilon\mathbb{B}, \ \forall\ \varepsilon > 0.$$

Therefore, for every $\varepsilon > 0$,

$$\begin{aligned} (1-\lambda)x + \lambda y + \varepsilon\mathbb{B} &\subset (1-\lambda)x + \lambda(F + \varepsilon\mathbb{B}) + \varepsilon\mathbb{B} \\ &= (1-\lambda)\left\{x + \frac{\varepsilon(1+\lambda)}{1-\lambda}\mathbb{B}\right\} + \lambda F, \ \forall\ \lambda \in [0,1). \end{aligned}$$

Because $x \in int\ F$, choosing $\varepsilon > 0$ sufficiently small such that

$$x + \frac{\varepsilon(1+\lambda)}{1-\lambda}\mathbb{B} \subset F,$$

which along with the convexity of F reduces the preceding relation to

$$(1-\lambda)x + \lambda y + \varepsilon\mathbb{B} \subset (1-\lambda)F + \lambda F \subset F, \ \forall\, \lambda \in [0,1).$$

Thus, $(1-\lambda)x + \lambda y \in int\ F$ for $\lambda \in [0,1)$ as desired.

(iii) For $x \in ri\ F$, by the definition of relative interior, the condition holds. Conversely, suppose that $x \in \mathbb{R}^n$ satisfies the condition. We claim that $x \in ri\ F$. By (i) there exists $\tilde{x} \in ri\ F$. If $x = \tilde{x}$, we are done. So assume that $x \neq \tilde{x}$. As $\tilde{x} \in ri\ F \subset F$, by the condition, there exists $\gamma > 1$ such that

$$y = x + (\gamma - 1)(x - \tilde{x}) \in F.$$

Therefore, for $\lambda = \dfrac{1}{\gamma} \in (0,1)$,

$$x = (1-\lambda)\tilde{x} + \lambda y,$$

which by the fact that $y \in F \subset cl\ F$ along with the line segment principle (ii) implies that $x \in ri\ F$, thereby establishing the result.

(iv) Because $ri\ F \subset cl\ F$, by (ii) we have that $ri\ F$ is convex. From (i), we know that there exist $x_1, x_2, \ldots, x_m \in F$ such that $aff\ \{x_1, x_2, \ldots, x_m\} = aff\ F$ and $co\ \{0, x_1, x_2, \ldots, x_m\} \subset ri\ F$. Therefore, $ri\ F$ has an affine hull the same as that of F.

By Proposition 2.3, $F + \varepsilon\mathbb{B}$ is convex for every $\varepsilon > 0$. Also, as intersection of convex sets is convex, $cl\ F$, which is the intersection of the collection of the sets $F + \varepsilon\mathbb{B}$ over $\varepsilon > 0$ is convex. Because $F \subset aff\ F$, $cl\ F \subset cl\ aff\ F = aff\ F$, and as $F \subset cl\ F$, $aff\ F \subset aff\ cl\ F$, which together implies that the affine hull of $cl\ F$ coincides with $aff\ F$. $\square$

In the result below we discuss the closure and relative interior operations. The proofs are from Bertsekas [12] and Rockafellar [97].

Proposition 2.15 *Consider nonempty convex sets $F, F_1, F_2 \subset \mathbb{R}^n$. Then the following hold:*

(i) $cl(ri\ F) = cl\ F$.

(ii) $ri(cl\ F) = ri\ F$.

(iii) $ri\ F_1 \cap ri\ F_2 \subset ri\ (F_1 \cap F_2)$ *and* $cl\ (F_1 \cap F_2) \subset cl\ F_1 \cap cl\ F_2$.
In addition if $ri\ F_1 \cap ri\ F_2 \neq \emptyset$,

$$ri\ F_1 \cap ri\ F_2 = ri\ (F_1 \cap F_2) \quad \text{and} \quad cl\ (F_1 \cap F_2) = cl\ F_1 \cap cl\ F_2.$$

(iv) Consider a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$*. Then*

$$L(cl\ F) \subset cl\ (LF) \quad \text{and} \quad L(ri\ F) = ri\ (LF).$$

(v) $ri\ (\alpha F) = \alpha\ ri\ F$ *for every* $\alpha \in \mathbb{R}$*.*

(vi) $ri\ (F_1 + F_2) = ri\ F_1 + ri\ F_2$ *and* $cl\ F_1 + cl\ F_2 \subset cl\ (F_1 + F_2)$*. If either* F_1 *or* F_2 *is bounded,*

$$cl\ F_1 + cl\ F_2 = cl\ (F_1 + F_2)$$

Proof. (i) Because $ri\ F \subset F$, it is obvious that

$$cl(ri\ F) \subset cl\ F.$$

Conversely, suppose that $y \in cl\ F$. We claim that $y \in cl(ri\ F)$. Consider any $x \in ri\ F$. By the line segment principle, Proposition 2.14 (ii), for every $\lambda \in [0, 1)$,

$$(1 - \lambda)x + \lambda y \in ri\ F.$$

Observe that the sequence $\{(1 - \lambda_k)x + \lambda_k y\} \subset ri\ F$ is such that as the limit $\lambda_k \to 1$, $(1 - \lambda_k)x + \lambda_k y \to y$, which implies that $y \in cl(ri\ F)$, as claimed. Hence the result.

(ii) We know that $F \subset cl\ F$ and by Proposition 2.14 (iv), $aff\ F = aff\ (cl\ F)$. Consider $x \in ri\ F$, which by the definition of relative interior along with the preceding facts imply that there exists $\varepsilon > 0$ such that

$$(x + \varepsilon\mathbb{B}) \cap aff\ (cl\ F) = (x + \varepsilon\mathbb{B}) \cap aff\ F \subset F \subset cl\ F,$$

thereby yielding that $x \in ri\ (cl\ F)$. Hence, $ri\ F \subset ri\ (cl\ F)$.

Conversely, suppose that $x \in ri\ (cl\ F)$. We claim that $x \in ri\ F$. By the nonemptiness of $ri\ F$, Proposition 2.14 (i), there exists $\tilde{x} \in ri\ F \subset cl\ F$. If in particular $x = \tilde{x}$, we are done. So assume that $x \neq \tilde{x}$. We can choose $\gamma > 1$, sufficiently close to 1 such that by applying the Prolongation Principle, Proposition 2.14 (iii),

$$y = x + (\gamma - 1)(x - \tilde{x}) \in cl\ F.$$

Therefore, for $\lambda = \dfrac{1}{\gamma} \in (0, 1)$,

$$x = (1 - \lambda)\tilde{x} + \lambda y,$$

which by the Line Segment Principle, Proposition 2.14 (ii), implies that $x \in ri\ F$, thereby leading to the requisite result.

(iii) Suppose that $x \in ri\ F_1 \cap ri\ F_2$ and $y \in F_1 \cap F_2$. By the Prolongation Principle, Proposition 2.14 (iii), there exist $\gamma_i > 1,\ i = 1, 2$ such that

$$x + (\gamma_i - 1)(x - y) \in F_i,\ i = 1, 2.$$

Choosing $\gamma = \min\{\gamma_1, \gamma_2\} > 1$, the above condition reduces to

$$x + (\gamma - 1)(x - y) \in F_1 \cap F_2,$$

which again by the Prolongation Principle leads to $x \in ri\ (F_1 \cap F_2)$. Thus,

$$ri\ F_1 \cap ri\ F_2 \subset ri\ (F_1 \cap F_2).$$

Because $F_1 \cap F_2 \subset cl\ F_1 \cap cl\ F_2$, it is obvious that $cl\ (F_1 \cap F_2) \subset cl\ F_1 \cap cl\ F_2$ as intersection of arbitrary closed sets is closed.

Assume that $ri\ F_1 \cap ri\ F_2$ is nonempty. Suppose that $x \in ri\ F_1 \cap ri\ F_2$ and $y \in cl\ F_1 \cap cl\ F_2$. By the Line Segment Principle, Proposition 2.14 (ii), for every $\lambda \in [0, 1)$,

$$(1 - \lambda)x + \lambda y \in ri\ F_1 \cap ri\ F_2.$$

Observe that the sequence $\{(1 - \lambda_k)x + \lambda_k y\} \subset F$ is such that as $\lambda_k \to 1$, $(1 - \lambda_k)x + \lambda_k y \to y$ and hence $y \in cl\ (ri\ F_1 \cap ri\ F_2)$. Therefore,

$$cl\ F_1 \cap cl\ F_2 \subset cl\ (ri\ F_1 \cap ri\ F_2) \subset cl\ (F_1 \cap F_2), \tag{2.7}$$

thereby yielding the desired equality, that is,

$$cl\ F_1 \cap cl\ F_2 = cl\ (F_1 \cap F_2).$$

Also from the inclusion (2.7),

$$cl\ (ri\ F_1 \cap ri\ F_2) = cl\ (F_1 \cap F_2).$$

By (ii), the above condition leads to

$$\begin{aligned} ri\ (ri\ F_1 \cap ri\ F_2) &= ri\ (cl\ (ri\ F_1 \cap ri\ F_2)) \\ &= ri\ (cl\ (F_1 \cap F_2)) = ri\ (F_1 \cap F_2), \end{aligned}$$

which implies that

$$ri\ F_1 \cap ri\ F_2 \supset ri\ (F_1 \cap F_2),$$

thus establishing the requisite result.

(iv) Suppose that $x \in cl\ F$, which implies there exists a sequence $\{x_k\} \subset F$ such that $x_k \to x$. Because L is a linear transformation, it is continuous. Therefore, $L(x_k) \to L(x)$, which implies $L(x) \in cl(LF)$ and hence $L(cl\ F) \subset cl(LF)$.

As $ri\ F \subset F$, on applying the linear transformation L, $L(ri\ F) \subset LF$ and thus $cl\ L(ri\ F) \subset cl\ (LF)$. Also, as $F \subset cl\ F$, proceeding as before which along with (i) and the closure inclusion yields

$$LF \subset L(cl\ F) = L(cl\ (ri\ F)) \subset cl\ L(ri\ F).$$

Therefore, $cl\ (LF) \subset cl\ L(ri\ F)$, which by earlier condition leads to $cl\ (LF) = cl\ L(ri\ F)$. By (ii),

$$ri\ (LF) = ri\ (cl\ (LF)) = ri\ (cl\ L(ri\ F)) = ri\ (L(ri\ F)),$$

thereby yielding $ri\ (LF) \subset L(ri\ F)$.

Conversely, suppose that $\bar{x} \in L(ri\ F)$, which implies that there exists $\tilde{x} \in ri\ F$ such that $\bar{x} = L(\tilde{x})$. Consider any $\bar{y} \in LF$ and corresponding $\tilde{y} \in F$ such that $\bar{y} = L(\tilde{y})$. By the Prolongation Principle, Proposition 2.14 (iii), there exists $\gamma > 1$ such that

$$(1-\gamma)\tilde{y} + \gamma\tilde{x} \in F,$$

which under the linear transformation leads to

$$(1-\gamma)\bar{y} + \gamma\bar{x} \in LF.$$

Because $\bar{y} \in LF$ was arbitrary, again applying the Prolongation Principle yields $\bar{x} \in ri\ (LF)$, that is, $L(ri\ F) \subset ri\ (LF)$, thereby establishing the desired equality.

(v) For arbitrary but fixed $\alpha \in \mathbb{R}$, define a linear transformation $L_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ given by $L_\alpha(x) = \alpha x$. Therefore, for a set F, $L_\alpha F = \alpha F$. For every $\alpha \in \mathbb{R}$, applying (iv) to $L_\alpha F$ leads to

$$\alpha\ ri\ F = L_\alpha(ri\ F) = ri\ (L_\alpha F) = ri\ (\alpha F)$$

and hence the result.

(vi) Define a linear transformation $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $L(x_1, x_2) = x_1 + x_2$ which implies $L(F_1, F_2) = F_1 + F_2$. Now applying (iv) to L yields

$$ri\ (F_1 + F_2) = ri\ F_1 + ri\ F_2 \quad \text{and} \quad cl\ F_1 + cl\ F_2 \subset cl\ (F_1 + F_2).$$

To establish the equality in the closure part, assume that F_1 is bounded. Suppose that $x \in cl(F_1 + F_2)$, which implies that there exist $\{x_i^k\} \subset F_i$, $i = 1, 2$, such that $x_1^k + x_2^k \to x$. Because F_1 is bounded, $\{x_1^k\}$ is a bounded sequence, which leads to the boundedness of $\{x_2^k\}$. By the Bolzano–Weierstrass Theorem, Proposition 1.3, the sequence $\{(x_1^k, x_2^k)\}$ has a subsequence converging to (x_1, x_2) such that $x_1 + x_2 = x$. As $x_i \in cl\ F_i$ for $i = 1, 2$, $x \in cl\ F_1 + cl\ F_2$, hence establishing the result $cl\ F_1 + cl\ F_2 = cl\ (F_1 + F_2)$. $\square$

Note that for the equality part in Proposition 2.15 (iii), the nonemptiness of $ri\ F_1 \cap ri\ F_2$ is required, otherwise the equality need not hold. We present an example from Bertsekas [12] to illustrate this fact. Consider the sets

$$F_1 = \{x \in \mathbb{R} : x \geq 0\} \quad \text{and} \quad F_2 = \{x \in \mathbb{R} : x \leq 0\}.$$

Therefore, $ri\ (F_1 \cap F_2) = \{0\} \neq \emptyset = ri\ F_1 \cap ri\ F_2$. For the closure part, consider

$$F_1 = \{x \in \mathbb{R} : x > 0\} \quad \text{and} \quad F_2 = \{x \in \mathbb{R} : x < 0\}.$$

Thus, $cl\ (F_1 \cap F_2) = \emptyset \neq \{0\} = cl\ F_1 \cap cl\ F_2$.

Also the boundedness assumption in (vi) for the closure equality is necessary. For instance, consider the sets

$$\begin{aligned} F_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 1,\ x_1 > 0,\ x_2 > 0\}, \\ F_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}. \end{aligned}$$

Here, both F_1 and F_2 are closed unbounded sets, whereas the sum

$$F_1 + F_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$$

is not closed. Thus $cl\ F_1 + cl\ F_2 = F_1 + F_2 \subsetneqq cl\ (F_1 + F_2)$.

As a consequence of Proposition 2.15, we have the following result from Rockafellar [97].

Corollary 2.16 *(i) Consider two convex sets F_1 and F_2 in $\mathbb{R}^n$. Then $cl\ F_1 = cl\ F_2$ if and only if $ri\ F_1 = ri\ F_2$. Equivalently,*

$$ri\ F_1 \subset F_2 \subset cl\ F_1.$$

(ii) Consider a convex set $F \subset \mathbb{R}^n$. Then any open set that meets $cl\ F$ also meets $ri\ F$.

(iii) Consider a convex set $F \subset \mathbb{R}^n$ and an affine set $H \subset \mathbb{R}^n$ containing a point from $ri\ F$. Then

$$ri\ (F \cap H) = ri\ F \cap H \quad \text{and} \quad cl(F \cap H) = cl\ F \cap H.$$

Proof. (i) Suppose that $cl\ F_1 = cl\ F_2$. Invoking Proposition 2.15 (ii),

$$ri\ F_1 = ri(cl\ F_1) = ri(cl\ F_2) = ri\ F_2. \tag{2.8}$$

Now assume that $ri\ F_1 = ri\ F_2$, which by Proposition 2.15 (i) implies that

$$cl\ F_1 = cl(ri\ F_1) = cl(ri\ F_2) = cl\ F_2. \tag{2.9}$$

Combining the relations (2.8) and (2.9) leads to

$$ri\ F_1 = ri\ F_2 \subset F_2 \subset cl\ F_2 = cl\ F_1,$$

thereby establishing the desired result.

(ii) Denote the open set by $\mathcal{O}$. Suppose that $\mathcal{O}$ meets $cl\ F$, that is, there exists $x \in \mathbb{R}^n$ such that $x \in \mathcal{O} \cap cl\ F$. By Proposition 2.15 (i), $cl\ F = cl(ri\ F)$, which implies

$$x \in \mathcal{O} \cap cl(ri\ F).$$

Because $x \in cl(ri\ F)$, there exists $\{x_k\} \subset ri\ F$ such that $x_k \to x$. Therefore, for k sufficiently large, one can choose $\bar{\varepsilon} > 0$ such that $x_k \in x + \bar{\varepsilon}\mathbb{B}$. Also, as $\mathcal{O}$ is an open set and $x \in \mathcal{O}$, there exists $\tilde{\varepsilon} > 0$ such that $x + \tilde{\varepsilon}\mathbb{B} \subset \mathcal{O}$. Define $\varepsilon = \min\{\bar{\varepsilon}, \tilde{\varepsilon}\}$. Thus for sufficiently large k,

$$x_k \in x + \varepsilon\mathbb{B} \subset \mathcal{O},$$

which along with the fact that $x_k \in ri\ F$ implies that $\mathcal{O}$ also meets $ri\ F$, hence proving the result.

(iii) Observe that for an affine set H, $ri\ H = H = cl\ H$. Therefore, by the given hypothesis,

$$ri\ F \cap H = ri\ F \cap ri\ H \neq \emptyset.$$

Thus, by Proposition 2.15 (iii),

$$ri(F \cap H) = ri\ F \cap H \quad \text{and} \quad cl(F \cap H) = cl\ F \cap H,$$

thereby completing the proof. $\square$

Before moving on to the various classes of convex sets, we would like to mention the concept of core of a set like the notions of closure and interior of a set from Borwein and Lewis [17].

Definition 2.17 The *core* of a set $F \subset \mathbb{R}^n$, denoted by *core* F, is defined as

$$core\ F = \{x \in F : \text{for every } d \in \mathbb{R}^n \text{ there exists } \lambda > 0 \\ \text{such that } x + \lambda d \in F\}.$$

It is obvious that $int\ F \subset core\ F$. For a convex set $F \subset \mathbb{R}^n$, $int\ F = core\ F$.

2.2.1 Convex Cones

Since we are interested in the study of convex optimization theory, a class of sets that plays an active role in this direction is the epigraphical set as discussed briefly in Chapter 1. From the definition it is obvious that epigraphical sets are unbounded. Thus it seems worthwhile to understand the class of unbounded convex sets for which one needs the idea of recession cones. But before that, we require the concept of cones.

Definition 2.18 A set $K \subset \mathbb{R}^n$ is said to be a *cone* if for every $x \in K$, $\lambda x \in K$ for every $\lambda \geq 0$. Therefore, for any set $F \subset \mathbb{R}^n$, the *cone generated* by F is denoted by *cone* F and is defined as

$$cone\ F = \bigcup_{\lambda \geq 0} \lambda F = \{z \in \mathbb{R}^n : z = \lambda x,\ x \in F,\ \lambda \geq 0\}.$$

Note that for a nonconvex set F, *cone* F may or may not be convex. For example, consider $F = \{(1,1),(2,2)\}$. Here,

$$cone\ F = \{z \in \mathbb{R}^2 : z = \lambda(1,1),\ \lambda \geq 0\},$$

which is convex. Now consider $F = \{(-1,1),(1,1)\}$. Observe that the cone generated by F comprises of two rays, that is,

$$cone\ F = \{z \in \mathbb{R}^2 : z = \lambda(-1,1) \text{ or } z = \lambda(1,1),\ \lambda \geq 0\}.$$

But we are interested in the convex scenarios, thereby moving on to the notion of the convex cone.

Definition 2.19 The set $K \subset \mathbb{R}^n$ is said to be *convex cone* if it is convex as well as a cone. Therefore, for any set $F \subset \mathbb{R}^n$, the *convex cone generated* by F is denoted by *cone co* F and is expressed as a set containing all *conic combinations* of the elements of the set F, that is,

$$cone\ co\ F = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \lambda_i x_i,\ x_i \in F,\ \lambda_i \geq 0,$$
$$i = 1,2,\ldots,m,\ m \geq 0\}.$$

The convex cone generated by the set F is the smallest convex cone containing F. Also, for a collection of convex sets $F_i \subset \mathbb{R}^n$, $i = 1,2,\ldots,m$, the convex cone generated by F_i, $i = 1,2,\ldots,m$ can be easily shown to be expressed as

$$cone\ co \bigcup_{i=1}^{m} F_i = \bigcup_{\lambda \in \mathbb{R}^m_+} \sum_{i=1}^{m} \lambda_i F_i.$$

Some of the important convex cones that play a pivotal role in convex optimization are the polar cone, tangent cone, and the normal cone. We shall discuss them later in the chapter. Before going back to the discussion of unbounded sets, we characterize the class of convex cones in the result below.

Theorem 2.20 *A cone $K \subset \mathbb{R}^n$ is convex if and only if $K + K \subset K$.*

Proof. Suppose that the cone K is convex. Consider $x, y \in K$. Because K is convex, for $\lambda = 1/2$,

$$(1-\lambda)x + \lambda y = \frac{1}{2}(x+y) \in K.$$

Also, as K is a cone, $x + y \in 2K \subset K$ which implies that $K + K \subset K$.

Conversely, suppose that $K + K \subset K$. Consider $x, y \in K$. Because K is a cone, for $\lambda \in [0, 1]$,

$$(1-\lambda)x \in K \quad \text{and} \quad \lambda y \in K,$$

which along with the assumption $K + K \subset K$ leads to

$$(1-\lambda)x + \lambda y \in K, \ \forall \ \lambda \in [0, 1].$$

As $x, y \in K$ were arbitrary, the cone K is convex, thus proving the result. $\square$

Now coming back to unbounded convex sets, a set can be thought to be unbounded if for any point in the set there exists a direction moving along which one still remains within the set. Such directions are known as the directions of recession and are independent of the point chosen.

Definition 2.21 For a convex set $F \subset \mathbb{R}^n$, $d \in \mathbb{R}^n$ is said to be the *direction of recession* of F if $x + \lambda d \in F$ for every $x \in F$ and for every $\lambda \geq 0$. The collection of all the directions of recession of a set $F \subset \mathbb{R}^n$ form a cone known as the *recession cone* of F and is denoted by 0^+F. Equivalently,

$$0^+F = \{d \in \mathbb{R}^n : F + d \subset F\}. \tag{2.10}$$

It is easy to observe that for any $d \in 0^+F$, d belongs to the set on the right-hand side of the relation (2.10) by choosing in particular $\lambda = 1$. Now suppose that $d \in \mathbb{R}^n$ belongs to the set on the right-hand side of the relation (2.10). Therefore, for any $x \in F$, $x + d \in F$. Invoking the condition iteratively, $x + kd \in F$ for $k \in \mathbb{N}$. Because F is convex, for any $\bar{\lambda} \in [0, 1]$,

$$(1-\bar{\lambda})x + \bar{\lambda}(x + kd) = x + \bar{\lambda}kd \in F, \ \forall \ k \in \mathbb{N}.$$

Denoting $\lambda = \bar{\lambda}k \geq 0$, the above condition reduces to $x + \lambda d \in F$ for every $\lambda \geq 0$. As this relation holds for any $x \in F$, $d \in 0^+F$, thereby establishing the relation (2.10).

Below we present some properties of the recession cone with proofs from Bertsekas [12].

Proposition 2.22 *Consider a closed convex set $F \subset \mathbb{R}^n$. Then the following holds:*

(i) *0^+F is a closed convex cone.*

(ii) *$d \in 0^+F$ if and only if there exists $x \in F$ such that $x + \lambda d \in F$ for every $\lambda \geq 0$.*

(iii) *The set F is bounded if and only if $0^+F = \{0\}$.*

Proof. (i) Suppose that $d \in 0^+F$, which implies that for every $x \in F$ and $\lambda \geq 0$, $x + \lambda d \in F$. Consider $\alpha > 0$. Denote $\bar{\lambda} = \lambda/\alpha \geq 0$. Then,

$$x + \bar{\lambda}(\alpha d) = x + \lambda d \in F,$$

which implies $\alpha d \in 0^+F$ for $\alpha > 0$. For $\alpha = 0$, it is trivial. Thus, 0^+F is a cone.

Suppose that $d_1, d_2 \in 0^+F$, which implies for every $x \in F$ and $\lambda \geq 0$,

$$x + \lambda d_i \in F, \ i = 1, 2.$$

By the convexity of F, for any $\tilde{\lambda} \in [0, 1]$,

$$(1 - \tilde{\lambda})(x + \lambda d_1) + \tilde{\lambda}(x + \lambda d_2) = x + \lambda((1 - \tilde{\lambda})d_1 + \tilde{\lambda}d_2) \in F,$$

which yields that $(1 - \tilde{\lambda})d_1 + \tilde{\lambda}d_2 \in 0^+F$, thus implying the convexity of 0^+F.

Finally, to establish the closedness of 0^+F, suppose that $d \in cl\ 0^+F$, which implies that there exists $\{d_k\} \subset 0^+F$. Therefore, for every $x \in F$ and every $\lambda \geq 0$, $x + \lambda d_k \in F$. Because F is closed,

$$x + \lambda d \in F, \ \forall\ x \in F, \ \forall\ \lambda \geq 0,$$

which implies that $d \in 0^+F$, thereby implying that 0^+F is closed.

(ii) If $d \in 0^+F$, then from the definition of recession cone itself, the condition is satisfied. Conversely, suppose that $d \in \mathbb{R}^n$ is such that there exists $x \in F$ satisfying

$$x + \lambda d \in F, \ \forall\ \lambda \geq 0.$$

Without loss of generality, assume that $d \neq 0$. Consider arbitrary $\tilde{x} \in F$. Because 0^+F is a cone, it suffices to prove that $\tilde{x} + d \in F$. Define

$$x_k = x + kd, \ k \in \mathbb{N},$$

which by the condition implies that $\{x_k\} \subset F$. If $\tilde{x} = x_k$ for some $k \in \mathbb{N}$, then again by the condition

$$\tilde{x} + d = x + (k + 1)d \in F$$

and thus we are done. So assume that $\tilde{x} \neq x_k$ for every k. Define

$$d_k = \frac{x_k - \tilde{x}}{\|x_k - \tilde{x}\|}\|d\|, \ \forall\ k \in \mathbb{N}.$$

Therefore, for $\tilde{\lambda} = \dfrac{\|y\|}{\|x_k - \tilde{x}\|} \geq 0$,

$$\tilde{x} + d_k = (1 - \tilde{\lambda})\tilde{x} + \tilde{\lambda}x_k,$$

which implies that $\tilde{x} + d_k$ lies on the line starting at $\tilde{x}$ and passing through x_k. Now consider

$$\begin{aligned}
\frac{d_k}{\|d\|} &= \frac{x_k - \tilde{x}}{\|x_k - \tilde{x}\|} \\
&= \frac{x_k - x}{\|x_k - \tilde{x}\|} + \frac{x - \tilde{x}}{\|x_k - \tilde{x}\|} \\
&= \frac{\|x_k - x\|}{\|x_k - \tilde{x}\|} \frac{x_k - x}{\|x_k - x\|} + \frac{x - \tilde{x}}{\|x_k - \tilde{x}\|} \\
&= \frac{\|x_k - x\|}{\|x_k - \tilde{x}\|} \frac{d}{\|d\|} + \frac{x - \tilde{x}}{\|x_k - \tilde{x}\|}.
\end{aligned}$$

By the construction of $\{x_k\}$, we know that it is an unbounded sequence. Therefore,

$$\frac{\|x_k - x\|}{\|x_k - \tilde{x}\|} = \frac{\|kd\|}{\|x - \tilde{x} + kd\|} \to 1 \quad \text{and} \quad \frac{x - \tilde{x}}{\|x_k - \tilde{x}\|} = \frac{x - \tilde{x}}{\|x - \tilde{x} + kd\|} \to 0,$$

which along with the preceding condition leads to $d_k \to d$. The vector $\tilde{x} + d_k \in (\tilde{x}, x_k)$ for every $k \in \mathbb{N}$ such that $\|x_k - \tilde{x}\| \geq \|d\|$, which by the convexity of F implies that $\tilde{x} + d_k \in F$. Therefore, $\tilde{x} + d_k \to \tilde{x} + d$, which by the closedness of F leads to $\tilde{x} + d \in F$. As $\tilde{x} \in F$ was arbitrarily chosen, $F + d \subset F$, thereby implying that $d \in 0^+F$.

(iii) Suppose that F is bounded. Consider $0 \neq d \in 0^+F$, which implies that for every $x \in F$,

$$x + \lambda d \in F, \ \forall \ \lambda \geq 0.$$

Therefore, as the limit $\lambda \to \infty$, $\|x + \lambda d\| \to \infty$, thereby contradicting the boundedness of F. Hence, $0^+F = \{0\}$.

Conversely, suppose that F is unbounded. Consider $x \in F$ and an unbounded sequence $\{x_k\} \subset F$. Define

$$d_k = \frac{x_k - x}{\|x_k - x\|}.$$

Observe that $\{d_k\}$ is a bounded sequence and thus by the Bolzano–Weierstrass Theorem, Proposition 1.3, has a convergent subsequence. Without loss of generality, assume that $d_k \to d$ and as $\|d_k\| = 1$, $\|d\| = 1$. For any fixed $\lambda \geq 0$, $x + \lambda d_k \in (x, x_k)$ for every $k \in \mathbb{N}$ such that $\|x_k - x\| \geq \lambda$. By the convexity of F, $x + \lambda d_k \in F$. Because $x + \lambda d_k \to x + \lambda d$, which by the closedness of F implies that

$$x + \lambda d \in F, \ \forall \ \lambda \geq 0.$$

Applying (ii) yields that $0 \neq d \in 0^+F$, thereby establishing the result. $\square$

Observe that if the set F is not closed, then the recession cone of F need not be closed. Also the equivalence in (ii) of the above proposition need not hold. To verify this claim, we present an example from Rockafellar [97]. Consider the set

$$F = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0\} \cup \{(0, 0)\},$$

which is not closed. Here the recession cone $0^+F = F$ and hence is not closed. Also $(1, 0) \notin 0^+F$ but $(1, 1)+\lambda(1, 0) \in F$ for every $\lambda \geq 0$, thereby contradicting the equivalence in (ii).

2.2.2 Hyperplane and Separation Theorems

An unbounded convex set that plays a pivotal role in the development of convex optimization is the hyperplane. A hyperplane divides the space into two half spaces. This property helps in the study of separation theorems, thus moving us a step ahead in the study of convex analysis.

Definition 2.23 A *hyperplane* $H \subset \mathbb{R}^n$ is defined as

$$H = \{x \in \mathbb{R}^n : \langle a, x\rangle = b\},$$

where $a \in \mathbb{R}^n$ with $a \neq 0$ and $b \in \mathbb{R}$. If $\bar{x} \in H$, then the hyperplane can be equivalently expressed as

$$H = \{x \in \mathbb{R}^n : \langle a, x\rangle = \langle a, \bar{x}\rangle\} = \bar{x} + \{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}.$$

Therefore, H is an affine set parallel to $\{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}$.

Definition 2.24 The hyperplane H divides the space into two half spaces, either closed or open. The *closed half spaces* associated with H are

$$H_{\leq} = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq b\} \quad \text{and} \quad H_{\geq} = \{x \in \mathbb{R}^n : \langle a, x\rangle \geq b\},$$

while the *open half spaces* associated with H are

$$H_{<} = \{x \in \mathbb{R}^n : \langle a, x\rangle < b\} \quad \text{and} \quad H_{>} = \{x \in \mathbb{R}^n : \langle a, x\rangle > b\}.$$

As already mentioned, the notion of separation is based on the fact that the hyperplane in $\mathbb{R}^n$ divides it into two parts. Before discussing the separation theorems, we first present types of separation that we will be using in our subsequent study of developing the optimality conditions for convex optimization problems.

Definition 2.25 Consider two convex sets F_1 and F_2 in $\mathbb{R}^n$. A hyperplane $H \subset \mathbb{R}^n$ is said to *separate* F_1 and F_2 if

$$\langle a, x_1\rangle \leq b \leq \langle a, x_2\rangle,\ \forall\ x_1 \in F_1,\ \forall\ x_2 \in F_2.$$

The separation is said to be *strict* if

$$\langle a, x_1\rangle \leq b < \langle a, x_2\rangle, \ \forall\ x_1 \in F_1, \ \forall\ x_2 \in F_2.$$

The separation is *proper* if

$$\sup_{x_1 \in F_1} \langle a, x_1\rangle \leq \inf_{x_2 \in F_2} \langle a, x_2\rangle \quad \text{and} \quad \inf_{x_1 \in F_1} \langle a, x_1\rangle < \sup_{x_2 \in F_2} \langle a, x_2\rangle.$$

In particular, if $F_1 = \{\bar{x}\}$ and $F_2 = F$ such that $\bar{x} \in cl\ F$, a hyperplane that separates $\{\bar{x}\}$ and F is called a *supporting hyperplane* to F at $\bar{x}$, that is,

$$\langle a, \bar{x}\rangle \leq \langle a, x\rangle, \ \forall\ x \in F.$$

The next obvious question is when will the separating hyperplane or the supporting hyperplane exist. In this respect we prove some existence results below. The proof is from Bertsekas [12].

Theorem 2.26 *(i) (Supporting Hyperplane Theorem) Consider a nonempty convex set $F \subset \mathbb{R}^n$ and $\bar{x} \notin ri\ F$. Then there exist $a \in \mathbb{R}^n$ with $a \neq 0$ and $b \in \mathbb{R}$ such that*

$$\langle a, \bar{x}\rangle \leq b \leq \langle a, x\rangle, \ \forall\ x \in F.$$

(ii) (Separation Theorem) Consider two nonempty convex sets F_1 and F_2 in $\mathbb{R}^n$ such that either $F_1 \cap F_2 = \emptyset$ or $F_1 \cap ri\ F_2 = \emptyset$. Then there exists a hyperplane in $\mathbb{R}^n$ separating them.

(iii) (Strict Separation Theorem) Consider two nonempty convex sets F_1 and F_2 in $\mathbb{R}^n$ such that $F_1 \cap F_2 = \emptyset$. Furthermore, if $F_1 - F_2$ is closed or F_1 is closed while F_2 is compact, then there exists a hyperplane in $\mathbb{R}^n$ strictly separating them. In particular, consider a nonempty closed convex set $F \subset \mathbb{R}^n$ and $\bar{x} \notin F$. Then there exist $a \in \mathbb{R}^n$ with $a \neq 0$ and $b \in \mathbb{R}$ such that

$$\langle a, \bar{x}\rangle < b \leq \langle a, x\rangle, \ \forall\ x \in F.$$

(iv) (Proper Separation Theorem) Consider a nonempty convex set $F \subset \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. There exists a hyperplane separating F and $\bar{x}$ properly if and only if

$$\bar{x} \notin ri\ F.$$

Further, consider two nonempty convex sets F_1 and F_2 in $\mathbb{R}^n$. Then $ri\ F_1 \cap ri\ F_2 = \emptyset$ if and only if there exists a hyperplane in $\mathbb{R}^n$ separating the sets properly.

Proof. (i) Consider the closure of F, $cl\ F$, which by Proposition 2.14 (iv) is also convex. Because $\bar{x} \notin ri\ F$, there exists a sequence $\{x_k\}$ such that $x_k \notin cl\ F$

and $x_k \to \bar{x}$. Denote the projection of x_k on $cl\ F$ by $\bar{x}_k$. By Proposition 2.52 (see Section 2.3), for every $k \in \mathbb{N}$,

$$\langle x_k - \bar{x}_k, x - \bar{x}_k \rangle \leq 0, \ \forall\ x \in cl\ F,$$

which implies for every $k \in \mathbb{N}$,

$$\begin{aligned} \langle \bar{x}_k - x_k, x \rangle &\geq \langle \bar{x}_k - x_k, \bar{x} \rangle \\ &= \langle \bar{x}_k - x_k, \bar{x} - x_k \rangle + \langle \bar{x}_k - x_k, x_k \rangle \\ &\geq \langle \bar{x}_k - x_k, x_k \rangle, \ \forall\ x \in cl\ F. \end{aligned}$$

Dividing the above inequality throughout by $\|\bar{x}_k - x_k\|$ and denoting $a_k = \dfrac{\bar{x}_k - x_k}{\|\bar{x}_k - x_k\|}$,

$$\langle a_k, x_k \rangle \leq \langle a_k, x \rangle, \ \forall\ x \in cl\ F, \ \forall\ k \in \mathbb{N}.$$

As $\|a_k\| = 1$ for every k, $\{a_k\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, $\{a_k\}$ has a convergent subsequence. Without loss of generality, assume that $a_k \to a$, where $a \neq 0$ with $\|a\| = 1$. Taking the limit as $k \to +\infty$ in the above inequality yields

$$\langle a, \bar{x} \rangle \leq \langle a, x \rangle, \ \forall\ x \in cl\ F.$$

Because $F \subset cl\ F$, the above inequality holds in particular for $x \in F$, that is,

$$\langle a, \bar{x} \rangle \leq b \leq \langle a, x \rangle, \ \forall\ x \in F,$$

where $b = \inf_{x \in cl\ F} \langle a, x \rangle$, thereby yielding the desired result. If $\bar{x} \in cl\ F$, then the hyperplane so obtained supports F at $\bar{x}$.

(ii) Define the set

$$F = F_1 - F_2 = \{x \in \mathbb{R}^n : x = x_1 - x_2, \ x_i \in F_i, \ i = 1, 2\}.$$

Suppose that either $F_1 \cap F_2 = \emptyset$ or $F_1 \cap ri\ F_2 = \emptyset$. Under both scenarios, $0 \notin ri\ F$. By the Supporting Hyperplane Theorem, that is (i), there exist $a \in \mathbb{R}^n$ with $a \neq 0$ such that

$$\langle a, x \rangle \geq 0, \ \forall\ x \in F,$$

which implies

$$\langle a, x_1 \rangle \geq \langle a, x_2 \rangle, \ \forall\ x_1 \in F_1, \ \forall\ x_2 \in F_2,$$

hence proving the requisite result.

(iii) We shall prove the result under the assumption that $F_2 - F_1$ is closed as by Proposition 2.15, the closedness of F_1 along with the compactness of F_2 imply that $F_1 - F_2$ is closed. As $F_1 \cap F_2 = \emptyset$, $0 \notin F_2 - F_1$. Suppose that $a \in F_2 - F_1$ is the projection of origin on $F_2 - F_1$. Therefore, there exist $\bar{x}_i \in F_i$, $i = 1, 2$, such that $a = \bar{x}_2 - \bar{x}_1$. Define $\bar{x} = \dfrac{\bar{x}_1 + \bar{x}_2}{2}$. Then the projection of $\bar{x}$ on $cl\ F_1$ is $\bar{x}_1$ while that on $cl\ F_2$ is $\bar{x}_2$. By Proposition 2.52,

$$\langle \bar{x} - \bar{x}_i, x_i - \bar{x}_i \rangle \leq 0, \ \forall\ x_i \in F_i,\ i = 1, 2,$$

which implies

$$\begin{aligned} \langle a, x_1 \rangle &\leq \langle a, \bar{x} \rangle - \frac{\|a\|^2}{2} < \langle a, \bar{x} \rangle, \ \forall\ x_1 \in F_1, \\ \langle a, \bar{x} \rangle &< \langle a, \bar{x} \rangle + \frac{\|a\|^2}{2} \leq \langle a, x_2 \rangle, \ \forall\ x_2 \in F_2. \end{aligned}$$

Denoting $b = \langle a, \bar{x} \rangle$, the above inequality leads to

$$\langle a, x_1 \rangle < b < \langle a, x_2 \rangle, \ \forall\ x_i \in F_i,\ i = 1, 2,$$

thus obtaining the strict separation result.

Now consider a closed convex set $F \subset \mathbb{R}^n$ with $\bar{x} \notin F$. Taking $F_1 = F$ and $F_2 = \{\bar{x}\}$ in the strict separation result, there exist $a \in \mathbb{R}^n$ with $a \neq 0$ and $b \in \mathbb{R}^n$ such that

$$\langle a, \bar{x} \rangle < b < \langle a, x \rangle, \ \forall\ x \in F.$$

Defining $\bar{b} = \inf_{x \in F} \langle a, x \rangle$, the above inequality yields

$$\langle a, \bar{x} \rangle < \bar{b} \leq \langle a, x \rangle, \ \forall\ x \in F,$$

as desired.

(iv) Suppose that there exists a hyperplane that separates F and $\bar{x}$ properly; that is, there exists $a \in \mathbb{R}^n$ with $a \neq 0$ such that

$$\langle a, \bar{x} \rangle \leq \inf_{x \in F} \langle a, x \rangle \quad \text{and} \quad \langle a, \bar{x} \rangle < \sup_{x \in F} \langle a, x \rangle.$$

We claim that $\bar{x} \notin ri\ F$. Suppose on the contrary that $\bar{x} \in ri\ F$. Therefore by the conditions of proper separation, $\langle a, . \rangle$ attains its minimum at $\bar{x}$ over F. By the assumption that $\bar{x} \in ri\ F$ implies that $\langle a, x \rangle = \langle a, \bar{x} \rangle$ for every $x \in F$, thereby violating the strict inequality. Hence the supposition was wrong and $\bar{x} \notin ri\ F$.

Conversely, suppose that $\bar{x} \notin ri\ F$. Consider the following two cases.

(a) $\bar{x} \notin aff\ F$: Because $aff\ F$ is a closed convex subset of $\mathbb{R}^n$, by the Strict Separation Theorem, that is (iii), there exists $a \in \mathbb{R}^n$ with $a \neq 0$ such that

$$\langle a, \bar{x} \rangle < \langle a, x \rangle, \ \forall\ x \in aff\ F.$$

As $F \subset aff\ F$, the above inequality holds for every $x \in F$ and hence

$$\langle a, \bar{x} \rangle \leq \inf_{x \in F} \langle a, x \rangle \quad \text{and} \quad \langle a, \bar{x} \rangle < \sup_{x \in F} \langle a, x \rangle,$$

thereby establishing the proper separation between F and $\bar{x}$.

(b) $\bar{x} \in aff\ F$: Consider a subspace C parallel to $aff\ F$ and define the *orthogonal complement* of C as

$$C^{\perp} = \{x^* \in \mathbb{R}^n : \langle x^*, x \rangle = 0,\ \forall\ x \in C\}.$$

Define $\tilde{F} = F + C^{\perp}$ and thus, by Proposition 2.15 (vi), $ri\ \tilde{F} = ri\ F + C^{\perp}$. We claim that $\bar{x} \notin ri\ \tilde{F}$. On the contrary, assume that $\bar{x} \in ri\ \tilde{F}$, which implies that there exists $x \in ri\ F$ such that $\bar{x} - x \in C^{\perp}$. As $\bar{x}, x \in aff\ F$, $\bar{x} - x \in C$. Therefore, $\|\bar{x} - x\|^2 = 0$, thereby yielding $\bar{x} = x \in ri\ F$, which is a contradiction. Therefore, $\bar{x} \notin ri\ \tilde{F}$. By the Supporting Hyperplane Theorem, that is (i), there exists $a \in \mathbb{R}^n$ with $a \neq 0$ such that

$$\langle a, \bar{x} \rangle \leq \langle a, \tilde{x} \rangle,\ \forall\ \tilde{x} \in \tilde{F},$$

which implies that

$$\langle a, \bar{x} \rangle \leq \langle a, x + y \rangle,\ \forall\ x \in F,\ \forall\ y \in C^{\perp}.$$

Suppose that $\langle a, \bar{y} \rangle \neq 0$ for some $\bar{y} \in C^{\perp}$. Without loss of generality, let $\langle a, \bar{y} \rangle > 0$. Consider $\tilde{x} = x + \alpha \bar{y}$. Therefore, as the limit $\alpha \to -\infty$, $\langle a, \tilde{x} \rangle \to -\infty$, thereby contradicting the above inequality. Thus,

$$\langle a, y \rangle = 0,\ \forall\ y \in C^{\perp}.$$

Now by Proposition 2.14, $ri\ \tilde{F}$ is nonempty and thus $\langle a, x \rangle$ is not constant over $\tilde{F}$. Thus, by the above condition on $C^{\perp}$,

$$\begin{aligned} \langle a, \bar{x} \rangle &< \sup_{\tilde{x} \in \tilde{F}} \langle a, \tilde{x} \rangle \\ &= \sup_{x \in F} \langle a, x \rangle + \sup_{y \in C^{\perp}} \langle a, y \rangle \ = \ \sup_{x \in F} \langle a, x \rangle, \end{aligned}$$

thereby establishing the proper separation between F and $\bar{x}$.

Thus, the equivalence between the proper separation of F and $\bar{x}$ and the fact that $\bar{x} \notin ri\ F$ is proved.

Consider the nonempty convex sets $F_1, F_2 \subset \mathbb{R}^n$. Define $F = F_2 - F_1$, which by Proposition 2.15 (v) and (vi) implies that $ri\ F = ri\ F_1 - ri\ F_2$.

Therefore, $ri\ F_1 \cap ri\ F_2 = \emptyset$ if and only if $0 \notin ri\ F$. By the proper separation result, $0 \notin ri\ F$ is equivalent to the existence of $a \in \mathbb{R}^n$ with $a \neq 0$ such that

$$0 \leq \inf_{x \in F} \langle a, x \rangle \quad \text{and} \quad 0 < \sup_{x \in F} \langle a, x \rangle.$$

By Proposition 1.7,

$$\sup_{x_1 \in F_1} \langle a, x_1 \rangle \leq \inf_{x_2 \in F_2} \langle a, x_2 \rangle \quad \text{and} \quad \inf_{x_1 \in F_1} \langle a, x_1 \rangle < \sup_{x_2 \in F_2} \langle a, x_2 \rangle,$$

thereby completing the proof. □

A consequence of the Separation Theorem is the following characterization of a closed convex set.

Theorem 2.27 *A closed convex set $F \subset \mathbb{R}^n$ is the intersection of closed half spaces containing it. Consequently for any $\tilde{F} \subset \mathbb{R}^n$, $cl\ co\ \tilde{F}$ is the intersection of all the closed half spaces containing $\tilde{F}$.*

Proof. Without loss of generality, we assume that $F \neq \mathbb{R}^n$, otherwise the result is trivial. For any $\bar{x} \notin F$, define $F_1 = \{\bar{x}\}$ and $F_2 = F$. Therefore by Theorem 2.26 (iii), there exist $(a, b) \in \mathbb{R}^n \times \mathbb{R}$ with $a \neq 0$ such that

$$\langle a, \bar{x} \rangle < b \leq \langle a, x \rangle, \ \forall\ x \in F,$$

which implies that a closed half space associated with the supporting hyperplane contains F and not $\bar{x}$. Thus the intersection of the closed half spaces containing F has no points that are not in F.

For any $\tilde{F} \subset \mathbb{R}^n$, taking $F = cl\ co\ \tilde{F}$ and applying the result for closed convex set F yields that $cl\ co\ \tilde{F}$ is the intersection of all the closed half spaces containing $\tilde{F}$. □

Another application of the separation theorem is the famous Helly's Theorem. We state the result from Rockafellar [97] without proof.

Proposition 2.28 *(Helly's Theorem) Consider a collection of nonempty closed convex sets $F_i,\ i \in I$ in $\mathbb{R}^n$, where I is an arbitrary index set. Assume that the sets F_i have no common direction of recession. If every subcollection consisting of $n+1$ or fewer sets has nonempty intersection, then $\bigcap_{i \in I} F_i$ is nonempty.*

The supporting hyperplanes through the boundary points of a set characterizes the convexity of the set that we present in the result below. The proof is from Schneider [103].

Proposition 2.29 *Consider a closed set $F \subset \mathbb{R}^n$ such that $int\ F$ is nonempty and through each boundary point of F there is a supporting hyperplane. Then F is convex.*

Proof. Suppose that F is not convex, which implies that there exist $x, y \in F$ such that $z \in [x, y]$ but $z \notin F$. Because $int\ F$ is nonempty, there exists some $a \in int\ F$ such that x, y and a are affinely independent. Also as F is closed, $[a, z)$ meets the boundary of F, say at $b \in F$. By the given hypothesis, there is a supporting hyperplane H to F through b with $a \notin H$. Therefore, H meets $aff\ \{x, y, a\}$ in a line and hence x, y and a must lie on the same side of the line, which is a contradiction. Hence, F is a convex set.

2.2.3 Polar Cones

From the previous discussions, we know that closed half spaces are closed convex sets and by Proposition 2.3 that arbitrary intersection of half spaces give rise to another closed convex set. One such class is of the polar cones.

Definition 2.30 Consider a set $F \subset \mathbb{R}^n$. The cone defined as

$$F^\circ = \{x^* \in \mathbb{R}^n : \langle x^*, x \rangle \leq 0,\ \forall\ x \in F\}$$

is called the *polar cone* of F. Observe that the elements of the polar cone make an obtuse angle with every element of the set. The cone $F^{\circ\circ} = (F^\circ)^\circ$ is called the *bipolar cone* of the set F.

Thus, the polar of the set F is a closed convex cone irrespective of whether F is closed convex or not. We present some properties of polar and bipolar cones.

Proposition 2.31 *(i) Consider two sets $F_1, F_2 \subset \mathbb{R}^n$ such that $F_1 \subset F_2$. Then $F_2^\circ \subset F_1^\circ$.*

(ii) Consider a nonempty set $F \subset \mathbb{R}^n$. Then

$$F^\circ = (cl\ F)^\circ = (co\ F)^\circ = (cone\ co\ F)^\circ.$$

(iii) Consider a nonempty set $F \subset \mathbb{R}^n$. Then

$$F^{\circ\circ} = cl\ cone\ co\ F.$$

If F is a convex cone, $F^{\circ\circ} = cl\ F$ and in addition if F is closed, $F^{\circ\circ} = F$.

(iv) Consider two cones $K_i \in \mathbb{R}^{n_i},\ i = 1, 2$. Then

$$(K_1 \times K_2)^\circ = K_1^\circ \times K_2^\circ.$$

(v) Consider two cones $K_1, K_2 \subset \mathbb{R}^n$. Then

$$(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ.$$

(vi) Consider two closed convex cones $K_1, K_2 \subset \mathbb{R}^n$. Then

$$(K_1 \cap K_2)^\circ = cl(K_1^\circ + K_2^\circ).$$

The closure is superfluous under the condition $K_1 \cap int\ K_2 \neq \emptyset$.

Proof. (i) Suppose that $x^* \in F_2^\circ$, which implies that

$$\langle x^*, x\rangle \leq 0, \ \forall\ x \in F_2.$$

Because $F_1 \subset F_2$, the above inequality leads to

$$\langle x^*, x\rangle \leq 0, \ \forall\ x \in F_1,$$

thereby showing that $F_2^\circ \subset F_1^\circ$.

(ii) As $F \subset cl\ F$, by (i) $(cl\ F)^\circ \subset F^\circ$. Conversely, suppose that $x^* \in F^\circ$. Consider $x \in cl\ F$, which implies that there exists $\{x_k\} \subset F$ such that $x_k \to x$. Because $x^* \in F^\circ$, by Definition 2.30,

$$\langle x^*, x_k\rangle \leq 0,$$

which implies that

$$\langle x^*, x\rangle \leq 0.$$

Because $x \in cl\ F$ was arbitrary, the above inequality holds for every $x \in cl\ F$ and hence $x^* \in (cl\ F)^\circ$, thus yielding $F^\circ = (cl\ F)^\circ$.

As $F \subset co\ F$, by (i) $(co\ F)^\circ \subset F^\circ$. Conversely, suppose that $x^* \in F^\circ$, which implies

$$\langle x^*, x\rangle \leq 0, \ \forall\ x \in F.$$

For any $\lambda \in [0,1]$,

$$\langle x^*, (1-\lambda)x + \lambda y\rangle \leq 0, \ \forall\ x, y \in F,$$

which implies that

$$\langle x^*, z\rangle \leq 0, \ \forall\ z \in co\ F.$$

Therefore, $x^* \in (co\ F)^\circ$, as desired.

Also, because $F \subset cone\ F$, again by (i) $(cone\ F)^\circ \subset F^\circ$. Conversely, suppose that $x^* \in F^\circ$. For any $\lambda \geq 0$,

$$\langle x^*, \lambda x\rangle \leq 0, \ \forall\ x \in F,$$

which implies that

$$\langle x^*, z\rangle \leq 0, \ \forall\ z \in cone\ F.$$

Therefore, $x^* \in (cone\ F)^\circ$, thereby yielding the desired result.

(iii) We shall first establish the case when F is a closed convex cone. Suppose that $x \in F$. By Definition 2.30 of F°,

$$\langle x^*, x\rangle \leq 0, \ \forall\ x^* \in F^\circ,$$

which implies that $x \in F^{\circ\circ}$. Therefore, $F \subset F^{\circ\circ}$.

Conversely, suppose that $\bar{x} \in F^{\circ\circ}$. We claim that $x \in F$. On the contrary, assume that $\bar{x} \notin F$. Because F is closed, by Theorem 2.26 (iii), there exist $a \in \mathbb{R}^n$ with $a \neq 0$ and $b \in \mathbb{R}$ such that

$$\langle a, \bar{x} \rangle < b \leq \langle a, x \rangle, \ \forall \ x \in F.$$

As F is a cone, $0 \in F$, which along with the above inequality implies that $b \leq 0$ and $\langle a, \bar{x} \rangle < 0$. We claim that $a \in -F^{\circ}$. If not, then there exists $\tilde{x} \in F$ such that $\langle a, \tilde{x} \rangle < 0$. Choosing $\tilde{\lambda} > 0$ such that $\langle a, \tilde{\lambda}\tilde{x} \rangle < b$. Again, as F is a cone, $\tilde{\lambda}\tilde{x} \in F$, thereby contradicting the fact that

$$b \leq \langle a, x \rangle, \ \forall \ x \in F.$$

Therefore, $a \in -F^{\circ}$. Because $\langle a, \bar{x} \rangle < 0$ for $\bar{x} \in F^{\circ\circ}$, it contradicts $a \in -F^{\circ}$. Thus we arrive at a contradiction and hence $F^{\circ\circ} \subset F$, thereby leading to the requisite result.

Now from (ii), it is obvious that

$$F^{\circ} = (cl \ cone \ co \ F)^{\circ}.$$

Therefore,

$$F^{\circ\circ} = (cl \ cone \ co \ F)^{\circ\circ},$$

which by the fact that *cl cone co* F is a closed convex cone yields that

$$F^{\circ\circ} = cl \ cone \ co \ F,$$

as desired. If F is a convex cone, from the above condition it is obvious that $F^{\circ\circ} = cl \ F$.

(iv) Suppose that $d = (d_1, d_2) \in (K_1 \times K_2)^{\circ}$, which implies that

$$\langle d, x \rangle \leq 0, \ \forall \ x \in K_1 \times K_2.$$

Therefore, for $x = (x_1, x_2) \in K_1 \times K_2$,

$$\langle d_1, x_1 \rangle + \langle d_2, x_2 \rangle \leq 0, \ \forall \ x_1 \in K_1, \ \forall \ x_2 \in K_2.$$

Because K_1 and K_2 are cones, $0 \in K_i, \ i = 1, 2$. In particular, for $x_2 = 0$, the above inequality reduces to

$$\langle d_1, x_1 \rangle \leq 0, \ \forall \ x_1 \in K_1,$$

which implies that $d_1 \in K_1^{\circ}$. Similarly it can be shown that $d_2 \in K_2^{\circ}$. Thus, $d \in K_1^{\circ} \times K_2^{\circ}$, thereby leading to $(K_1 \times K_2)^{\circ} \subset K_1^{\circ} \times K_2^{\circ}$.

Conversely, suppose that $d_i \in K_i^{\circ}, \ i = 1, 2$, which implies

$$\langle d_i, x_i \rangle \leq 0, \ \forall \ x_i \in K_i, \ i = 1, 2.$$

Therefore,

$$\langle (d_1, d_2), (x_1, x_2) \rangle \leq 0, \ \forall \ (x_1, x_2) \in K_1 \times K_2,$$

which yields $(d_1, d_2) \in (K_1 \times K_2)^\circ$, that is, $K_1^\circ \times K_2^\circ \subset (K_1 \times K_2)^\circ$, thereby proving the result.

(v) Suppose that $x^* \in (K_1 + K_2)^\circ$, which implies that for $x_i \in K_i, i = 1, 2$,

$$\langle x^*, x_1 + x_2 \rangle \leq 0, \ \forall \ x_1 \in K_1, \ \forall \ x_2 \in K_2.$$

Because K_1 and K_2 are cones, $0 \in K_i, \ i = 1, 2$, which reduces the above inequality to

$$\langle x^*, x_i \rangle \leq 0, \ \forall \ x_i \in K_i, \ i = 1, 2.$$

Therefore, $x^* \in K_1^\circ \cap K_2^\circ$, thereby leading to $(K_1 + K_2)^\circ \subset K_1^\circ \cap K_2^\circ$.

Conversely, suppose that $x^* \in K_1^\circ \cap K_2^\circ$, which implies that

$$\langle x^*, x_i \rangle \leq 0, \ \forall \ x_i \in K_i, \ i = 1, 2.$$

Thus, for $x = x_1 + x_2 \in K_1 + K_2$, the above inequality leads to

$$\langle x^*, x \rangle \leq 0, \ \forall \ x \in K_1 + K_2,$$

which implies that $x^* \in (K_1 + K_2)^\circ$, thus yielding the desired result.

(vi) Replacing K_i by $K_i^\circ, \ i = 1, 2$, in (iv) along with (iii) leads to

$$(K_1^\circ + K_2^\circ)^\circ = K_1 \cap K_2.$$

Again by (iii), the above condition becomes

$$cl \ (K_1^\circ + K_2^\circ) = (K_1 \cap K_2)^\circ,$$

thereby yielding the requisite result. □

Similar to the polar cone, we have the notion of a positive polar cone.

Definition 2.32 Consider a set $F \subset \mathbb{R}^n$. The *positive polar cone* to the set F is defined as

$$F^+ = \{x^* \in \mathbb{R}^n : \langle x^*, x \rangle \geq 0, \ \forall \ x \in F\}.$$

Observe that $F^+ = (-F)^\circ = -F^\circ$.

The notion of polarity will play a major role in the study of tangent and normal cones that are polar to each other. These cones are important in the development of convex optimization.

2.2.4 Tangent and Normal Cones

In the analysis of a constrained optimization problem, we try to look at the local behavior of the function in the neighboring feasible points. To move from one point to another feasible point, we need a direction that leads to the notion of feasible directions.

Definition 2.33 Let $F \subset \mathbb{R}^n$ and $\bar{x} \in F$. A vector $d \in \mathbb{R}^n$ is said to be a *feasible direction* of F at $\bar{x}$ if there exists $\bar{\alpha} > 0$ such that

$$\bar{x} + \alpha d \in F, \ \forall \ \alpha \in [0, \bar{\alpha}].$$

It is easy to observe that the set of all feasible directions forms a cone. For a convex set F, the set of feasible directions at $\bar{x}$ is of the form $\alpha(x - \bar{x})$ where $\alpha \in [0, 1]$ and $x \in F$. However, in case F is nonconvex, the set of feasible directions may reduce to singleton $\{0\}$. For example, consider the nonconvex set $F = \{(-1, 1), (1, 1)\}$. The only feasible direction possible is $\{0\}$. This motivates us to introduce the concept of tangent cones that would provide local information of the set F at a point even when feasible direction is just zero. The notion of tangent cones may be considered a generalization of the tangent concept in a smooth scenario to that in a nonsmooth case.

Definition 2.34 Consider a set $F \subset \mathbb{R}^n$ and $\bar{x} \in F$. A vector $d \in \mathbb{R}^n$ is said to be a *tangent* to F at $\bar{x}$ if there exist $\{d_k\} \subset \mathbb{R}^n$ with $d_k \to d$ and $\{t_k\} \subset \mathbb{R}_+$ with $t_k \to 0$ such that

$$\bar{x} + t_k d_k \in F, \ \forall \ k \in \mathbb{N}.$$

Observe that if d is a tangent, then so is λd for $\lambda \geq 0$. Thus, the collection of all tangents form a cone known as the *tangent cone* denoted by $T_F(\bar{x})$ and given by

$$T_F(\bar{x}) = \{d \in \mathbb{R}^n : \text{there exist } d_k \to d, \ t_k \downarrow 0 \text{ such that} \\ \bar{x} + t_k d_k \in F, \ \forall \ k \in \mathbb{N}\}.$$

In the above definition, denote $x_k = \bar{x} + t_k d_k \in F$. Taking the limit as $k \to +\infty$, $t_k \to 0$, and $d_k \to d$, which implies that $t_k d_k \to 0$, thereby leading to $x_k \to \bar{x}$. Also from construction,

$$\frac{x_k - \bar{x}}{t_k} = d_k \to d.$$

Thus, the tangent cone can be equivalently expressed as

$$T_F(\bar{x}) = \{d \in \mathbb{R}^n : \text{there exist } \{x_k\} \subset F, \ x_k \to \bar{x}, \ t_k \downarrow 0 \text{ such that} \\ \frac{x_k - \bar{x}}{t_k} \to d\}.$$

Figure 2.4 is a representation of the tangent cone to a convex set F. Next we present some properties of the tangent cone. The proofs are from Hiriart-Urruty and Lemaréchal [63].

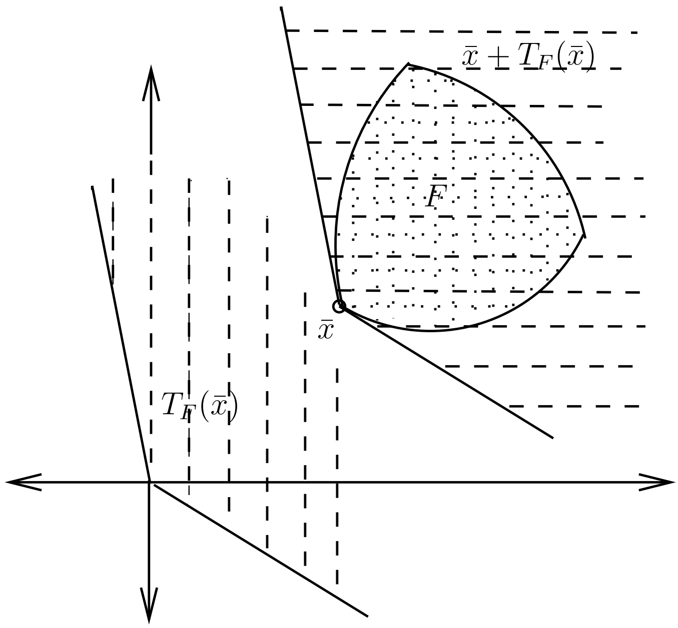

FIGURE 2.4: Tangent cone.

Theorem 2.35 *Consider a set $F \subset \mathbb{R}^n$ and $\bar{x} \in F$. Then the following hold:*

(i) $T_F(\bar{x})$ is closed.

(ii) If F is convex, $T_F(\bar{x})$ is the closure of the cone generated by $F - \{\bar{x}\}$, that is,

$$T_F(\bar{x}) = cl\ cone(F - \bar{x})$$

and hence convex.

Proof. (i) Suppose that $\{d_k\} \subset T_F(\bar{x})$ such that $d_k \to d$. Because $d_k \in T_F(\bar{x})$, there exist $\{x_k^r\} \subset F$ with $x_k^r \to \bar{x}$ and $\{t_k^r\} \subset \mathbb{R}_+$ with $t_k^r \to 0$ such that

$$\frac{x_k^r - \bar{x}}{t_k^r} \to d_k, \ \forall\ k \in \mathbb{N}.$$

For a fixed k, one can always find $\bar{r}$ such that

$$\|\frac{x_k^r - \bar{x}}{t_k^r} - d_k\| < \frac{1}{k}, \ \forall\ r \geq \bar{r}.$$

Taking the limit as $k \to +\infty$, one can generate a sequence $\{x_k\} \subset F$ with $x_k \to \bar{x}$ and $\{t_k\} \subset \mathbb{R}_+$ with $t_k \to 0$ such that

$$\frac{x_k - \bar{x}}{t_k} \to d.$$

Thus, $d \in T_F(\bar{x})$, thereby establishing that $T_F(\bar{x})$ is closed.

(ii) Suppose that $d \in T_F(\bar{x})$, which implies that there exist $\{x_k\} \subset F$ with $x_k \to \bar{x}$ and $\{t_k\} \subset \mathbb{R}_+$ with $t_k \to 0$ such that

$$\frac{x_k - \bar{x}}{t_k} \to d.$$

Observe that $x_k - \bar{x} \in F - \bar{x}$. As $t_k > 0$, $1/t_k > 0$, which implies that

$$\frac{x_k - \bar{x}}{t_k} \in \mathit{cone}\ (F - \bar{x}),$$

thereby implying that $d \in \mathit{cl}\ \mathit{cone}\ (F - \bar{x})$. Hence

$$T_F(\bar{x}) \subset \mathit{cl}\ \mathit{cone}\ (F - \bar{x}). \tag{2.11}$$

Conversely, consider an arbitrary but fixed element $x \in F$. Define a sequence

$$x_k = \bar{x} + \frac{1}{k}(x - \bar{x}), \ \ k \in \mathbb{N}.$$

By the convexity of F, it is obvious that $\{x_k\} \subset F$. Taking the limit as $k \to +\infty$, $x_k \to \bar{x}$, then by construction

$$k(x_k - \bar{x}) = x - \bar{x}.$$

Denoting $t_k = \dfrac{1}{k} > 0$, $t_k \to 0$ such that $\dfrac{(x_k - \bar{x})}{t_k} \to x - \bar{x}$, which implies that $x - \bar{x} \in T_F(\bar{x})$. Because $x \in F$ is arbitrary, $F - \bar{x} \subset T_F(\bar{x})$. As $T_F(\bar{x})$ is a cone, *cone* $(F - \bar{x}) \subset T_F(\bar{x})$. By (i), $T_F(\bar{x})$ is closed, which implies

$$\mathit{cl}\ \mathit{cone}\ (F - \bar{x}) \subset T_F(\bar{x}).$$

The above inclusion along with the reverse inclusion (2.11) yields the desired equality.

Because F is convex, the set $F - \bar{x}$ is also convex. Invoking Proposition 2.14 (iv) implies that $T_F(\bar{x})$ is a convex set. $\square$

We now move on to another conical approximation of a convex set that is the normal cone that plays a major role in establishing the optimality conditions.

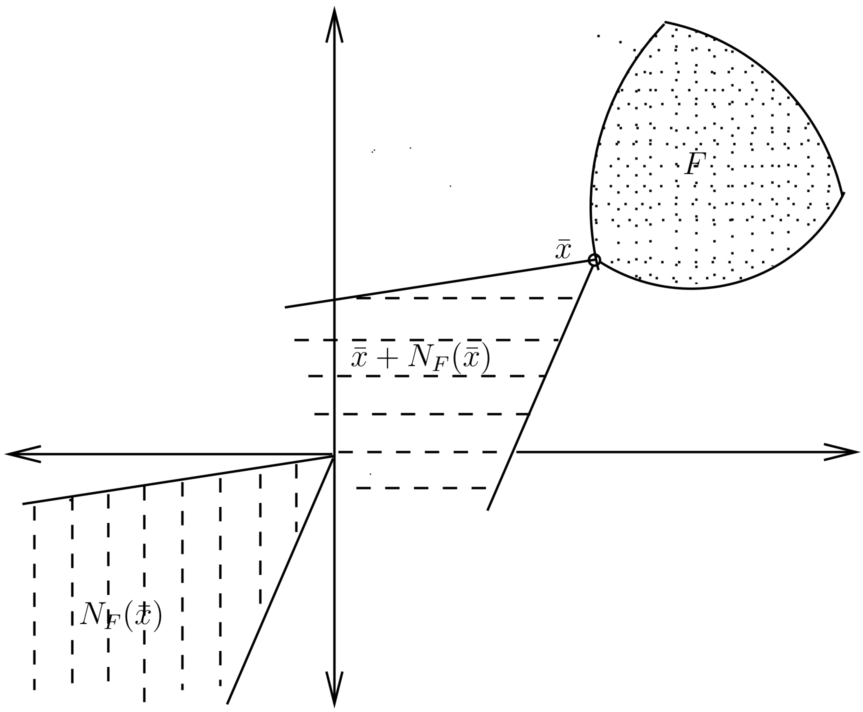

FIGURE 2.5: Normal cone.

Definition 2.36 Consider a convex set $F \subset \mathbb{R}^n$ and $\bar{x} \in F$. A vector $d \in \mathbb{R}^n$ is *normal* to F at $\bar{x}$ if

$$\langle d, x - \bar{x} \rangle \leq 0, \ \forall \ x \in F.$$

Observe that if d is a normal, then so is λd for $\lambda \geq 0$. The collection of all normals forms the cone called *normal cone* and is denoted by $N_F(\bar{x})$.

For a convex set, the relation between the tangent cone and the normal cone is given by the proposition below.

Proposition 2.37 *Consider a convex set $F \subset \mathbb{R}^n$. Then $T_F(\bar{x})$ and $N_F(\bar{x})$ are polar to each other, that is,*

$$N_F(\bar{x}) = (T_F(\bar{x}))^\circ \quad \textit{and} \quad T_F(\bar{x}) = (N_F(\bar{x}))^\circ.$$

Proof. Suppose that $d \in N_F(\bar{x})$, which implies that

$$\langle d, x - \bar{x} \rangle \leq 0, \ \forall \ x \in F.$$

Observe that for $x \in F$, $x - \bar{x} \in F - \bar{x}$, which implies that $d \in (F - \bar{x})^\circ$. By Proposition 2.31 (ii) along with the convexity of F and hence of $F - \bar{x}$, and Theorem 2.35 (ii),

$$d \in (cl\ cone\ (F - \bar{x}))^\circ = (T_F(\bar{x}))^\circ,$$

thereby implying that $N_F(\bar{x}) \in (T_F(\bar{x}))^\circ$.

Conversely, suppose that $d \in (T_F(\bar{x}))^\circ$. As $F - \bar{x} \subset T_F(\bar{x})$, by Proposition 2.31 (i), $(T_F(\bar{x}))^\circ \subset (F - \bar{x})^\circ$, which implies that

$$\langle d, x - \bar{x} \rangle \leq 0, \ \forall\ x \in F,$$

that is, $d \in N_F(\bar{x})$. Therefore, $N_F(\bar{x}) = (T_F(\bar{x}))^\circ$ as desired.

For a convex set F, $T_F(\bar{x})$ is a closed convex cone. Therefore, by Proposition 2.31 (iii),

$$(N_F(\bar{x}))^\circ = (T_F(\bar{x}))^{\circ\circ} = T_F(\bar{x}),$$

thereby yielding the requisite result. □

Figure 2.5 is a representation of the normal cone to a convex set F. Observe that the normal cone is polar to the tangent cone in Figure 2.4. Now we present some simple examples for tangent cones and normal cones.

Example 2.38 (i) For a convex set $F \subset \mathbb{R}^n$, it can be easily observed that $T_F(x) = \mathbb{R}^n$ for every $x \in int\ F$ and by polarity, $N_F(x) = \{0\}$ for every $x \in int\ F$.
(ii) For a closed convex cone $K \subset \mathbb{R}^n$, by Theorem 2.35 (ii) it is obvious that $T_K(0) = K$ while by Proposition 2.37, $N_K(0) = K^\circ$. Also, for $0 \neq x \in K$, from the definition of normal cone,

$$N_K(x) = \{d \in \mathbb{R}^n : d \in K^\circ,\ \langle d, x \rangle = 0\}.$$

(iii) Consider the closed convex set $F \subset \mathbb{R}^n$ given by

$$F = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i,\ i = 1, 2, \ldots, m\}$$

and define the *active index set* $I(x) = \{i \in \{1, 2, \ldots, m\} : \langle a_i, x \rangle = b_i\}$. The set F is called a *polyhedral set*, which we will discuss in the next section. Then

$$\begin{aligned} T_F(x) &= \{d \in \mathbb{R}^n : \langle a_i, d \rangle \leq 0,\ \forall\ i \in I(x)\}, \\ N_F(x) &= cone\ co\ \{a_i : i \in I(x)\}. \end{aligned}$$

Before moving on to discuss the polyhedral sets, we present some results on the tangent and normal cones.

Proposition 2.39 *(i) Consider two closed convex sets $F_i \subset \mathbb{R}^n,\ i = 1, 2$. Let $\bar{x} \in F_1 \cap F_2$. Then*

$$T(\bar{x}; F_1 \cap F_2) \subset T(\bar{x}; F_1) \cap T(\bar{x}; F_2),$$
$$N(\bar{x}; F_1 \cap F_2) \supset N(\bar{x}; F_1) + N(\bar{x}; F_2).$$

If in addition, $ri\ F_1 \cap ri\ F_2 \neq \emptyset$, the above inclusions hold as equality.

(ii) Consider two closed convex sets $F_i \subset \mathbb{R}^{n_i},\ i = 1, 2$. Let $\bar{x}_i \in F_i,\ i = 1, 2$. Then

$$T((\bar{x}_1, \bar{x}_2); F_1 \times F_2) = T(\bar{x}_1; F_1) \times T(\bar{x}_2; F_2),$$
$$N((\bar{x}_1, \bar{x}_2); F_1 \times F_2) = N(\bar{x}_1; F_1) \times N(\bar{x}_2; F_2).$$

(iii) Consider two closed convex sets $F_i \subset \mathbb{R}^n,\ i = 1, 2$. Let $\bar{x}_i \in F_i,\ i = 1, 2$. Then

$$T(\bar{x}_1 + \bar{x}_2; F_1 + F_2) = cl\ (T(\bar{x}_1; F_1) + T(\bar{x}_2; F_2)),$$
$$N(\bar{x}_1 + \bar{x}_2; F_1 + F_2) = N(\bar{x}_1; F_1) \cap N(\bar{x}_2; F_2).$$

Proof. (i) We first establish the result for the normal cone and then use it to derive the result for the tangent cone. Suppose that $d_i \in N_{F_i}(\bar{x}),\ i = 1, 2$, which implies that

$$\langle d_i, x_i - \bar{x} \rangle \leq 0,\ \forall\ x_i \in F_i,\ i = 1, 2.$$

For any $x \in F_1 \cap F_2$, the above inequality is still valid for $i = 1, 2$. Therefore,

$$\langle d_1 + d_2, x - \bar{x} \rangle \leq 0,\ \forall\ x \in F_1 \cap F_2,$$

which implies that $d_1 + d_2 \in N_{F_1 \cap F_2}(\bar{x})$. Because $d_i \in N_{F_i}(\bar{x})$, $i = 1, 2$, were arbitrarily chosen, $N_{F_1}(\bar{x}) + N_{F_2}(\bar{x}) \subset N_{F_1 \cap F_2}(\bar{x})$.

By Propositions 2.31 (i), (v), and 2.37,

$$T_{F_1 \cap F_2}(\bar{x}) \subset (N_{F_1}(\bar{x}) + N_{F_2}(\bar{x}))^\circ = T_{F_1}(\bar{x}) \cap T_{F_2}(\bar{x}),$$

as desired. We shall prove the equality part as an application of the subdifferential sum rule, Theorem 2.91.

(ii) Suppose that $d = (d_1, d_2) \in N_{F_1 \times F_2}(\bar{x}_1, \bar{x}_2)$, which implies

$$\langle (d_1, d_2), (x_1, x_2) - (\bar{x}_1, \bar{x}_2) \rangle \leq 0,\ \forall\ (x_1, x_2) \in F_1 \times F_2,$$

that is,

$$\langle d_1, x_1 - \bar{x}_1 \rangle + \langle d_2, x_2 - \bar{x}_2 \rangle \leq 0,\ \forall\ x_1 \in F_1,\ \forall\ x_2 \in F_2.$$

The above inequality holds in particular for $x_2 = \bar{x}_2$, thereby reducing it to

$$\langle d_1, x_1 - \bar{x}_1 \rangle \leq 0,\ \forall\ x_1 \in F_1,$$

which by Definition 2.36 implies that $d_1 \in N_{F_1}(\bar{x}_1)$. Similarly, it can be shown that $d_2 \in N_{F_2}(\bar{x}_2)$. Because $(d_1, d_2) \in N_{F_1 \times F_2}(\bar{x}_1, \bar{x}_2)$ was arbitrary, $N_{F_1 \times F_2}(\bar{x}_1, \bar{x}_2) \subset N_{F_1}(\bar{x}_1) \times N_{F_2}(\bar{x}_2)$.

Conversely, consider $d_1 \in N_{F_1}(\bar{x}_1)$ and $d_2 \in N_{F_2}(\bar{x}_2)$, which implies that $(d_1, d_2) \in N_{F_1}(\bar{x}_1) \times N_{F_2}(\bar{x}_2)$. Therefore,

$$\langle d_i, x_i - \bar{x}_i \rangle \leq 0, \ \forall \ x_i \in F_i, \ i = 1, 2,$$

which leads to

$$\langle (d_1, d_2), (x_1, x_2) - (\bar{x}_1, \bar{x}_2) \rangle \leq 0, \ \forall \ (x_1, x_2) \in F_1 \times F_2,$$

thereby yielding that $(d_1, d_2) \in N_{F_1 \times F_2}(\bar{x}_1, \bar{x}_2)$. As $d_i \in N_{F_i}(\bar{x}_i), \ i = 1, 2$, were arbitrary, $N_{F_1 \times F_2}(\bar{x}_1, \bar{x}_2) \supset N_{F_1}(\bar{x}_1) \times N_{F_2}(\bar{x}_2)$, thereby leading to the desired result. The result on the tangent cone can be obtained by applying Propositions 2.31 (iv) and 2.37.

(iii) Suppose that $d \in N_{F_1+F_2}(\bar{x}_1 + \bar{x}_2)$, which leads to

$$\langle d, x_1 - \bar{x}_1 \rangle + \langle d, x_2 - \bar{x}_2 \rangle \leq 0, \ \forall \ x_1 \in F_1, \ \forall \ x_2 \in F_2.$$

In particular, for $x_2 = \bar{x}_2$, the above inequality reduces to

$$\langle d, x_1 - \bar{x}_1 \rangle \leq 0, \ \forall \ x_1 \in F_1,$$

that is, $d \in N_{F_1}(\bar{x}_1)$. Similarly, $d \in N_{F_2}(\bar{x})$. Because $d \in N_{F_1+F_2}(\bar{x}_1 + \bar{x}_2)$ was arbitrary, $N_{F_1+F_2}(\bar{x}_1 + \bar{x}_2) \subset N_{F_1}(\bar{x}_1) \cap N_{F_2}(\bar{x}_2)$.

Conversely, consider $d \in N_{F_1}(\bar{x}_1) \cap N_{F_2}(\bar{x}_2)$. Therefore,

$$\langle d, x_i - \bar{x}_i \rangle \leq 0, \ \forall \ x_i \in F_i, \ i = 1, 2,$$

which implies that

$$\langle d, (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2) \rangle \leq 0, \ \forall \ x_1 \in F_1, \ \forall \ x_2 \in F_2.$$

This leads to $d \in N_{F_1+F_2}(\bar{x}_1 + \bar{x}_2)$. As $d \in N_{F_1}(\bar{x}_1) \cap N_{F_2}(\bar{x}_2)$ was arbitrary, $N_{F_1}(\bar{x}_1) \cap N_{F_2}(\bar{x}_2) \subset N_{F_1+F_2}(\bar{x}_1 + \bar{x}_2)$, thus establishing the desired result. The result on tangent cone can be obtained by applying Propositions 2.31 (vi) and 2.37. □

2.2.5 Polyhedral Sets

As discussed in the beginning, finite intersection of closed half spaces generate a class of convex sets known as the polyhedral sets. Here, we discuss briefly this class of sets.

Definition 2.40 A set $P \subset \mathbb{R}^n$ is said to be a *polyhedral set* if it is nonempty and is expressed as

$$P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i, \ i = 1, 2, \ldots, m\},$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$. Obviously, P is a convex set.

Polyhedral sets play an important role in the study of linear programming problems. A polyhedral set can also be considered a finite intersection of closed half spaces and hyperplane. Any hyperplane $\langle a, x \rangle = b$ can be further segregated into two half spaces, $\langle a, x \rangle \leq b$ and $\langle -a, x \rangle \leq -b$, and thus can be expressed as the form in the definition. If in the above definition of polyhedral sets, $b_i = 0,\ i = 1, 2, \ldots, m$, we get the notion of polyhedral cones.

Definition 2.41 A polyhedral set P is a *polyhedral cone* if and only if it can be expressed as the intersection of finite collection of closed half spaces whose supporting hyperplane pass through the origin. Equivalently, the polyhedral cone P is given by

$$P = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq 0,\ i = 1, 2, \ldots, m\},$$

where $a_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, m$.

Next we state some operations on the polyhedral sets and cones. For proofs, the readers are advised to refer to Rockafellar [97].

Proposition 2.42 *(i) Consider polyhedral set (cone) $P \subset \mathbb{R}^n$ and a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$. Then $A(C)$ as well as $A^{-1}(C)$ are polyhedral sets (cones).*

(ii) Consider polyhedral sets (cones) $F_i \subset \mathbb{R}^{n_i},\ i = 1, 2, \ldots, m$. Then the Cartesian product $F_1 \times F_2 \times \ldots \times F_m$ is a polyhedral set (cone).

(iii) Consider polyhedral sets (cones) $F_i \subset \mathbb{R}^n,\ i = 1, 2, \ldots, m$. Then the intersection $\bigcap_{i=1}^{m} F_i$ and the sum $\sum_{i=1}^{m} F_i$ are also polyhedral sets (cones).

With the notion of polyhedral sets, another concept that comes into the picture is that of a finitely generated set.

Definition 2.43 A set $F \subset \mathbb{R}^n$ is a *finitely generated set* if and only if there exist $x_i \in \mathbb{R}^n,\ i = 1, 2, \ldots, m$, such that for a fixed integer $j, 0 \leq j \leq m$, F is given by

$$F = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{j} \lambda_i x_i + \sum_{i=j+1}^{m} \lambda_i x_i,\ \lambda_i \geq 0,\ i = 1, 2, \ldots, m,$$
$$\sum_{i=1}^{j} \lambda_i = 1\}$$

where $\{x_1, x_2, \ldots, x_m\}$ are the *generators of the set.* For a *finitely generated cone*, it is the same set with $j = 0$ and then $\{x_1, x_2, \ldots, x_m\}$ are the *generators of the cone.*

Below we mention some characterizations and properties of polyhedral sets and finitely generated cones. The results are stated without proofs. For more details on polyhedral sets, one can refer to Bertsekas [12], Rockafellar [97], and Wets [111].

Proposition 2.44 *(i) A set (cone) is polyhedral if and only if it is finitely generated.*

(ii) The polar of a polyhedral convex set is polyhedral.

(iii) Let $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$. Then the finitely generated cone

$$F = cone\ co\{x_1, x_2, \ldots, x_m\}$$

is closed and its polar cone is a polyhedral cone given by

$$F^\circ = \{x \in \mathbb{R}^n : \langle x_i, x\rangle \leq 0,\ i = 1, 2, \ldots, m\}.$$

With all these background on convex sets, we move on to the study of convex functions.

2.3 Convex Functions

We devote this section to the study of convex functions and their properties. We also look into some special class of convex functions, namely the sublinear functions. We begin by formally defining the convex functions. But before that, let us recall some notions.

Definition 2.45 Consider a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. The *domain* of the function ϕ is defined as

$$dom\ \phi = \{x \in \mathbb{R}^n : \phi(x) < +\infty\}.$$

The *epigraph* of the function ϕ is given by

$$epi\ \phi = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \phi(x) \leq \alpha\}.$$

Observe that the notion of epigraph holds for domain points only. The function is *proper* if $\phi(x) > -\infty$ for every $x \in \mathbb{R}^n$ and *dom* ϕ is nonempty. A function is said to be *improper* if there exists $\hat{x} \in \mathbb{R}^n$ such that $\phi(\hat{x}) = -\infty$.

Definition 2.46 A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be *convex* if for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y).$$

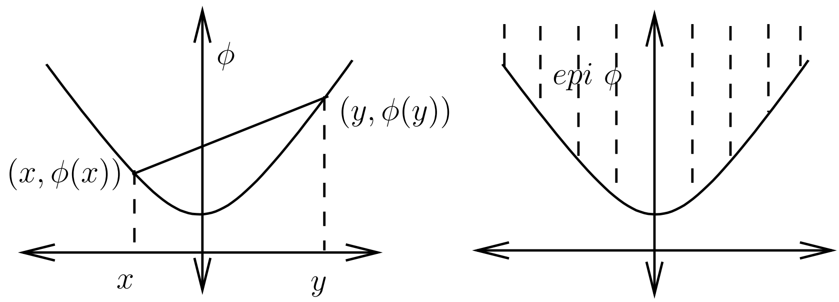

FIGURE 2.6: Graph and epigraph of convex function.

If ϕ is a convex function, then the function $\psi : \mathbb{R}^n \to \mathbb{R}$ defined as $\psi = -\psi$ is said to be a *concave function.*

Definition 2.47 A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be *strictly convex* if for distinct $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$ we have

$$\phi((1 - \lambda)x + \lambda y) < (1 - \lambda)\phi(x) + \lambda\phi(y).$$

The proposition given below is an equivalent characterization of a convex set in terms of its epigraph mentioned in Chapter 1.

Proposition 2.48 *Consider a proper function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. ϕ is convex if and only if epi ϕ is a convex set on $\mathbb{R}^n \times \mathbb{R}$.*

Proof. Suppose ϕ is convex. Consider $(x_i, \alpha_i) \in epi\ \phi,\ i = 1, 2$, which implies that $\phi(x_i) \leq \alpha_i,\ i = 1, 2$. This along with the convexity of ϕ yields that for every $\lambda \in [0, 1]$,

$$\phi((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)\phi(x_1) + \lambda\phi(x_2) \leq (1 - \lambda)\alpha_1 + \lambda\alpha_2.$$

Thus, $((1 - \lambda)x_1 + \lambda x_2, (1 - \lambda)\alpha_1 + \lambda\alpha_2) \in epi\ f$. Because (x_i, α_i), $i = 1, 2$, were arbitrary, thereby leading to the convexity of *epi* ϕ on $\mathbb{R}^n \times \mathbb{R}$.

Conversely, suppose that *epi* ϕ is convex. Consider $x_1, x_2 \in dom\ \phi$. It is obvious that $(x_i, \phi(x_i)) \in epi\ \phi,\ i = 1, 2$. By the convexity of *epi* ϕ, for every $\lambda \in [0, 1]$,

$$(1 - \lambda)(x_1, \phi(x_1)) + \lambda(x_2, \phi(x_2)) \in epi\ \phi,$$

which implies that

$$\phi((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)\phi(x_1) + \lambda\phi(x_2),\ \forall\ \lambda \in [0, 1],$$

thereby implying the convexity of ϕ, and thus establishing the result. □

Figure 2.6 represents the graph and epigraph of a convex function. Observe that the epigraph is a convex set. Another alternate characterization of the convex function is in terms of the strict epigraph set. So next we state the notion of strict epigraph and present the equivalent characterization.

Definition 2.49 The *strict epigraph* of the function ϕ is given by

$$epi_s\ \phi = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \phi(x) < \alpha\}.$$

Proposition 2.50 *Consider a proper function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. ϕ is convex if and only if $epi_s\ \phi$ is a convex set on $\mathbb{R}^n \times \mathbb{R}$.*

Proof. The necessary part, that is, the convexity of ϕ implies that *epi s* ϕ is convex, can be worked along the lines of proof of Proposition 2.48.

Conversely, suppose that $epi_s\ \phi$ is convex. Consider $x_1, x_2 \in dom\ \phi$ and $\alpha_i \in \mathbb{R},\ i = 1, 2$ such that $\phi(x_i) < \alpha_i,\ i = 1, 2$. Therefore, $(x_i, \alpha_i) \in epi_s\ \phi$, $i = 1, 2$. By the convexity of $epi_s\ \phi$, for every $\lambda \in [0, 1]$,

$$(1 - \lambda)(x_1, \alpha_1) + \lambda(x_2, \alpha_2) \in epi_s\ \phi,$$

which implies that

$$\phi((1 - \lambda)x_1 + \lambda x_2) < (1 - \lambda)\alpha_1 + \lambda\alpha_2,\ \forall\ \lambda \in [0, 1].$$

As the above inequality holds for every $\alpha_i > \phi(x_i),\ i = 1, 2$, taking the limit as $\alpha_i \to \phi(x_i),\ i = 1, 2$, the above condition becomes

$$\phi((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)\phi(x_1) + \lambda\phi(x_2),\ \forall\ \lambda \in [0, 1].$$

Because x_1 and x_2 were arbitrarily chosen, the above inequality leads to the convexity of ϕ and hence the result. □

The definitions presented above are for extended-valued functions. These definitions can also be given for a function to be convex over a convex set $F \subset \mathbb{R}^n$ as ϕ is convex when *dom* ϕ is restricted to F. However, in this book, we will be considering real-valued functions unless otherwise specified.

Next we state Jensen's inequality for the proper convex functions. The proof can be worked out using the induction and the readers are advised to do so.

Theorem 2.51 *(Jensen's Inequality) Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Let $x_i \in dom\ \phi$ and $\lambda_i \geq 0$ for $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$. Then ϕ is convex if and only if*

$$\phi(\sum_{i=1}^{m} \lambda_i x_i) \leq \sum_{i=1}^{m} \lambda_i \phi(x_i)$$

for every such collection of x_i and λ_i, $i = 1, 2, \ldots, m$.

Consider a set $F \subset \mathbb{R}^n$. The *indicator function*, $\delta_F : \mathbb{R}^n \to \bar{\mathbb{R}}$, to the set F is defined as

$$\delta_F(x) = \begin{cases} 0, & x \in F, \\ +\infty, & \text{otherwise}. \end{cases}$$

It can be easily shown that δ_F is lsc and convex if and only if F is closed and convex, respectively. Also, for sets $F_1, F_2 \subset \mathbb{R}^n$,

$$\delta_{F_1 \cap F_2}(x) = \delta_{F_1}(x) + \delta_{F_2}(x).$$

The indicator function plays an important role in the study of optimality conditions by converting a constrained problem into an unconstrained one. Consider a constrained programming problem

$$\min \ f(x) \quad \text{subject to} \quad x \in C,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $C \subset \mathbb{R}^n$. Then the associated unconstrained problem is

$$\min \ f_0(x) \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

where $f_0 : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a function given by $f_0(x) = f(x) + \delta_C(x)$, that is,

$$f_0(x) = \begin{cases} f(x), & x \in C, \\ +\infty, & \text{otherwise}. \end{cases}$$

We will look into this aspect more when we study the derivations of optimality condition for the convex programming problem (CP) presented in Chapter 1 in the subsequent chapters.

For a set $F \subset \mathbb{R}^n$, the *distance function*, $d_F : \mathbb{R}^n \to \mathbb{R}$, to F from a point $\bar{x}$ is defined as

$$d_F(\bar{x}) = \inf_{x \in F} \|x - \bar{x}\|.$$

For a convex set F, the distance function d_F is a convex function. If the infimum is attained, say at $\tilde{x} \in F$, that is,

$$\inf_{x \in F} \|x - \bar{x}\| = \|\tilde{x} - \bar{x}\|,$$

then $\tilde{x}$ is said to be a *projection* of $\bar{x}$ to F and denoted by $proj_F(\bar{x})$. Below we present an important result on projection.

Proposition 2.52 *Consider a closed convex set $F \subset \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. Then $\tilde{x} \in proj_F(\bar{x})$ if and only if*

$$\langle \bar{x} - \tilde{x}, x - \tilde{x} \rangle \leq 0, \ \forall \ x \in F. \tag{2.12}$$

Proof. Suppose that the inequality (2.12) holds for $\tilde{x} \in F$ and $\bar{x} \in \mathbb{R}^n$. For any $x \in F$, consider

$$\begin{aligned} \|x - \bar{x}\|^2 &= \|x - \bar{x}\|^2 + \|\tilde{x} - \bar{x}\|^2 - 2\langle \bar{x} - \tilde{x}, x - \tilde{x}\rangle \\ &\geq \|\tilde{x} - \bar{x}\|^2 - 2\langle \bar{x} - \tilde{x}, x - \tilde{x}\rangle, \ \forall\, x \in F. \end{aligned}$$

Because (2.12) is assumed to hold, the above condition leads to

$$\|x - \bar{x}\|^2 \geq \|\tilde{x} - \bar{x}\|^2, \ \forall\, x \in F,$$

thereby implying that $\tilde{x} \in proj_F(\bar{x})$.

Conversely, suppose that $\tilde{x} \in proj_F(\bar{x})$. Consider any $x \in F$ and for $\alpha \in [0, 1]$, define

$$x_\alpha = (1 - \alpha)\tilde{x} + \alpha x \in F.$$

Therefore,

$$\begin{aligned} \|\bar{x} - x_\alpha\|^2 &= \|(1 - \alpha)(\bar{x} - \tilde{x}) + \alpha(\bar{x} - x)\|^2 \\ &= (1 - \alpha)^2\|\bar{x} - \tilde{x}\|^2 + \alpha^2\|\bar{x} - x\|^2 + 2\alpha(1 - \alpha)\langle \bar{x} - \tilde{x}, \bar{x} - x\rangle. \end{aligned}$$

Observe that as a function of α, $\|\bar{x} - x_\alpha\|^2$ has a point of minimizer over $[0, 1]$ at $\alpha = 0$. Thus,

$$\nabla_\alpha\{\|\bar{x} - x_\alpha\|^2\} \mid_{\alpha=0} \geq 0,$$

which implies

$$2\left(-\|\bar{x} - \tilde{x}\|^2 + \langle \bar{x} - x, \bar{x} - \tilde{x}\rangle\right) \geq 0.$$

The above inequality leads to

$$-\langle \bar{x} - \tilde{x}, \bar{x} - \tilde{x}\rangle + \langle \bar{x} - x, \bar{x} - \tilde{x}\rangle = \langle \bar{x} - \tilde{x}, \tilde{x} - x\rangle \geq 0, \ \forall\, x \in F,$$

thereby yielding (2.12) and hence completing the proof. □

Another class of functions that is also convex in nature are the *sublinear functions* and *support functions*. These classes of functions will be discussed in the next subsection. But before that we present some operations on the convex functions that again belong to the class of convex functions itself.

Proposition 2.53 *(i) Consider proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, m$. Then $\phi = \sum_{i=1}^{m} \alpha_i\phi_i$ is also a convex function.*

(ii) Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and a nondecreasing proper convex function $\psi : \mathbb{R} \to \bar{\mathbb{R}}$. Then the composition function defined as $(\psi \circ \phi)(x) = \psi(\phi(x))$ is a convex function provided $\psi(+\infty) = +\infty$.

(iii) Consider a family of proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i \in I$, where I is an arbitrary index set. Then $\phi = \sup_{i \in I} \phi_i$ is a convex function.

(iv) Consider a convex set $F \subset \mathbb{R}^{n+1}$. Then $\phi(x) = \inf\{\alpha \in \mathbb{R} : (x, \alpha) \in F\}$ is convex.

Proof. (i) By Definition 2.46 of convexity, for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0,1]$,

$$\phi_i((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi_i(x) + \lambda\phi_i(y), \ i = 1, 2, \ldots, m.$$

As $\alpha_i \geq 0, \ i = 1, 2, \ldots, m$, multiplying the above inequality by α_i and adding them leads to

$$\sum_{i=1}^{m} \alpha_i\phi_i((1-\lambda)x + \lambda y) \leq (1-\lambda)\sum_{i=1}^{m} \alpha_i\phi_i(x) + \lambda\sum_{i=1}^{m} \alpha_i\phi_i(y),$$

thereby yielding the convexity of $\sum_{i=1}^{m} \alpha_i\phi_i$.

(ii) By the convexity of ϕ, for every $x, y \in \mathbb{R}^n$ and for every $\lambda \in [0,1]$,

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y).$$

Because ψ is nondecreasing convex function, for every $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \psi(\phi((1-\lambda)x + \lambda y)) &\leq \psi((1-\lambda)\phi(x) + \lambda\phi(y)) \\ &\leq (1-\lambda)\psi(\phi(x)) + \lambda\psi(\phi(y)), \ \forall \ \lambda \in [0,1]. \end{aligned}$$

Thus, $(\psi \circ \phi)$ is a convex function.

(iii) Observe that $epi \ \phi = \bigcap_{i \in I} epi \ \phi_i$, which on applying Proposition 2.3 (i) leads to the convexity of $epi \ \phi$. Now invoking Proposition 2.48 yields the convexity of ϕ.

(iv) Consider any arbitrary $\varepsilon > 0$. Then for any $(x_i, \alpha_i) \in F, \ i = 1, 2$,

$$\alpha_i \leq \phi(x_i) + \varepsilon, \ i = 1, 2.$$

By the convexity of F, for any $\lambda \in [0,1]$,

$$\phi((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)\alpha_1 + \lambda\alpha_2 \leq (1-\lambda)\phi(x_1) + \lambda\phi(x_2) + \varepsilon.$$

Because the above condition holds for every $\varepsilon > 0$, taking the limit as $\varepsilon \to 0$, the above condition reduces to

$$\phi((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)\phi(x_1) + \lambda\phi(x_2), \ \forall \ \lambda \in [0,1],$$

thereby leading to the convexity of ϕ. $\square$

The proof of (iv) is from Hiriart-Urruty and Lemaréchal [63]. These properties play an important role in convex analysis. From the earlier discussions we have that a constrained problem can be equivalently expressed as an unconstrained problem using the indicator function. Under the convexity assumptions as in the convex programming problem (CP) and using (i) of the above proposition, one has that $f_0(x) = (f + \delta_C)(x)$ is a convex function, thereby reducing (CP) to an unconstrained convex programming problem that then leads to the KKT optimality conditions under some assumptions, as we shall see in Chapter 3.

The property (ii) of Proposition 2.53 leads to the formulation of *conjugate functions*. We will discuss this class of functions later in this chapter as it will also play a pivotal in the study of convex optimization theory.

Next we define infimal convolution or simply inf-convolution on convex functions. The motivation for this operation comes from the sum of epigraph and the infimum operation as in (iv) of the above proposition. Consider two proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$. Then by Proposition 2.3 (ii), the set $F = epi\ \phi_1 + epi\ \phi_2$ is a convex set in $\mathbb{R}^n \times \mathbb{R}$. Explicitly, F is expressed as

$$F = \{(x_1 + x_2, \alpha_1 + \alpha_2) \in \mathbb{R}^n \times \mathbb{R} : (x_i, \alpha_i) \in epi\ \phi_i,\ i = 1, 2\}.$$

Then by (iv) of Proposition 2.53, the function

$$\phi(x) = \inf\{\alpha_1 + \alpha_2 : (x_1 + x_2, \alpha_1 + \alpha_2) \in F,\ x_1 + x_2 = x\}$$

is a convex function. This function ϕ can be reduced to the form known as the inf-convolution of ϕ_1 and ϕ_2 as defined below.

Definition 2.54 Consider proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$. Then the *infimal convolution* or *inf-convolution* of ϕ_1 and ϕ_2 is denoted by $\phi_1 \,\Box\, \phi_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$ and defined as

$$\begin{aligned}(\phi_1 \,\Box\, \phi_2)(\bar{x}) &= \inf\{\phi_1(x_1) + \phi_2(x_2) : x_i \in \mathbb{R}^n, i = 1, 2,\ x_1 + x_2 = \bar{x}\} \\ &= \inf\{\phi_1(x) + \phi_2(\bar{x} - x) : x \in \mathbb{R}^n\}.\end{aligned}$$

A simple consequence for the inf-convolution is the distance function. Consider a convex set $F \subset \mathbb{R}^n$. Then the distance function $\phi(x) = d_F(x)$ can be expressed as

$$\phi(x) = (\phi_1 \,\Box\, \phi_2)(x),$$

where $\phi_1(x) = \|x\|$ and $\phi_2(x) = \delta_F(x)$.

As it turns out, the inf-convolution of convex functions is again convex. To verify this claim, we will need the following result on strict epigraph sum. The proof appears in Moreau [89] but here we present its proof and that of the proposition to follow from Attouch, Buttazzo, and Michaille [3].

Proposition 2.55 *Consider two proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$. Then*

$$epi_s(\phi_1 \,\Box\, \phi_2) = epi_s\phi_1 + epi_s\phi_2. \tag{2.13}$$

Consequently,

$$cl\ epi\ (\phi_1 \,\Box\, \phi_2) = cl\ (epi\ \phi_1 + epi\ \phi_2). \tag{2.14}$$

Proof. Consider $(x, \alpha) \in epi_s\ (\phi_1 \,\Box\, \phi_2)$, which implies

$$(\phi_1 \,\Box\, \phi_2)(x) < \alpha.$$

The above inequality holds if and only if there exist $x_1, x_2 \in \mathbb{R}^n$ with $x_1 + x_2 = x$ such that

$$\phi_1(x_1) + \phi_2(x_2) < \alpha.$$

This is equivalent to the existence of $x_1, x_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $x_1 + x_2 = x$ and $\alpha_1 + \alpha_2 = \alpha$ such that $\phi_i(x_i) < \alpha_i,\ i = 1, 2$, thereby establishing (2.13).

By Definition 2.49 of strict epigraph, it is obvious that for any function ϕ,

$$epi_s\ \phi \subset epi\ \phi \quad \text{and} \quad cl\ epi_s\ \phi = epi\ \phi,$$

which along with the strict epigraph condition implies that

$$\begin{aligned} epi\ (\phi_1 \,\Box\, \phi_2) &= cl\ epi_s\ (\phi_1 \,\Box\, \phi_2) \\ &= cl\ (epi_s\ \phi_1 + epi_s\ \phi_2) \subset cl\ (epi\ \phi_1 + epi\ \phi_2). \end{aligned} \tag{2.15}$$

Now suppose that $(x_i, \alpha_i) \in epi\ \phi_i,\ i = 1, 2$, which along with the definition of inf-convolution implies that

$$(\phi_1 \,\Box\, \phi_2)(x_1 + x_2) \leq \phi_1(x_1) + \phi_2(x_2) \leq \alpha_1 + \alpha_2.$$

Therefore, $(x_1 + x_2, \alpha_1 + \alpha_2) \in epi\ (\phi_1 \,\Box\, \phi_2)$. Because $(x_i, \alpha_i) \in epi\ \phi_i$, $i = 1, 2$, were arbitrary,

$$epi\ \phi_1 + epi\ \phi_2 \subset epi\ (\phi_1 \,\Box\, \phi_2).$$

Taking closure on both sides of the above relation along with the condition (2.15) yields the condition (2.14), as desired. $\square$

Using this proposition, we now move on to show that the inf-convolution of proper convex functions is also convex.

Proposition 2.56 *Consider two proper convex functions $\phi_1, \phi_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then $\phi_1 \,\Box\, \phi_2$ is also convex.*

Proof. From Proposition 2.55,

$$epi_s(\phi_1 \,\Box\, \phi_2) = epi_s\ \phi_1 + epi_s\ \phi_2.$$

As ϕ_1 and ϕ_2 are convex functions, by Proposition 2.50, $epi_s\ \phi_1$ and $epi_s\ \phi_2$ are convex sets. This along with the above condition implies that $epi_s\ (\phi_1 \,\Box\, \phi_2)$, is convex which again by the characterization of convex functions, Proposition 2.50, leads to the convexity of $\phi_1 \,\Box\, \phi_2$. $\square$

An application of inf-convolution can be seen in the following property of indicator function. For convex sets F_1 and F_2 in $\mathbb{R}^n$, the indicator function of the sum of the sets is

$$\delta_{F_1+F_2} = \delta_{F_1} \,\Box\, \delta_{F_2}.$$

The importance of inf-convolution will be discussed in the study of conjugate functions later in this chapter. For more on inf-convolution, the readers may refer to Strömberg [107].

As we discussed that the inf-convolution is motivated by taking $F = epi\ \phi_1 + epi\ \phi_2$, similarly the notion of convex hull of a function is motivated by taking $F = co\ epi\ \phi$. Below we define this concept.

Definition 2.57 The *convex hull* of a nonconvex function ϕ is denoted as $co\ \phi$ and is obtained from Proposition 2.53 (iv) with $F = co\ epi\ \phi$. Therefore, by Theorem 2.7, $(x, \alpha) \in F$ if and only if there exist $(x_i, \alpha_i) \in epi\ \phi$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$\begin{aligned} (x, \alpha) &= \lambda_1(x_1, \alpha_1) + \lambda_2(x_2, \alpha_2) + \ldots + \lambda_m(x_m, \alpha_m) \\ &= (\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m, \lambda_1\alpha_1 + \lambda_2\alpha_2 + \ldots + \lambda_m\alpha_m). \end{aligned}$$

Because $\phi(x_i) \leq \alpha_i,\ i = 1, 2, \ldots, m$, Proposition 2.53 (iv) leads to

$$\begin{aligned} co\ \phi(x) = \inf\{\lambda_1\phi(x_1) + \lambda_2\phi(x_2) + \ldots + \lambda_m\phi(x_m) \in \mathbb{R} :& \\ \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m = x&, \\ \lambda_i \geq 0, i = 1, 2, \ldots, m, \ \sum \lambda_i = 1\}&. \end{aligned}$$

It is the greatest convex function majorized by ϕ. If ϕ is convex, $co\ \phi = \phi$. The *convex hull* of an arbitrary collection of functions $\{\phi_i : i \in I\}$ is denoted by $co\ \bigcup_{i \in I} \phi_i$ and is the convex hull of the pointwise infimum of the collection, that is,

$$co\ \bigcup_{i \in I} \phi_i = co\ \inf_{i \in I} \phi_i.$$

It is a function obtained from Proposition 2.53 (iv) by taking

$$F = co \bigcup_{i \in I} epi\ \phi_i.$$

It is the greatest convex function majorized by every $\phi_i,\ i = 1, 2, \ldots, m$.

The *closed convex hull* of a function ϕ is denoted by $cl\ co\ \phi$ and defined as

$$cl\ co\ \phi(x') = \sup\{\langle \xi, x' \rangle - \alpha : \langle \xi, x \rangle - \alpha \leq \phi(x),\ \forall\ x \in \mathbb{R}^n\}.$$

Similar to closure of a function, $cl\ co\ \phi$ satisfies the condition

$$epi\ cl\ co\ \phi = cl\ co\ epi\ \phi.$$

For more details on the convex hull and the closed convex hull of a function, readers are advised to refer to Hiriart-Urruty and Lemaréchal [62, 63].

Now before moving on with the properties of convex functions, we briefly discuss an important class of convex functions, namely sublinear and support functions, which as we will see later in the chapter are important in the study of convex analysis.

2.3.1 Sublinear and Support Functions

Definition 2.58 A proper function $p : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be a *sublinear function* if and only if p is subadditive and positively homogeneous, that is,

$$\begin{array}{ll} p(x_1 + x_2) \leq p(x_1) + p(x_2),\ \forall\ x_1, x_2 \in \mathbb{R}^n & \text{(subadditive property)} \\ p(\lambda x) = \lambda p(x),\ \forall\ x \in \mathbb{R}^n,\ \forall\ \lambda > 0 & \text{(positively homogeneous property)} \end{array}$$

From the positive homogeneity property, $p(0) = \lambda p(0)$ for every $\lambda > 0$, which is satisfied for $p(0) = 0$ as well as $p(0) = +\infty$. Most sublinear functions satisfy $p(0) = 0$. As p is proper, *dom* p is nonempty. So if $p(x) < +\infty$, then by the positive homogeneity property, $p(tx) < +\infty$, which implies that *dom* p is a cone. Observe that as p is positively homogeneous, for $x, y \in \mathbb{R}^n$ and any $\lambda \in (0, 1)$,

$$p((1-\lambda)x) = (1-\lambda)p(x) \quad \text{and} \quad p(\lambda y) = \lambda p(y).$$

By the subadditive property of p,

$$\begin{aligned} p((1-\lambda)x + \lambda y) &\leq p((1-\lambda)x) + p(\lambda y) \\ &= (1-\lambda)p(x) + \lambda p(y),\ \forall\ \lambda \in (0, 1). \end{aligned}$$

The inequality holds as equality for $\lambda = 0$ and $\lambda = 1$. Because $x, y \in \mathbb{R}^n$ were arbitrary, p is convex. Therefore, a sublinear function is a particular class of convex functions and hence *dom* p is convex. Next we present a proposition that gives the geometric characterization of sublinear functions. For the proof, we will also need the equivalent form of positive homogeneity from Hiriart-Urruty and Lemaréchal [63] according to which

$$p(\lambda x) \leq \lambda p(x),\ \forall\ x \in \mathbb{R}^n,\ \forall\ \lambda > 0.$$

Note that if p is positively homogeneous, then the above condition holds trivially. Conversely, if the above inequality holds, then for any $\lambda > 0$,

$$p(x) = p(\lambda^{-1}\lambda x) \leq \frac{1}{\lambda} p(\lambda x),\ \forall\ x \in \mathbb{R}^n,$$

which along with the preceding inequality yields that p is positively homogeneous.

Theorem 2.59 *Consider a proper function $p : \mathbb{R}^n \to \bar{\mathbb{R}}$. p is a sublinear function if and only if its epigraph, epi p, is a convex cone in $\mathbb{R}^n \times \mathbb{R}$.*

Proof. Suppose that p is sublinear. From the above discussion, p is a convex function as well and thus *epi* p is convex. Consider $(x, \alpha) \in$ *epi* p, which implies that $p(x) \leq \alpha$. By the positively homogeneous property

$$p(\lambda x) = \lambda p(x) \leq \lambda \alpha, \ \lambda > 0,$$

which implies that $\lambda(x, \alpha) = (\lambda x, \lambda \alpha) \in$ *epi* p for every $\lambda > 0$. Also, $(0, 0) \in$ *epi* p. Thus, *epi* p is a cone.

Conversely, suppose that *epi* p is a convex cone. By Theorem 2.20, for any $(x_i, \alpha_i) \in$ *epi* $p, \ i = 1, 2$,

$$(x_1 + x_2, \alpha_1 + \alpha_2) \in epi\ p.$$

In particular for $\alpha_i = p(x_i), \ i = 1, 2$, the above condition leads to

$$p(x_1 + x_2) \leq p(x_1) + p(x_2).$$

Because $x_1, x_2 \in \mathbb{R}^n$ are arbitrarily chosen, the above inequality implies that p is subadditive. Also, as *epi* p is a cone, any $(x, \alpha) \in$ *epi* p implies that $\lambda(x, \alpha) \in$ *epi* p for every $\lambda > 0$. In particular, for $\alpha = p(x)$,

$$p(\lambda x) \leq \lambda p(x), \ \forall \ \lambda > 0,$$

which is an equivalent definition for positive homogeneity, as discussed before. Hence, p is a sublinear function. □

Sublinear functions are particular class of convex functions. For a convex cone $K \subset \mathbb{R}^n$, the indicator function δ_K and the distance function d_K are also sublinear functions. An important class of sublinear functions is that of support functions. We will discuss the support functions in brief.

Definition 2.60 Consider a set $F \subset \mathbb{R}^n$. The *support function*, $\sigma_F : \mathbb{R}^n \to \bar{\mathbb{R}}$, to F at $\bar{x} \in \mathbb{R}^n$ is defined as

$$\sigma_F(\bar{x}) = \sup_{x \in F} \langle \bar{x}, x \rangle.$$

From Proposition 1.7 (ii) and (iii), it is obvious that a support function is sublinear. As it is the supremum of linear functions that are continuous, support functions are lsc. For a closed convex cone $K \subset \mathbb{R}^n$,

$$\sigma_K(\bar{x}) = \begin{cases} 0, & \langle \bar{x}, x \rangle \leq 0, \ \forall \ x \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

which is nothing but the indicator function of the polar cone K°. Equivalently,

$$\sigma_K = \delta_{K^\circ} \quad \text{and} \quad \delta_K = \sigma_{K^\circ}.$$

Next we present some properties of the support functions, the proofs of which are from Burke and Deng [22], Hiriart-Urruty and Lemaréchal [63], and Rockafellar [97].

Proposition 2.61 *(i) Consider two convex sets F_1 and F_2 in $\mathbb{R}^n$. Then*

$$F_1 \subset F_2 \quad \Longrightarrow \quad \sigma_{F_1}(x) \leq \sigma_{F_2}(x), \ \forall\, x \in \mathbb{R}^n.$$

(ii) For a set $F \subset \mathbb{R}^n$, one has

$$\sigma_F = \sigma_{cl\ F} = \sigma_{co\ F} = \sigma_{cl\ co\ F}.$$

(iii) Consider a convex set $F \subset \mathbb{R}^n$. Then $\bar{x} \in cl\ F$ if and only if

$$\langle x^*, \bar{x} \rangle \leq \sigma_F(x^*), \ \forall\, x^* \in \mathbb{R}^n.$$

(iv) For convex sets $F_1, F_2 \subset \mathbb{R}^n$, $cl\ F_1 \subset cl\ F_2$ if and only if

$$\sigma_{F_1}(x^*) \leq \sigma_{F_2}(x^*), \ \forall\, x^* \in \mathbb{R}^n.$$

(v) Let $F_1, F_2 \subset \mathbb{R}^n$ be convex sets and $K \subset \mathbb{R}^n$ be a closed convex cone. Then

$$\begin{aligned} \sigma_{F_1}(x) \leq \sigma_{F_2}(x), \ \forall\, x \in K \quad &\Longleftrightarrow \quad \sigma_{F_1}(x) \leq \sigma_{F_2+K^\circ}(x), \ \forall\, x \in \mathbb{R}^n \\ &\Longleftrightarrow \quad F_1 \subset cl(F_2 + K^\circ). \end{aligned}$$

(vi) The support function of a set $F \subset \mathbb{R}^n$ is finite everywhere if and only if F is bounded.

Proof. (i) By Proposition 1.7 (i), it is obvious that for $F_1 \subset F_2$,

$$\sup_{x_1 \in F_1} \langle x, x_1 \rangle \leq \sup_{x_2 \in F_2} \langle x, x_2 \rangle, \ \forall\, x \in \mathbb{R}^n,$$

thereby leading to the desired result.

(ii) As $\langle x, . \rangle$ is linear and hence continuous over $\mathbb{R}^n$, then on taking supremum over F,

$$\sigma_F(x) = \sigma_{cl\ F}(x), \ \forall\, x \in \mathbb{R}^n.$$

Because $F \subset co\ F$, by (i),

$$\sigma_F(x) \leq \sigma_{co\ F}(x), \ \forall\, x \in \mathbb{R}^n.$$

Also, for any $x' \in co\ F$, by Carathéodory Theorem, Theorem 2.8, there exist $x'_i \in F$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, n+1$, satisfying $\sum_{i=1}^{n+1} \lambda_i = 1$ such that $x' = \sum_{i=1}^{n+1} \lambda_i x'_i$. Therefore,

$$\langle x, x' \rangle = \sum_{i=1}^{n+1} \lambda_i \langle x, x'_i \rangle \leq \sum_{i=1}^{n+1} \lambda_i \sigma_F(x) = \sigma_F(x).$$

Because $x' \in co\ F$ was arbitrary, the above inequality holds for every $x' \in co\ F$ and hence

$$\sigma_{co\ F}(x) \leq \sigma_F(x), \ \forall\, x \in \mathbb{R}^n,$$

thus yielding the equality as desired. These relations also imply that $\sigma_F = \sigma_{cl\ co\ F}$.

(iii) Invoking Theorem 2.27, the desired result holds.

(iv) By (i) and (ii), $cl\ F_1 \subset cl\ F_2$ implies that

$$\sigma_{F_1}(x^*) \leq \sigma_{F_2}(x^*),\ \forall\ x^* \in \mathbb{R}^n.$$

Conversely, suppose that the above inequality holds, which implies for every $x \in cl\ F_1$,

$$\langle x^*, x\rangle \leq \sigma_{F_2}(x^*),\ \forall\ x^* \in \mathbb{R}^n.$$

Therefore, by (iii), $x \in cl\ F_2$. Because $x \in cl\ F_1$ was arbitrary, $cl\ F_1 \subset cl\ F_2$, thereby completing the proof.

(v) Consider $x \in K$. As $F_2 \subset F_2 + K^\circ$, by (i) along with Proposition 1.7 and the definition of polar cone leads to

$$\sigma_{F_2}(x) \leq \sigma_{F_2+K^\circ}(x) = \sigma_{F_2}(x) + \sigma_{K^\circ}(x) \leq \sigma_{F_2}(x),$$

that is, $\sigma_{F_2}(x) = \sigma_{F_2+K^\circ}(x)$ for $x \in K$. Now if $x \notin K$, there exists $z \in K^\circ$ such that $\langle z, x\rangle > 0$. Consider $y \in F_2$. Therefore, as the limit $\lambda \to +\infty$, $\langle y + \lambda z, x\rangle \to +\infty$ which implies $\sigma_{F_2+K^\circ}(x) = +\infty$. Thus, establishing the first equivalence. The second equivalence can be obtained by (ii) and (iv).

(vi) Suppose that F is bounded, which implies that there exists $M > 0$ such that

$$\|x'\| \leq M,\ \forall\ x' \in F.$$

Therefore, by the Cauchy–Schwarz Inequality, Proposition 1.1,

$$\langle x, x'\rangle \leq \|x\|\|x'\| \leq \|x\|M,\ \forall\ x' \in F,$$

which implies that $\sigma_F(x) \leq \|x\|M$ for every $x \in \mathbb{R}^n$. Thus, σ_F is finite everywhere.

Conversely, suppose that σ_F is finite everywhere. In the next section, we will present a result establishing the local Lipschitz property and hence continuity of the convex function, Theorem 2.72. This leads to the local boundedness. Therefore there exists M such that

$$\langle x, x'\rangle \leq \sigma_F(x) \leq M,\ \forall\ (x, x') \in \mathbb{B} \times F.$$

If $x' \neq 0$, taking $x = \dfrac{x'}{\|x'\|}$, the above inequality leads to $\|x'\| \leq M$ for every $x' \in F$, thereby establishing the boundedness of F and hence proving the result. □

As mentioned earlier, the support function is lsc and sublinear. Similarly, a closed sublinear function can be viewed as a support function. We end this subsection by presenting this important result to assert the preceding statement. The proof is again due to Hiriart-Urruty and Lemaréchal [63].

Theorem 2.62 *For a proper lsc sublinear function $\sigma : \mathbb{R}^n \to \bar{\mathbb{R}}$, there exists a linear function minorizing σ. In fact, σ is the supremum of the linear function minorizing it; that is, σ is the support function of the closed convex set given by*

$$F_\sigma = \{x \in \mathbb{R}^n : \langle x, d\rangle \leq \sigma(d), \ \forall\ d \in \mathbb{R}^n\}.$$

Proof. Because sublinear functions are convex, σ is a proper lsc convex function. As we will discuss in one of the later sections, every proper lsc convex function can be represented as a pointwise supremum of affine functions majorized by it, Theorem 2.100 and there exists $(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\langle x, d\rangle - \alpha \leq \sigma(d), \ \forall\ d \in \mathbb{R}^n.$$

As $\sigma(0) = 0$, the preceding inequality leads to $\alpha \geq 0$. By the positive homogeneity of σ,

$$\langle x, d\rangle - \frac{\alpha}{\lambda} \leq \sigma(d), \ \forall\ d \in \mathbb{R}^n, \ \forall\ \lambda > 0.$$

Taking the limit as $\lambda \to +\infty$,

$$\langle x, d\rangle \leq \sigma(d), \ \forall\ d \in \mathbb{R}^n,$$

that is, σ is minorized by linear functions.

As mentioned in the beginning, convex functions are supremum of affine functions, which for sublinear functions can be restricted to linear functions. Therefore, by Theorem 2.100,

$$\sigma(d) = \sup_{x \in F_\sigma} \langle x, d\rangle$$

and hence σ is the support function of F_σ. □

After discussing these classes of convex functions, we move on to discuss the nature of convex functions.

2.3.2 Continuity Property

We have already discussed the operations that preserve convexity of the functions. Now we shall study the continuity, Lipschitzian and differentiability properties of the function. But before doing so, let us recall proper functions.

A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is *proper* if $\phi(x) > -\infty$ for every $x \in \mathbb{R}^n$ and *dom* ϕ is nonempty, that is, *epi* ϕ is nonempty and contains no vertical lines. A function that is not proper is called an *improper function.* We know that for a convex function, the epigraph is a convex set. If ϕ is an improper convex function such that there exists $\bar{x} \in ri\ dom\ \phi$ such that $\phi(\bar{x}) = -\infty$, then the convexity of *epi* ϕ is broken unless $\phi(x) = -\infty$ for every $x \in ri\ dom\ \phi$. Such

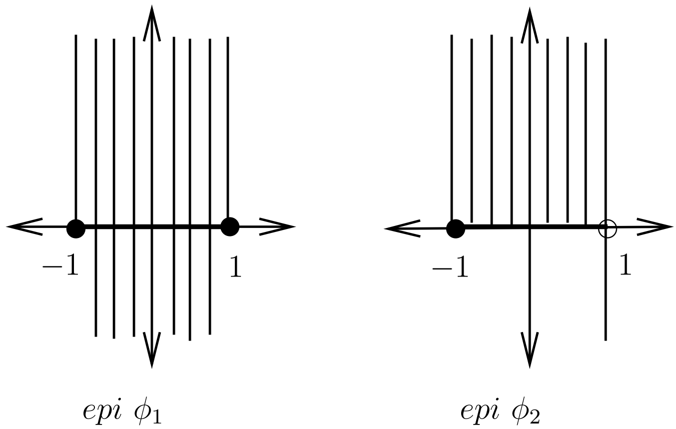

FIGURE 2.7: Epigraphs of improper functions ϕ_1 and ϕ_2.

functions can however have finite values at the boundary points. For example, consider $\phi_1 : \mathbb{R} \to \bar{\mathbb{R}}$ given by

$$\phi_1(x) = \begin{cases} -\infty, & |x| < 1, \\ 0, & |x| = 1, \\ +\infty, & |x| > 1. \end{cases}$$

Here, ϕ_1 is an improper convex function with finite values at boundary points of the domain $x = -1$ and $x = 1$. Also it is obvious that ϕ cannot have a finite value on *ri dom* ϕ and $-\infty$ at a boundary point. For better understanding, suppose that $x \in$ *ri odm* ϕ such that $\phi(x) > -\infty$ and let y be a boundary point of *dom* ϕ with $\phi(y) = -\infty$. By the convexity of ϕ,

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y), \ \forall \ \lambda \in (0,1),$$

which implies that for $(1-\lambda)x + \lambda y \in$ *ri dom* ϕ,

$$\phi((1-\lambda)x + \lambda y) = -\infty.$$

This contradicts the convexity of the epigraph. This aspect can be easily visualized by modifying the previous example as follows. Define an improper function $\phi_2 : \mathbb{R} \to \bar{\mathbb{R}}$ as

$$\phi_2(x) = \begin{cases} -\infty, & x = 1, \\ 0, & -1 \leq x < 1, \\ +\infty, & |x| > 1. \end{cases}$$

Obviously ϕ_2 cannot be convex as *epi* ϕ_2 is not convex as in Figure 2.7. These discussions can be stated as the following result from Rockafellar [97].

Proposition 2.63 *Consider an improper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then $\phi(x) = -\infty$ for every $x \in ri\ dom\ \phi$. Thus ϕ is necessarily infinite except perhaps at the boundary point of dom ϕ. Moreover, an lsc improper convex function can have no finite values.*

As discussed in Chapter 1, the continuity of a function plays an important role in the study of its bounds and hence in optimization problems. Before discussing the continuity property of convex functions we shall present some results on interior of the epigraph of a convex function and closure of a convex function.

Proposition 2.64 *Consider a convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ such that ri dom ϕ is nonempty. Then ri epi ϕ is also nonempty and given by*

$$ri\ epi\ \phi = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in ri\ dom\ \phi,\ \phi(x) < \alpha\}.$$

Equivalently, $(\bar{x}, \bar{\alpha}) \in ri\ epi\ \phi$ if and only if $\bar{\alpha} > \limsup_{x \to \bar{x}} \phi(x)$.

Proof. To obtain the result for *ri epi* ϕ, it is sufficient to derive it for *int epi* ϕ, that is,

$$int\ epi\ \phi = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in int\ dom\ \phi,\ \phi(x) < \alpha\}.$$

By Definition 2.12, for $(\bar{x}, \bar{\alpha}) \in int\ epi\ \phi$, there exists $\varepsilon > 0$ such that

$$(\bar{x}, \bar{\alpha}) + \varepsilon \mathbb{B} \subset epi\ \phi,$$

which implies that $\bar{x} \in int\ dom\ \phi$ along with $\phi(\bar{x}) < \bar{\alpha}$. As $(\bar{x}, \bar{\alpha}) \in int\ epi\ \phi$ is arbitrary,

$$int\ epi\ \phi \subset \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in int\ dom\ \phi,\ \phi(x) < \alpha\}.$$

Now suppose that $\bar{x} \in int\ dom\ \phi$ and $\phi(\bar{x}) \leq \bar{\alpha}$. Consider $x_1, x_2, \ldots, x_m \in dom\ \phi$ such that $\bar{x} \in int\ F$ where $F = co\ \{x_1, x_2, \ldots, x_m\}$. Define

$$\gamma = \max_{1,2,\ldots,m} \{\phi(x_i)\}.$$

By the convexity of F, for any $x \in F$ there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, satisfying $\sum_{i=1}^m \lambda_i = 1$ such that

$$x = \sum_{i=1}^m \lambda_i x_i.$$

Because ϕ is convex,

$$\phi(x) \leq \sum_{i=1}^m \lambda_i \phi(x_i) \leq \sum_{i=1}^m \lambda_i \gamma = \gamma.$$

Therefore, the open set

$$\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in int\ F,\ \gamma < \alpha\} \subset epi\ \phi.$$

In particular, for any $\alpha > \gamma$, $(\bar{x}, \alpha) \in int\ epi\ \phi$. Thus, $(\bar{x}, \bar{\alpha})$ can be considered as lying on the interior of line segment passing through the points $(\bar{x}, \alpha) \in int\ epi\ \phi$, which by the line segment principle, Proposition 2.14, $(\bar{x}, \bar{\alpha}) \in int\ epi\ \phi$. Because $(\bar{x}, \bar{\alpha})$ is arbitrary,

$$int\ epi\ \phi \supset \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in int\ dom\ \phi,\ \phi(x) < \alpha\},$$

thereby leading to the requisite result.

Now we move on to prove the equivalent part for *ri epi* ϕ. Suppose that $(\bar{x}, \bar{\alpha}) \in ri\ epi\ \phi$. Therefore, by the earlier characterization one can always find an $\varepsilon > 0$ such that

$$\bar{x} \in ri\ dom\ \phi \quad \text{and} \quad \sup_{x \in \mathbb{B}_\varepsilon(\bar{x})} \phi(x) < \bar{\alpha}.$$

Taking the limit as $\varepsilon \to 0$ along with Definition 1.5 of limit supremum,

$$\limsup_{x \to \bar{x}} \phi(x) < \bar{\alpha}.$$

Conversely, suppose that for $(\bar{x}, \bar{\alpha})$ the strict inequality condition holds which implies

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{B}_\varepsilon(\bar{x})} \phi(x) < \bar{\alpha}.$$

Therefore, there exists $\varepsilon > 0$ such that

$$\sup_{x \in \mathbb{B}_\varepsilon(\bar{x})} \phi(x) < \bar{\alpha},$$

which yields $\phi(\bar{x}) < \bar{\alpha}$ with $\bar{x} \in ri\ dom\ \phi$, thereby proving the equivalent result. Note that this equivalence can be established for *int epi* ϕ as well. □

Note that the above result can also be obtained for the relative interior of the epigraph as it is nothing but the interior relative to the affine hull of the epigraph. As a consequence of the above characterization of *ri* F, we have the following result from Rockafellar [97].

Corollary 2.65 *Consider* $\alpha \in \mathbb{R}$ *and* $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ *to be a proper convex function such that for some* $x \in dom\ \phi$, $\phi(x) < \alpha$. *Then actually* $\phi(x) < \alpha$ *for some* $x \in ri\ dom\ \phi$.

Proof. Define a hyperplane H as

$$H = \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : \mu < \alpha\}.$$

Because for some $x \in \mathbb{R}^n$, $\phi(x) < \alpha$, in particular for $\mu = \phi(x)$, we have that H meets *epi* ϕ. Invoking Corollary 2.16 (ii), H also meets *ri epi* ϕ, which by Proposition 2.64 implies that there exists $x \in ri\ dom\ \phi$ such that $\phi(x) < \alpha$, thereby yielding the desired result. $\square$

Recall that in the previous chapter the closure of a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ was defined as

$$cl\ \phi(\bar{x}) = \liminf_{x \to \bar{x}} \phi(x),\ \forall\ \bar{x} \in \mathbb{R}^n,$$

which is a bit complicated to compute. In case of a convex function, it is much easier to compute and is presented in the next proposition. The proof is from Rockafellar [97].

Proposition 2.66 *Consider a proper convex function* $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. *Then cl* ϕ *agrees with* ϕ *in ri dom* ϕ *and for* $\hat{x} \in ri\ dom\ \phi$,

$$cl\ \phi(x) = \lim_{\lambda \to 1} \phi((1-\lambda)\hat{x} + \lambda x),\ \forall\ x \in \mathbb{R}^n.$$

Proof. From Definition 1.11 of closure of a function, $cl\ \phi$ is lsc and $cl\ \phi \leq \phi$. Therefore, by the lower semicontinuity of $cl\ \phi$,

$$\liminf_{\lambda \to 1} (cl\ \phi)((1-\lambda)\hat{x} + \lambda x) = cl\ \phi(x) \leq \liminf_{\lambda \to 1} \phi((1-\lambda)\hat{x} + \lambda x).$$

To prove the result, we will establish the following inequality

$$cl\ \phi(x) \geq \limsup_{\lambda \to 1} \phi((1-\lambda)\hat{x} + \lambda x).$$

Consider $\alpha \in \mathbb{R}$ such that $cl\ \phi(x) \leq \alpha$, which implies that

$$(x, \alpha) \in epi\ cl\ \phi = cl\ epi\ \phi.$$

Consider any $(\hat{x}, \hat{\alpha}) \in ri\ epi\ \phi$. Applying the Line Segment Principle, Proposition 2.14,

$$(1-\lambda)(\hat{x}, \hat{\alpha}) + \lambda(x, \alpha) \in ri\ epi\ \phi,\ \forall\ \lambda \in [0, 1).$$

By Proposition 2.64,

$$\phi((1-\lambda)\hat{x} + \lambda x) < (1-\lambda)\hat{\alpha} + \lambda\alpha,\ \forall\ \lambda \in [0, 1).$$

Taking the limit superior as $\lambda \to 1$, the above inequality leads to

$$\limsup_{\lambda \to 1} \phi((1-\lambda)\hat{x} + \lambda x) \leq \limsup_{\lambda \to 1} (1-\lambda)\hat{\alpha} + \lambda\alpha = \alpha.$$

In particular, taking $\alpha = cl\ \phi(x)$ in the above inequality yields the desired result.

In the relation

$$cl\ \phi(x) = \lim_{\lambda \to 1} \phi((1-\lambda)\hat{x} + \lambda x),$$

in particular, taking $x = \hat{x} \in ri\ dom\ \phi$ leads to $cl\ \phi(\hat{x}) = \phi(\hat{x})$. Because $\hat{x} \in ri\ dom\ \phi$ is arbitrary, $cl\ \phi = \phi$ on $ri\ dom\ \phi$, that is, $cl\ \phi$ agrees with ϕ in $ri\ dom\ \phi$. □

Next we present some results from Rockafellar [97] on closure and relative interior.

Proposition 2.67 *Consider a proper convex function* $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ *and let* $\alpha \in \mathbb{R}$ *such that* $\alpha > \inf_{x \in \mathbb{R}^n} \phi(x)$*. Then the level sets*

$$\{x \in \mathbb{R}^n : \phi(x) \leq \alpha\} \quad \textit{and} \quad \{x \in \mathbb{R}^n : \phi(x) < \alpha\}$$

have the same closure and relative interior, namely

$$\{x \in \mathbb{R}^n : cl\ \phi(x) \leq \alpha\} \quad \textit{and} \quad \{x \in \mathbb{R}^n : x \in ri\ dom\ \phi,\ \phi(x) < \alpha\},$$

respectively.

Proof. Define a hyperplane $H = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathbb{R}^n\}$ in $\mathbb{R}^{n+1}$. Applying Corollary 2.65 and Proposition 2.64, H intersects $ri\ epi\ \phi$, which implies that

$$ri\ H \cap ri\ epi\ \phi = H \cap ri\ epi\ \phi \neq \emptyset.$$

Now consider

$$H \cap epi\ \phi = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \phi(x) \leq \alpha\}.$$

Invoking Corollary 2.16 (iii),

$$cl(H \cap epi\ \phi) = cl\ H \cap cl\ epi\ \phi = H \cap epi\ cl\ \phi, \tag{2.16}$$

$$ri(H \cap epi\ \phi) = ri\ H \cap ri\ epi\ \phi = H \cap ri\ epi\ \phi. \tag{2.17}$$

The projection of these sets in $\mathbb{R}^n$ are, respectively,

$$cl\ \{x \in \mathbb{R}^n : \phi(x) \leq \alpha\} = \{x \in \mathbb{R}^n : cl\ \phi(x) \leq \alpha\},$$
$$ri\ \{x \in \mathbb{R}^n : \phi(x) \leq \alpha\} = \{x \in \mathbb{R}^n : x \in ri\ dom\ \phi,\ \phi(x) < \alpha\}.$$

The latter relation implies that

$$ri\ \{x \in \mathbb{R}^n : \phi(x) \leq \alpha\} \subset \{x \in \mathbb{R}^n : \phi(x) < \alpha\} \subset \{x \in \mathbb{R}^n : \phi(x) \leq \alpha\}.$$

Therefore, by Corollary 2.16 (ii), $\{x \in \mathbb{R}^n : \phi(x) < \alpha\}$ has the same closure and relative interior as $\{x \in \mathbb{R}^n : \phi(x) \leq \alpha\}$. □

Proposition 2.68 *Consider proper convex functions* $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$. *If every* ϕ_i, $i = 1, 2, \ldots, m$, *is lsc and* $\phi_1 + \phi_2 + \ldots + \phi_m \not\equiv +\infty$, *then* $\phi_1 + \phi_2 + \ldots + \phi_m$ *is a proper lsc convex function. If* ϕ_i, $i = 1, 2, \ldots, m$, *are not all lsc but ri dom* $\phi_1 \cap$ *ri dom* $\phi_2 \cap \ldots \cap$ *ri dom* ϕ_m *is nonempty, then*

$$cl\ (\phi_1 + \phi_2 + \ldots + \phi_m) = cl\ \phi_1 + cl\ \phi_2 + \ldots + cl\ \phi_m.$$

Proof. Define $\phi = \phi_1 + \phi_2 + \ldots + \phi_m$ and assume

$$\hat{x} \in ri\ dom\ \phi = ri\ (\bigcap_{i=1}^{m} dom\ \phi_i).$$

By Proposition 2.66, for every $x \in \mathbb{R}^n$,

$$cl\ \phi(x) = \lim_{\lambda \to 1} \phi((1-\lambda)\hat{x} + \lambda x) = \lim_{\lambda \to 1} \sum_{i=1}^{m} \phi_i((1-\lambda)\hat{x} + \lambda x). \tag{2.18}$$

If ϕ_i, $i = 1, 2, \ldots, m$, are all lsc, then the above condition becomes

$$cl\ \phi(x) = \phi_1(x) + \phi_2(x) + \ldots + \phi_m(x),\ \forall\ x \in \mathbb{R}^n$$

and thus $cl\ \phi = \phi$.

Suppose that ϕ_i, $i = 1, 2, \ldots, m$, are not all lsc. If

$$\bigcap_{i=1}^{m} ri\ dom\ \phi_i \neq \emptyset,$$

by Proposition 2.15 (iii),

$$\bigcap_{i=1}^{m} ri\ dom\ \phi_i = ri\ \bigcap_{i=1}^{m} dom\ \phi_i = ri\ dom\ \phi.$$

Therefore,

$$\hat{x} \in ri\ dom\ \phi_i,\ i = 1, 2, \ldots, m.$$

Again by Proposition 2.66,

$$cl\ \phi_i(x) = \lim_{\lambda \to 1} \phi_i((1-\lambda)\hat{x} + \lambda x),\ i = 1, 2, \ldots, m.$$

Therefore, the condition (2.18) becomes

$$cl\ \phi(x) = cl\ \phi_1(x) + cl\ \phi_2(x) + \ldots + cl\ \phi_m(x),\ \forall\ x \in \mathbb{R}^n,$$

thereby completing the proof. □

Using the above propositions, one can prove the continuity property of the convex functions.

Theorem 2.69 *A proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is continuous on ri dom ϕ.*

Proof. By Proposition 2.66, *cl* ϕ agrees with ϕ in *ri dom* ϕ, which implies that ϕ is lsc on *ri dom* ϕ. Now suppose that $\bar{x} \in$ *ri dom* ϕ. For any α such that $(\bar{x}, \alpha) \in$ *ri epi* ϕ, by Proposition 2.64,

$$\limsup_{x \to \bar{x}} \phi(x) < \alpha.$$

Taking the limit as $\alpha \to \phi(\bar{x})$, the preceding condition becomes

$$\limsup_{x \to \bar{x}} \phi(x) \leq \phi(\bar{x}),$$

thereby implying the upper semicontinuity of ϕ at $\bar{x}$. Because $\bar{x} \in$ *ri dom* ϕ is arbitrary, ϕ is usc on *ri dom* ϕ. Thus ϕ is continuous on *ri dom* ϕ, thereby yielding the desired result. $\square$

Before moving on to discuss the derivative property of a convex function, we shall discuss its Lipschitzian property. For that we first define Lipschitz and locally Lipschitz functions.

Definition 2.70 A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be *Lipschitz* if there exists $L > 0$ such that

$$|\phi(x) - \phi(y)| \leq L \, \|x - y\|, \ \forall \ x, y \in \mathbb{R}^n.$$

The positive number L is called the *Lipschitz constant* of ϕ, or ϕ is said to be Lipschitz with constant L.

Definition 2.71 Consider a function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Then ϕ is said to be *locally Lipschitz* if there exist $L_{\bar{x}} > 0$ and a neighborhood $\mathcal{N}(\bar{x})$ of $\bar{x}$ such that

$$|\phi(x) - \phi(y)| \leq L_{\bar{x}} \, \|x - y\|, \ \forall \ x, y \in \mathcal{N}(\bar{x}).$$

It is a well known that a Lipschitz function is continuous but the converse need not hold. From Theorem 2.69, we know that a convex function function is continuous in the relative interior of its domain. In the results to follow, we show that local boundedness of a convex function implies that the function is continuous as well as is locally Lipschitz. The result is from Attouch, Buttazzo, and Michaille [3].

Theorem 2.72 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in$ dom ϕ. For some $\varepsilon > 0$ such that*

$$\sup_{x \in \mathbb{B}_\varepsilon(\bar{x})} \phi(x) = M < +\infty.$$

Then ϕ is continuous at $\bar{x}$. Moreover, ϕ is Lipschitz continuous on every ball $\mathbb{B}_{\varepsilon'}(\bar{x})$ with $\varepsilon' < \varepsilon$ and

$$|\phi(x) - \phi(y)| \leq \frac{2M}{\varepsilon - \varepsilon'} \|x - y\|, \ \forall \ x, y \in \mathbb{B}_{\varepsilon'}(\bar{x}).$$

Proof. Without loss of generality, by translation (that is, by considering the function $\phi(x+\bar{x})-\phi(\bar{x})$), the problem reduces to the case when $\bar{x}=0$ and $\phi(0)=0$. Therefore, the local boundedness in the neighborhood of $\bar{x}=0$ reduces to

$$\sup_{x\in\mathbb{B}_\varepsilon(0)}\phi(x)=M<+\infty.$$

Consider an arbitrary $\delta\in(0,1]$ and $x\in\mathbb{B}_{\delta\varepsilon}(0)$. Now expressing

$$x=(1-\delta)0+\delta(\frac{1}{\delta}x),$$

where $\frac{1}{\delta}x\in\mathbb{B}_\varepsilon(0)$. The convexity of ϕ along with the local boundedness condition leads to

$$\phi(x)\leq(1-\delta)\phi(0)+\delta\phi(\frac{1}{\delta}x)\leq\delta M.$$

Rewriting

$$0=\frac{1}{1+\delta}x+\frac{\delta}{1+\delta}(\frac{-1}{\delta}x),$$

where $\frac{1}{\delta}x\in\mathbb{B}_\varepsilon(0)$. Again, the convexity of ϕ yields

$$0=\phi(0)\leq\frac{1}{\delta}\phi(x)+\frac{\delta}{1+\delta}\phi(\frac{-1}{\delta}x)\leq\frac{1}{1+\delta}\phi(x)+\frac{\delta M}{1+\delta},$$

which along with the previous condition on $\phi(x)$ implies that

$$-\delta M\leq\phi(x)\leq\delta M.$$

Because $x\in\mathbb{B}_{\delta\varepsilon}(0)$ is arbitrary,

$$|\phi(x)|\leq\delta M,\ \forall\ x\in\mathbb{B}_{\delta\varepsilon}(0),$$

thereby establishing the continuity of ϕ at 0.

In the above discussion, in particular for $\delta=1$,

$$|\phi(x+\bar{x})-\phi(\bar{x})|\leq M,\ \forall\ x\in\mathbb{B}_\varepsilon(0).$$

Consider arbitrary $x,y\in\mathbb{B}_{\varepsilon'}(\bar{x})$ with $x\neq y$. Denoting $\delta=\varepsilon-\varepsilon'>0$,

$$z=x+\frac{\delta}{\|x-y\|}(x-y)\quad\text{and}\quad\lambda=\frac{\|x-y\|}{\delta+\|x-y\|}.$$

Observe that

$$\begin{aligned}\|z-\bar{x}\| &= \|(y-\bar{x})++\frac{\delta}{\|x-y\|}(x-y)\|\\ &\leq \|y-\bar{x}\|+\frac{\delta}{\|x-y\|}\|x-y\|\\ &= \varepsilon'+\delta=\varepsilon,\end{aligned}$$

which implies $z \in \mathbb{B}_\varepsilon(\bar{x})$. Also

$$\|x-y\|z = (\delta + \|x-y\|)x - \delta y,$$

which implies that

$$x = (1-\lambda)y + \lambda z, \ \forall \ \lambda \in (0,1).$$

By the convexity of ϕ,

$$\phi(x) \leq (1-\lambda)\phi(y) + \lambda\phi(x) = \phi(y) + \lambda(\phi(z) - \phi(y)),$$

which leads to

$$\phi(x) - \phi(y) \leq \lambda(\phi(z) - \phi(y)) \leq \lambda|\phi(z) - \phi(y)|.$$

Observe that

$$|\phi(z) - \phi(y)| \leq |\phi(z) - \phi(\bar{x})| + |\phi(y) - \phi(\bar{x})| \leq 2M,$$

as $z \in \mathbb{B}_\varepsilon(\bar{x})$ and $y \in \mathbb{B}_{\varepsilon'}(\bar{x}) \subset \mathbb{B}_\varepsilon(\bar{x})$. Therefore,

$$\phi(x) - \phi(y) \leq \frac{\|x-y\|}{\delta + \|x-y\|} 2M \leq \frac{2M}{\delta}\|x-y\|.$$

Interchanging the roles of x and y yields

$$|\phi(x) - \phi(y)| \leq \frac{2M}{\delta}\|x-y\|,$$

thereby establishing the result. □

In the above result, we showed that if a proper convex function is locally bounded at a point, then it is locally Lipschitz at that point. As a matter of fact, it is more than that which is presented in the result below, the proof of which is along the lines of Hiriart-Urruty and Lemaréchal [63].

Theorem 2.73 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then ϕ is locally Lipschitz on ri dom ϕ.*

Proof. Similar to the proof of Proposition 2.14 (i), consider $n+1$ linearly independent vectors $x_1, x_2, \ldots, x_{n+1} \in$ *dom* ϕ such that $\bar{x} \in ri\ co\ \{x_1, x_2, \ldots, x_{n+1}\} \subset dom\ \phi$. Now consider $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon(\bar{x}) \subset co\ \{x_1, x_2, \ldots, x_{n+1}\}$. For any arbitrary $x \in \mathbb{B}_\varepsilon(\bar{x})$, there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, n+1$, satisfying $\sum_{i=1}^{n+1} \lambda_i = 1$ such that

$$x = \sum_{i=1}^{n+1} \lambda_i x_i.$$

By the convexity of ϕ,

$$\phi(x) \leq \sum_{i=1}^{n+1} \lambda_i \phi(x_i) \leq \max_{1,2,\ldots,n+1} \phi(x_i) = M < +\infty.$$

Because $x \in \mathbb{B}_\varepsilon(\bar{x})$ is arbitrary, the above condition holds for every $x \in \mathbb{B}_\varepsilon(\bar{x})$. Therefore, by Theorem 2.72, for $\varepsilon' < \varepsilon$,

$$|\phi(x) - \phi(y)| \leq \frac{2M}{\varepsilon - \varepsilon'} \|x - y\|, \ \forall\, x, y \in \mathbb{B}_{\varepsilon'}(\bar{x}),$$

thus proving that ϕ is locally Lipschitz at $\bar{x} \in ri\ dom\ \phi$. Because $\bar{x} \in ri\ dom\ \phi$ is arbitrary, ϕ is locally Lipschitz on $ri\ dom\ \phi$. $\square$

2.3.3 Differentiability Property

After discussing the continuity and the Lipschitzian property of a convex function, we will now make a move toward studying its differentiability nature. In general, a convex function need not be differentiable on the whole of $\mathbb{R}^n$. For instance, consider the convex function $\phi(x) = |x|$. It is differentiable everywhere except $x = 0$, which is the point of minimizer if we minimize this function over the whole of $\mathbb{R}$. Another example of nonsmooth convex function that appears naturally is the max-function. Consider $\phi(x) = \max\{x, x^2\}$. As we know from Proposition 2.53, the supremum of convex functions is convex, so ϕ is convex. Here both x and x^2 are differentiable over $\mathbb{R}$ but ϕ is not differentiable at $x = 0$ and $x = 1$. Again for the unconstrained minimization of ϕ over $\mathbb{R}$, the point of minimizer is $\bar{x} = 0$. So how is one supposed to study the optimality at a point if the function is not differentiable there. This means the notion of differentiability must be replaced by some other concept so as to facilitate nonsmooth convex functions. For a differentiable function we know that both left-sided as well as right-sided derivatives exist and are equal. In case of a convex function, the right-sided derivative always exists. So in the direction to replace differentiability, we first introduce the concept of one-sided directional derivative or simply directional derivative.

Definition 2.74 For a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$, the *directional derivative* of ϕ at $\bar{x} \in dom\ \phi$ in the direction $d \in \mathbb{R}^n$ is defined as

$$\phi'(\bar{x}, d) = \lim_{\lambda \downarrow 0} \frac{\phi(\bar{x} + \lambda d) - \phi(\bar{x})}{\lambda},$$

provided $+\infty$ and $-\infty$ are allowed as limits.

Before we move on to present the result on the existence of directional derivatives of a convex function, we present a result from Rockafellar and Wets [101].

Proposition 2.75 *(Slope Inequality) Consider a function $\phi : I \to \mathbb{R}$ where $I \subset \mathbb{R}$ denotes an interval. Then ϕ is convex on I if and only if for arbitrary points $x < z < y$ in I,*

$$\frac{\phi(z) - \phi(x)}{z - x} \leq \frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(y) - \phi(z)}{y - z}. \tag{2.19}$$

Consequently, $\psi(y) = \dfrac{\phi(y) - \phi(x)}{y - x}$ is nondecreasing on I for every $y \in I \backslash \{x\}$. Moreover, if ϕ is differentiable over an open interval $I \subset \mathbb{R}$, then $\nabla \phi$ is nondecreasing on I.

Proof. We know that the convexity of ϕ on I is equivalent to

$$\phi(z) \leq \left(\frac{y - z}{y - x}\right) \phi(x) + \left(\frac{z - x}{y - x}\right) \phi(y), \ \forall \ x < z < y \text{ in } I.$$

The above inequality leads to

$$\begin{aligned} \phi(z) - \phi(x) \quad &\leq \quad \left(\frac{y - z}{y - x} - 1\right) \phi(x) + \frac{z - x}{y - x} \phi(y) \\ &= \quad (z - x) \left(\frac{\phi(y) - \phi(x)}{y - x}\right), \end{aligned}$$

as desired. The other inequalities can be established similarly, thereby leading to (2.19).

Conversely, suppose that $x < z < y$, which implies that there exists $\lambda \in (0, 1)$ such that $z = (1 - \lambda)x + \lambda y$. Substituting $z = (1 - \lambda)x + \lambda y$ in (2.19) leads to

$$\frac{\phi((1 - \lambda)x + \lambda y) - \phi(x)}{\lambda(y - x)} \leq \frac{\phi(y) - \phi(x)}{y - x},$$

that is,

$$\phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\phi(x) + \lambda\phi(y).$$

Because x and y were arbitrarily chosen, the above inequality holds for any $x, y \in I$ and any $\lambda \in [0, 1]$ (the above inequality holds trivially for $\lambda = 0$ and $\lambda = 1$). Hence, ϕ is a convex function.

Suppose that $y_1, y_2 \in I$ such that $y_i \neq x, \ i = 1, 2$, and $y_1 < y_2$. Consider the following cases:

$$x < y_1 < y_2, \quad y_1 < x < y_2 \quad \text{ and } \quad y_1 < y_2 < x.$$

Suppose that $x < y_1 < y_2$. In particular, for $z = y_1$ and $y = y_2$ in the inequality (2.19) yields

$$\psi(y_1) = \frac{\phi(y_1) - \phi(x)}{y_1 - x} \leq \frac{\phi(y_2) - \phi(x)}{y_2 - x} = \psi(y_2).$$

Applying (2.19) to the remaining two cases leads to the fact that $\psi(y) = \dfrac{\phi(y) - \phi(x)}{y - x}$ is nondecreasing.

Suppose that ϕ is convex, which implies (2.19) holds. As ϕ is differentiable, for $x_1, x_2 \in I$ with $x_1 < x_2$,

$$\nabla\phi(x_1) \leq \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} = \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2} \leq \nabla\phi(x_2),$$

thereby establishing the result. □

If $\phi : \mathbb{R} \to \bar{\mathbb{R}}$ is a proper convex function, *dom* ϕ may be considered an interval I. Then from the nondecreasing property of ψ in the above proposition, the right-sided derivative of ϕ, ϕ'_+, exists at $\bar{x}$ provided both $-\infty$ and $+\infty$ values are allowed and is defined as

$$\phi'_+(\bar{x}) = \lim_{x \downarrow \bar{x}} \frac{\phi(x) - \phi(\bar{x})}{x - \bar{x}}.$$

If $\dfrac{\phi(x) - \phi(\bar{x})}{x - \bar{x}}$ has a finite lower bound,

$$\lim_{x \downarrow \bar{x}} \frac{\phi(x) - \phi(\bar{x})}{x - \bar{x}} = \inf_{x > \bar{x},\ x \in I} \frac{\phi(x) - \phi(\bar{x})}{x - \bar{x}},$$

because $\dfrac{\phi(x) - \phi(\bar{x})}{x - \bar{x}}$ is nondecreasing on I. In case $\dfrac{\phi(x) - \phi(\bar{x})}{x - \bar{x}}$ does not have a finite lower bound,

$$\lim_{x \downarrow \bar{x}} \frac{\phi(x) - \phi(\bar{x})}{x - \bar{x}} = \inf_{x > \bar{x},\ x \in I} \frac{\phi(x) - \phi(\bar{x})}{x - \bar{x}} = -\infty,$$

and for the case when $I = \{\bar{x}\}$,

$$\inf_{x > \bar{x},\ x \in I} \frac{\phi(x) - \phi(\bar{x})}{x - \bar{x}} = +\infty$$

as $\{x \in \mathbb{R} : x > \bar{x},\ x \in I\} = \emptyset$. Thus,

$$\phi'_+(\bar{x}) = \inf_{x > \bar{x},\ x \in I} \frac{\phi(x) - \phi(\bar{x})}{x - \bar{x}}.$$

Theorem 2.76 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in dom\ \phi$. Then for every $d \in \mathbb{R}^n$, the directional derivative $\phi'(\bar{x}, d)$ exists with $\phi'(\bar{x}, 0) = 0$ and*

$$\phi'(\bar{x}, d) = \inf_{\lambda > 0} \frac{\phi(\bar{x} + \lambda d) - \phi(\bar{x})}{\lambda}.$$

Moreover, $\phi'(\bar{x}, d)$ is a sublinear function in d for every $d \in \mathbb{R}^n$.

Proof. Define $\psi : \mathbb{R} \to \bar{\mathbb{R}}$ given by

$$\psi(\lambda) = \phi(\bar{x} + \lambda d).$$

As $\bar{x} \in dom\ \phi$, $\psi(0) = \phi(\bar{x}) < +\infty$, which along with the convexity of ϕ implies that ψ is a proper convex function. Now consider $\varphi : \mathbb{R} \to \bar{\mathbb{R}}$ defined as

$$\varphi(\lambda) = \frac{\psi(\lambda) - \psi(0)}{\lambda} = \frac{\phi(\bar{x} + \lambda d) - \phi(\bar{x})}{\lambda}.$$

By Proposition 2.75, φ is nondecreasing when $\lambda > 0$. Then by the discussion preceding the theorem, $\psi'_+(0)$ exists and

$$\psi'_+(0) = \lim_{\lambda \to 0} \varphi(\lambda) = \inf_{\lambda > 0} \varphi(\lambda),$$

as desired.

Suppose that $d \in \mathbb{R}^n$ and $\alpha > 0$. Then

$$\begin{aligned} \phi'(\bar{x}, \alpha d) &= \lim_{\lambda \to 0} \frac{\phi(\bar{x} + \lambda \alpha d) - \phi(\bar{x})}{\lambda} \\ &= \lim_{\lambda \to 0} \alpha \frac{\phi(\bar{x} + \lambda \alpha d) - \phi(\bar{x})}{\lambda \alpha} \\ &= \alpha \lim_{\lambda' \to 0} \frac{\phi(\bar{x} + \lambda' d) - \phi(\bar{x})}{\lambda'} = \alpha \phi'(\bar{x}, d), \end{aligned}$$

which implies that $\phi'(x, .)$ is positively homogeneous.

Suppose that $d_1, d_2 \in \mathbb{R}^n$ and $\alpha \in [0, 1]$, by the convexity of ϕ,

$$\begin{aligned} \phi(\bar{x} + \lambda((1-\alpha)d_1 + \alpha d_2)) - \phi(\bar{x}) \leq (1-\alpha)(\phi(\bar{x} + \lambda d_1) - \phi(\bar{x})) \\ +\alpha(\phi(\bar{x} + \lambda d_2) - \phi(\bar{x})). \end{aligned}$$

Dividing both sides by $\lambda > 0$ and taking the limit as $\lambda \to 0$, the above inequality reduces to

$$\phi'(\bar{x}, ((1-\alpha)d_1 + \alpha d_2)) \leq (1-\alpha)\phi'(\bar{x}, d_1) + \alpha \phi(\bar{x}, d_2), \ \forall \ \alpha \in [0, 1].$$

In particular for $\alpha = 1/2$ and applying the positive homogeneity property, the above condition yields

$$\phi'(\bar{x}, (d_1 + d_2)) \leq \phi'(\bar{x}, d_1) + \phi(\bar{x}, d_2).$$

Because $d_1, d_2 \in \mathbb{R}^n$ were arbitrary, the above inequality implies that $\phi'(\bar{x}, .)$ is subadditive, which along with positive homogeneity implies that $\phi'(\bar{x}, .)$ is sublinear. $\square$

For a differentiable convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, the following relation holds between the directional derivative and the gradient of the function ϕ

$$\phi'(\bar{x}, d) = \langle \nabla \phi(\bar{x}), d \rangle, \ \forall \ d \in \mathbb{R}^n.$$

But in absence of differentiability, can one have such a relation for the directional derivative? The answer is yes. The notion that replaces the gradient in the above condition is the subgradient.

Definition 2.77 Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in dom\ \phi$. Then $\xi \in \mathbb{R}^n$ is said to be the *subgradient* of the function ϕ at $\bar{x}$ if

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n.$$

The collection of all such vectors constitute the *subdifferential* of ϕ at $\bar{x}$ and is denoted by $\partial\phi(\bar{x})$. For $\bar{x} \notin dom\ \phi$, $\partial\phi(\bar{x})$ is empty.

For a differentiable function, its gradient at any point acts as a tangent to the graph of the function at that point. In a similar way, from the definition above it can be seen that the affine function $\phi(\bar{x}) + \langle \xi, x - \bar{x} \rangle$ is a supporting hyperplane to the epigraph of ϕ at $(\bar{x}, \phi(\bar{x}))$ with the slope ξ. In fact, at the point of nondifferentiability, there can be an infinite number of such supporting hyperplanes and the collection of the slopes of each of these hyperplanes forms the subdifferential.

Recall the indicator function to the convex set $F \subset \mathbb{R}^n$. Obviously $\delta_F : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a proper convex function. Now from the above definition, the subdifferential of δ_F at $\bar{x} \in F$ is given by

$$\begin{aligned} \partial\delta_F(\bar{x}) &= \{\xi \in \mathbb{R}^n : \delta_F(x) - \delta_F(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n\} \\ &= \{\xi \in \mathbb{R}^n : 0 \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ x \in F\}, \end{aligned}$$

which is nothing but the normal cone to the set F at $\bar{x}$. Therefore, for a convex set F, $\partial\delta_F = N_F$.

Consider the norm function $\phi(x) = \|x\|, \ x \in \mathbb{R}^n$. Observe that ϕ is a convex function. At $\bar{x} = 0$, ϕ is not differentiable and $\partial\phi(\bar{x}) = \mathbb{B}$.

Like the relation between the directional derivative and gradient, we are interested in deriving a relationship between the directional derivative and the subdifferential, which we establish in the next result.

Theorem 2.78 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in dom\ \phi$. Then*

$$\partial\phi(\bar{x}) = \{\xi \in \mathbb{R}^n : \phi'(\bar{x}, d) \geq \langle \xi, d \rangle, \ \forall\ d \in \mathbb{R}^n\}.$$

Proof. Suppose that $\xi \in \partial\phi(\bar{x})$, which by Definition 2.77 implies that

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n.$$

In particular, for $x = \bar{x} + \lambda d$ with $\lambda > 0$, the above condition reduces to

$$\frac{\phi(\bar{x} + \lambda d) - \phi(\bar{x})}{\lambda} \geq \langle \xi, d \rangle, \ \forall\ d \in \mathbb{R}^n.$$

Taking the limit as $\lambda \to 0$ leads to

$$\phi'(\bar{x}, d) \geq \langle \xi, d \rangle, \ \forall \ d \in \mathbb{R}^n,$$

as desired.

Conversely, suppose that $\xi \in \mathbb{R}^n$ satisfies

$$\phi'(\bar{x}, d) \geq \langle \xi, d \rangle, \ \forall \ d \in \mathbb{R}^n.$$

By the alternate definition of $\phi'(\bar{x}, d)$ from Theorem 2.76 leads to

$$\frac{\phi(\bar{x} + \lambda d) - \phi(\bar{x})}{\lambda} \geq \langle \xi, d \rangle, \ \forall \ d \in \mathbb{R}^n.$$

In particular, for $\lambda \in [0, 1]$ and $d = x - \bar{x}$, which along with the convexity of ϕ leads to

$$\phi(x) - \phi(\bar{x}) \geq \frac{\phi(\bar{x} + \lambda(x - \bar{x})) - \phi(\bar{x})}{\lambda} \geq \langle \xi, x - \bar{x} \rangle, \ \forall \ x \in \mathbb{R}^n,$$

which implies that $\xi \in \partial\phi(\bar{x})$, thereby establishing the result. □

The result below from Rockafellar [97] shows that actually the directional derivative is the support function of the subdifferential set.

Theorem 2.79 *Consider a proper convex function* $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ *and* $\bar{x} \in dom\ \phi$. *Then*

$$cl\ \phi'(\bar{x}, d) = \sup_{\xi \in \partial\phi(\bar{x})} \langle \xi, d \rangle = \sigma_{\partial\phi(\bar{x})}(d), \ \forall \ d \in \mathbb{R}^n.$$

However, if $\bar{x} \in ri\ dom\ \phi$,

$$\phi'(\bar{x}, d) = \sigma_{\partial\phi(\bar{x})}(d), \ \forall \ d \in \mathbb{R}^n$$

and if $\bar{x} \in int\ dom\ \phi$, $\phi'(\bar{x}, d)$ *is finite for every* $d \in \mathbb{R}^n$.

Proof. Because $\phi'(\bar{x}, .)$ is sublinear, combining Theorems 2.62 and 2.78 leads to

$$cl\ \phi'(\bar{x}, d) = \sigma_{\partial\phi(\bar{x})}(d).$$

If $\bar{x} \in ri\ dom\ \phi$, the domain of $\phi'(\bar{x}, .)$ is an affine set that is actually a subspace parallel to the affine hull of *dom* ϕ. By sublinearity, $\phi'(\bar{x}, 0) = 0$, it is not identically $-\infty$ on the affine set. Therefore, by Proposition 2.63, *cl* $\phi'(\bar{x}, .)$ and hence $\phi'(\bar{x}, .)$ is a proper function. By Proposition 2.66, *cl* $\phi'(\bar{x}, .)$ agrees with $\phi'(\bar{x}, .)$ on the affine set and hence is closed, thereby leading to the desired condition. For $\bar{x} \in int\ dom\ \phi$, the domain of $\phi'(\bar{x}, .)$ is $\mathbb{R}^n$ and hence it is finite everywhere. □

As mentioned earlier for a differentiable convex function, for every $d \in \mathbb{R}^n$, $\phi'(\bar{x}, d) = \langle \nabla\phi(\bar{x}), d \rangle$. So the question is: for a differentiable convex function, how are the gradient and the subdifferential related? We discuss this aspect in the result below.

Proposition 2.80 *Consider a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ differentiable at $\bar{x}$ with gradient $\nabla\phi(\bar{x})$. Then the unique subgradient of ϕ at $\bar{x}$ is the gradient, that is, $\partial\phi(\bar{x}) = \{\nabla\phi(\bar{x})\}$.*

Proof. For a differentiable convex function ϕ,

$$\phi'(\bar{x}, d) = \langle \nabla\phi(\bar{x}), d\rangle, \ \forall\ d \in \mathbb{R}^n.$$

By Theorem 2.79, for every $\xi \in \partial\phi(\bar{x})$,

$$\langle \nabla\phi(\bar{x}) - \xi, d\rangle \geq 0, \ \forall\ d \in \mathbb{R}^n.$$

Because the above condition holds for every $d \in \mathbb{R}^n$, it reduces to

$$\langle \nabla\phi(\bar{x}) - \xi, d\rangle = 0, \ \forall\ d \in \mathbb{R}^n,$$

which leads to $\nabla\phi(\bar{x}) = \xi$. As $\xi \in \partial\phi(\bar{x})$ is arbitrary, the subdifferential is a singleton with $\partial\phi(\bar{x}) = \{\nabla\phi(\bar{x})\}$. $\square$

From the above theorem, we have the following result, which gives the equivalent characterization of a differentiable convex function.

Theorem 2.81 *Consider a differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$. Then ϕ is convex if and only if*

$$\phi(y) - \phi(x) \geq \langle \nabla\phi(x), y - x\rangle, \ \forall\ x, y \in \mathbb{R}^n.$$

Observe that in Theorem 2.79 we defined the relation between the directional derivative and the support function of the subdifferential for point $\bar{x}$ in the relative interior of the domain. The reason for this is the fact that at the boundary of the domain, the subdifferential may be an empty set. For a clear view into this aspect, we consider the following example from Bertsekas [12]. Let $\phi : \mathbb{R} \to \bar{\mathbb{R}}$ be a proper convex function given by

$$\phi(x) = \begin{cases} -\sqrt{x}, & 0 \leq x \leq 1, \\ +\infty, & \text{otherwise}. \end{cases}$$

The subdifferential of ϕ is

$$\partial\phi(x) = \begin{cases} \dfrac{-1}{2\sqrt{x}}, & 0 < x < 1, \\ [-1/2, +\infty), & x = 1, \\ \emptyset, & x \leq 0 \text{ or } x > 1. \end{cases}$$

Note that the subdifferential is empty at the boundary point $x = 0$. Also at the other boundary point $x = 1$, it is unbounded. But the subdifferential may also turn out to be unbounded at a point in the relative interior of the domain.

For example, consider the following proper convex function $\phi : \mathbb{R} \to \bar{\mathbb{R}}$ defined as

$$\phi(x) = \begin{cases} 0, & x = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

Observe that at $x = 0$, $\partial\phi(x) = \mathbb{R}$, which is unbounded even though 0 is in the relative interior of the domain. Based on these illustrations, we have the following result from Rockafellar [97] and Attouch, Buttazzo, and Michaille [3].

Proposition 2.82 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in dom\ \phi$. Then $\partial\phi(\bar{x})$ is closed and convex. For $\bar{x} \in ri\ dom\ \phi$, the subdifferential $\partial\phi(\bar{x})$ is nonempty. Furthermore, if $\bar{x} \in int\ dom\ \phi$, $\partial\phi(\bar{x})$ is nonempty and compact. Moreover, if ϕ is continuous at $\bar{x} \in dom\ \phi$, then $\partial\phi(\bar{x})$ is compact.*

Proof. Suppose that $\{\xi_k\} \subset \partial\phi(\bar{x})$ such that $\xi_k \to \xi$. By Definition 2.77 of subdifferential,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi_k, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n.$$

Taking the limit as $k \to +\infty$, the above inequality leads to

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n,$$

which implies that $\xi \in \partial\phi(\bar{x})$, thereby yielding the closedness of $\partial\phi(\bar{x})$.

Consider $\xi_1, \xi_2 \in \partial\phi(\bar{x})$, which implies that for $i = 1, 2$,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n.$$

Therefore, for any $\lambda \in [0, 1]$,

$$\phi(x) - \phi(\bar{x}) \geq \langle (1 - \lambda)\xi_1 + \lambda\xi_2, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n,$$

which implies $(1 - \lambda)\xi_1 + \lambda\xi_2 \in \partial\phi(\bar{x})$. Because ξ_1, ξ_2 were arbitrary, $\partial\phi(\bar{x})$ is convex.

From the proof of Theorem 2.79, for $\bar{x} \in ri\ dom\ \phi$, $\phi'(\bar{x}, .)$ is the support function of $\partial\phi(\bar{x})$, which is proper. Hence, $\partial\phi(\bar{x})$ is nonempty.

Again by Theorem 2.79, for $\bar{x} \in int\ dom\ \phi$, $\phi'(\bar{x}, .)$ is finite everywhere. Because it is a support of $\partial\phi(\bar{x})$, by Proposition 2.61, $\partial\phi(\bar{x})$ is bounded and hence compact.

Now suppose that ϕ is continuous at $\bar{x} \in dom\ \phi$. We have already seen that $\partial\phi$ is always closed and convex. Therefore to establish that $\partial\phi(\bar{x})$ is compact, we only need to show that it is bounded. By the continuity of ϕ at $\bar{x}$, it is bounded in the neighborhood of $\bar{x}$. Thus, there exist $\varepsilon > 0$ and $M \geq 0$ such that

$$\phi(\bar{x} + \varepsilon d) \leq M, \ \forall\ d \in \mathbb{B}.$$

Consider $\xi \in \partial\phi(\bar{x})$, which implies that

$$\langle \xi, x - \bar{x} \rangle \leq \phi(x) - \phi(\bar{x}), \ \forall\, x \in \mathbb{R}^n.$$

In particular, for any $d \in \mathbb{B}$, the above inequality along with the boundedness of ϕ in the neighborhood of $\bar{x}$ leads to

$$\langle \xi, \varepsilon d \rangle \leq \phi(\bar{x} + \varepsilon d) - \phi(\bar{x}) \leq M + |\phi(\bar{x})|,$$

which implies that

$$\langle \xi, d \rangle \leq \frac{1}{\varepsilon}(M + |\phi(\bar{x})|), \ \forall\, d \in \mathbb{B}.$$

Therefore,

$$\|\xi\| \leq \frac{1}{\varepsilon}(M + |\phi(\bar{x})|).$$

Because $\xi \in \partial\phi(\bar{x})$ was arbitrary, $\partial\phi(\bar{x})$ is bounded and hence compact. $\square$

If we consider a real-valued convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, then *int dom* $\phi = \mathbb{R}^n$ and therefore, the above result reduces to the following.

Proposition 2.83 *Consider a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$. Then the subdifferential $\partial\phi(x)$ is nonempty, convex, and compact for every $x \in \mathbb{R}^n$.*

With the discussion on subdifferentials, we present some properties of the subdifferential as x varies by treating it as a multifunction or set-valued mapping $x \mapsto \partial\phi(x)$ starting with some of the fundamental continuity results of the subdifferential mapping.

Theorem 2.84 *(Closed Graph Theorem) Consider a proper lsc convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. If for sequences $\{x_k\}, \{\xi_k\} \subset \mathbb{R}^n$ such that $\xi_k \in \partial\phi(x_k)$ with $x_k \to \bar{x}$ and $\xi_k \to \bar{\xi}$, then $\bar{\xi} \in \partial\phi(\bar{x})$. This means gph $\partial\phi$ is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Because $\xi_k \in \partial\phi(x_k)$ with $(x_k, \xi_k) \to (\bar{x}, \bar{\xi})$, then from Definition 2.77 of subdifferential,

$$\phi(x) - \phi(x_k) \geq \langle \xi_k, x - x_k \rangle, \ \forall\, x \in \mathbb{R}^n.$$

Taking the limit infimum as $k \to +\infty$, which along with the lower semicontinuity of ϕ reduces the above condition to

$$\phi(x) - \phi(\bar{x}) \geq \langle \bar{\xi}, x - \bar{x} \rangle, \ \forall\, x \in \mathbb{R}^n,$$

thereby implying that $\bar{\xi} \in \partial\phi(\bar{x})$ and thus establishing that *gph* $\partial\phi$ is closed, as desired. $\square$

From the above theorem one may note that the normal cone to a convex set $F \subset \mathbb{R}^n$ is also graph closed as it is nothing but the subdifferential of the convex indicator function δ_F, that is, $N_F = \partial \delta_F$.

In general we know that the arbitrary union of closed sets need not be closed. But in the proposition below from Bertsekas [12] and Rockafellar [97] we have that the union of the subdifferential over a compact set is compact.

Proposition 2.85 *Consider a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ and a compact set $F \in \mathbb{R}^n$. Then the set $\partial\phi(F) = \bigcup_{x \in F} \partial\phi(x)$ is nonempty and compact.*

Proof. Because F is a nonempty subset of *dom* $\phi = \mathbb{R}^n$, by Proposition 2.82, $\partial\phi(F)$ is nonempty.

We claim that $\partial\phi(F)$ is closed. Consider a sequence $\{\xi_k\} \subset \partial\phi(F)$ such that $\xi_k \to \bar{\xi}$. As $\xi_k \in \partial\phi(F)$ for $k \in \mathbb{N}$, there exist $x_k \in F$ such that $\xi_k \in \partial\phi(x_k),\ k \in \mathbb{N}$. By the compactness of F, $\{x_k\}$ is a bounded sequence that by the Bolzano–Weierstrass Theorem, Proposition 1.3, has a convergent subsequence. Without loss of generality, suppose that $x_k \to \bar{x}$, which by the closedness of F implies that $\bar{x} \in F$. Invoking the Closed Graph Theorem, Theorem 2.84, $\bar{\xi} \in \partial\phi(\bar{x}) \subset \partial\phi(F)$. Thus, $\partial\phi(F)$ is closed.

Now to establish the compactness of $\partial\phi(F)$, we will establish the boundedness of $\partial\phi(F)$. On the contrary, suppose that there exist a bounded sequence $\{x_k\} \subset F$ and an unbounded sequence $\{\xi_k\} \subset \mathbb{R}^n$ such that $\xi_k \in \partial\phi(x_k)$. Define $\eta_k = \dfrac{\xi_k}{\|\xi_k\|}$, which is a bounded sequence. Because $\{x_k\}$ and $\{\eta_k\}$ are bounded sequences, by the Bolzano–Weierstrass Theorem, have a convergent subsequence. As $\xi_k \in \partial\phi(x_k)$, by Definition 2.77 of subdifferential,

$$\phi(x_k + \eta_k) - \phi(x_k) \geq \langle \xi_k, \eta_k \rangle = \|\xi_k\|.$$

By Theorem 2.69, ϕ is continuous on $\mathbb{R}^n$, which along with the convergence of $\{x_k\}$ and $\{\eta_k\}$ yields that $\phi(x_k + \eta_k) - \phi(x_k)$ is bounded. Therefore, by the above inequality, $\{\xi_k\}$ is a bounded sequence, thereby contradicting our assumption. Thus, $\partial\phi(F)$ is a bounded set and hence compact. □

Theorem 2.86 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then $\partial\phi$ is usc on int dom ϕ. Moreover, if $\phi : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function, then it is continuously differentiable.*

Proof. By Proposition 2.82, $\partial\phi(\bar{x})$ is nonempty and compact if and only if $\bar{x} \in$ *int dom* ϕ. By Theorem 2.84, $\partial\phi$ is graph closed. Therefore, from the discussion on set-valued mappings in Chapter 1, $\partial\phi$ is usc on *int dom* ϕ.

As for a single-valued map, the notion of upper semicontinuity coincides with that of continuity and by Proposition 2.80, for a differentiable convex function $\partial\phi = \{\nabla\phi\}$, ϕ is continuously differentiable. □

Below we state another important characteristic of the subdifferential without proof. For more details on the treatment of $\partial\phi$ as a multifunction, one may refer to Rockafellar [97].

Theorem 2.87 *Consider a closed proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then the subdifferential $\partial\phi$ is a maximal monotone where by monotonicity we mean that for any $x_1, x_2 \in \mathbb{R}^n$,*

$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geq 0, \ \forall\ \xi_i \in \partial\phi(x_i), \ i = 1, 2,$$

and maximal monotone map in the sense that its graph is not properly contained in the graph of any other monotone map.

Similar to the standard Mean Value Theorem, Theorem 1.18, we present the Mean Value Theorem for convex functions in terms of the subdifferential.

Theorem 2.88 *Consider a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$. Then for $x, y \in \mathbb{R}^n$, there exists $z \in (x, y)$ such that*

$$\phi(y) - \phi(x) \in \langle \partial\phi(z), y - x \rangle,$$

where $\langle \partial\phi(z), y - x \rangle = \{\langle \xi, y - x \rangle : \xi \in \partial\phi(z)\}$.

Proof. Consider the function $\psi : [0, 1] \to \mathbb{R}$ defined by

$$\psi(\lambda) = \phi(x + \lambda(y - x)) - \phi(x) + \lambda(\phi(x) - \phi(y)).$$

Because ϕ is real-valued and by Theorem 2.69 it is continuous on $\mathbb{R}^n$, hence ψ is a real-valued continuous function on [0,1]. Observe that $\psi(0) = 0 = \psi(1)$. Also, by the convexity of ϕ,

$$\psi(\lambda) \leq (1 - \lambda)\phi(x) + \lambda\phi(y) - \phi(x) + \lambda(\phi(x) - \phi(y)) = 0, \ \forall\ \lambda \in [0, 1].$$

Thus, ψ attains its maximum at $\lambda = 0$ and $\lambda = 1$ and hence there exists $\bar{\lambda} \in (0, 1)$ at which ψ attains its minimum over $[0, 1]$. Therefore,

$$\psi'(\bar{\lambda}, d) \geq 0, \ \forall\ d \in \mathbb{R}.$$

Denote $z = x + \bar{\lambda}(y - x) \in (x, y)$. Therefore,

$$\begin{aligned}
\psi'(\bar{\lambda}, d) &= \lim_{\lambda \downarrow 0} \frac{\psi(\bar{\lambda} + \lambda d) - \psi(\bar{\lambda})}{\lambda} \\
&= \lim_{\lambda \downarrow 0} \frac{\phi(x + (\bar{\lambda} + \lambda d)(y - x)) - \phi(x + \bar{\lambda}(y - x))}{\lambda} + d(\phi(x) - \phi(y)) \\
&= \phi'(z, d(y - x)) + d(\phi(x) - \phi(y)), \ \forall\ d \in \mathbb{R},
\end{aligned}$$

which implies that

$$\phi'(z, d(y - x)) \geq d(\phi(y) - \phi(x)), \ \forall\ d \in \mathbb{R}.$$

In particular, taking $d = 1$ in the above condition leads to

$$\phi(y) - \phi(x) \leq \phi'(z, y - x),$$

whereas taking $d = -1$ yields

$$-\phi'(z, x - y) \leq \phi(y) - \phi(x).$$

Combining the preceding inequalities imply

$$-\phi'(z, x - y) \leq \phi(y) - \phi(x) \leq \phi'(z, y - x),$$

which by Theorem 2.79 becomes

$$\inf_{\xi \in \partial\phi(z)} \langle \xi, y - x \rangle = - \sup_{\xi \in \partial\phi(z)} \langle \xi, x - y \rangle \leq \phi(y) - \phi(x) \leq \sup_{\xi \in \partial\phi(z)} \langle \xi, y - x \rangle.$$

By Proposition 2.83, $\partial\phi(z)$ is compact, which along with the continuity of $\langle \xi, y - x \rangle$ implies that there exists $\bar{\xi} \in \partial\phi(z)$ such that

$$\phi(y) - \phi(x) = \langle \bar{\xi}, y - x \rangle \in \langle \partial\phi(z), y - x \rangle,$$

thereby completing the proof. $\square$

We have discussed the various continuity and differentiability behaviors of convex functions but in most cases these properties were restricted to the interior or relative interior of the domain of the function. As seen in the discussion preceding Proposition 2.82, the subdifferential set may be empty at the boundary of the domain. To overcome this flaw of the subdifferential of a convex function, we have the notion of *ε-subdifferentials*, which have the nonemptiness property throughout the domain of the function. We will discuss this notion in a later section in the chapter.

As we are interested in the convex optimization problem, we first give the optimality condition for the unconstrained convex programming problem

$$\min\ f(x) \quad \text{subject to} \quad x \in \mathbb{R}^n, \tag{CP_u}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

Theorem 2.89 *Consider the unconstrained convex programming problem (CP_u). Then $\bar{x} \in \mathbb{R}^n$ is the point of minimizer of (CP_u) if and only if $0 \in \partial f(\bar{x})$.*

Proof. Suppose that $\bar{x} \in \mathbb{R}^n$ is a point of minimizer of (CP_u), which implies that

$$f(x) - f(\bar{x}) \geq 0,\ \forall\ x \in \mathbb{R}^n.$$

By Definition 2.77 of subdifferential, $0 \in \partial f(\bar{x})$. The converse can be proved by again employing the definition of the subdifferential. $\square$

Now recall the constrained convex programming problem presented in Chapter 1:

$$\min\ f(x) \quad \text{subject to} \quad x \in C, \tag{CP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and C is a convex subset of $\mathbb{R}^n$. Recall the important property of convex optimization discussed in Section 1.3 that makes its study useful is that every local minimizer is also a global minimizer. The next result provides an alternative proof to this fact.

Theorem 2.90 *Consider a convex set $C \subset \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper convex function. Then the point of local minimum is a point of global minimum. If in addition f is strictly convex, there exists at most one global point of minimum.*

Proof. Suppose that $\bar{x} \in \mathbb{R}^n$ is a point of local minimum of f over C. We claim that $\bar{x}$ is a point of global minimum. On the contrary, assume that $\bar{x}$ is not a point of global minimum. Thus there exists $\tilde{x} \in C$ such that $f(\tilde{x}) < f(\bar{x})$. By the convexity of f, for every $\lambda \in (0, 1)$,

$$f((1-\lambda)\bar{x} + \lambda\tilde{x}) \leq (1-\lambda)f(\bar{x}) + \lambda f(\tilde{x}) < f(\bar{x}). \tag{2.20}$$

Also by the convexity of C, $(1-\lambda)\bar{x} + \lambda\tilde{x} \in C$. Taking λ sufficiently small, $(1-\lambda)\bar{x} + \lambda\tilde{x}$ is in the neighborhood of $\bar{x}$, which by the inequality (2.20) implies that

$$f((1-\lambda)\bar{x} + \lambda\tilde{x}) < f(\bar{x}),$$

which contradicts that $\bar{x}$ is a point of local minimum. Hence, $\bar{x}$ is a point of global minimum of f over C.

Suppose that f is a strictly convex function with $\bar{x}$ and $\bar{y}$ as the points of global minimum. Let $f(\bar{x}) = f(\bar{y}) = f_{min}$, say. We claim that $\bar{x} = \bar{y}$. On the contrary, assume that $\bar{x} \neq \bar{y}$. By Definition 2.47 of strict convexity, for every $\lambda \in (0, 1)$,

$$f((1-\lambda)\bar{x} + \lambda\bar{y}) < (1-\lambda)f(\bar{x}) + \lambda f(\bar{y}) = f_{min}. \tag{2.21}$$

By the convexity of C, $(1-\lambda)\bar{x} + \lambda\bar{y} \in (\bar{x}, \bar{y}) \subset C$. Now the strict inequality (2.21) contradicts the fact that $\bar{x}$ and $\bar{y}$ are the points of global minimizers of f over C, which is a contradiction. Thus, $\bar{x} = \bar{y}$, thereby implying that minimizing a strictly convex function f over a convex set C has at most one point of global minimum. $\square$

As discussed earlier in this chapter, the above problem can be converted into the unconstrained convex programming problem of the form (CP_u) with the objective function f replaced by $f + \delta_C$. From the above theorem, $\bar{x}$ is the point of minimizer of (CP) if and only if

$$0 \in \partial(f + \delta_C)(\bar{x}).$$

To express the above inclusion explicitly in terms of the subdifferentials of the objective function f and the indicator function δ_C, one needs the calculus rules for the subdifferentials. Thus, following this path we shall now discuss the subdifferential calculus rules.

2.4 Subdifferential Calculus

As we have already seen that subdifferentials play a pivotal role in the convex analysis. It replaces the role of derivative in case of nondifferentiable convex functions. So it is obvious to look into the matter as to whether or not the differential calculus is carried over to subdifferential calculus. As we proceed in this direction, one will see that it does satisfy results similar to standard calculus but under certain assumptions. We begin our journey of subdifferential calculus with the sum rule.

Theorem 2.91 *(Moreau–Rockafellar Sum Rule) Consider two proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}},\ i = 1, 2$. Suppose that ri dom $\phi_1 \cap$ ri dom $\phi_2 \neq \emptyset$. Then*

$$\partial(\phi_1 + \phi_2)(x) = \partial\phi_1(x) + \partial\phi_2(x)$$

for every $x \in dom(\phi_1 + \phi_2)$.

Proof. We first show that

$$\partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}) \subset \partial(\phi_1 + \phi_2)(\bar{x}). \tag{2.22}$$

Suppose that $\xi_i \in \partial\phi_i(\bar{x}),\ i = 1, 2$. By the definition of a subdifferential,

$$\phi_i(x) - \phi_i(\bar{x}) \geq \langle \xi_i, x - \bar{x}\rangle,\ \forall\ x \in \mathbb{R}^n,\ i = 1, 2.$$

Therefore,

$$(\phi_1 + \phi_2)(x) - (\phi_1 + \phi_2)(\bar{x}) \geq \langle \xi_1 + \xi_2, x - \bar{x}\rangle,\ \forall\ x \in \mathbb{R}^n,$$

which implies that $(\xi_1 + \xi_2) \in \partial(\phi_1 + \phi_2)(\bar{x})$, thereby establishing (2.22).

To obtain the result, we will now prove the reverse inclusion, that is,

$$\partial(\phi_1 + \phi_2)(\bar{x}) \subset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}). \tag{2.23}$$

Suppose that $\xi \in \partial(\phi_1 + \phi_2)(\bar{x})$. Define two convex functions

$$\psi_1(x) = \phi_1(x + \bar{x}) - \phi_1(\bar{x}) - \langle \xi, x\rangle \quad \text{and} \quad \psi_2(x) = \phi_2(x + \bar{x}) - \phi_2(\bar{x}).$$

Here, $\psi_1(0) = \psi_2(0) = 0$. Observe that $\xi \in \partial(\phi_1 + \phi_2)(\bar{x})$ which by the above constructed functions is equivalent to

$$(\psi_1 + \psi_2)(x) \geq 0,\ \forall\ x \in \mathbb{R}^n,$$

that is, $0 \in \partial(\psi_1 + \psi_2)(0)$. Thus, without loss of generality, consider $\bar{x} = 0$, $\xi = 0$, and $\phi_1(0) = \phi_2(0) = 0$ such that

$$0 \in \partial(\phi_1 + \phi_2)(0),$$

which implies

$$(\phi_1 + \phi_2)(x) \geq (\phi_1 + \phi_2)(0) = 0, \ \forall\, x \in \mathbb{R}^n,$$

that is, $\phi_1(x) \geq -\phi_2(x)$ for every $x \in \mathbb{R}^n$. Define

$$F_1 = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \phi_1(x) \leq \alpha\}$$

and
$$F_2 = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \leq -\phi_2(x)\}.$$

Observe that both F_1 and F_2 are closed convex sets, where by Proposition 2.64,

$$ri\ F_1 = ri\ epi\ \phi_1 = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : x \in ri\ dom\ \phi_1,\ \phi_1(x) < \alpha\}.$$

As $\phi_1(x) \geq -\phi_2(x)$, we have

$$ri\ F_1 \cap F_2 = \emptyset$$

with $(0, 0) \in F_1 \cap F_2$. Therefore, by the separation theorem, Theorem 2.26 (ii), there exists $(x^*, \alpha^*) \in \mathbb{R}^n \times \mathbb{R}$ with $(x^*, \alpha^*) \neq (0, 0)$ such that

$$\begin{aligned}
\langle x^*, x\rangle + \alpha^* \alpha \geq 0,\ \forall\ (x, \alpha) \in F_1,\\
\langle x^*, x\rangle + \alpha^* \alpha \leq 0,\ \forall\ (x, \alpha) \in F_2.
\end{aligned}$$

By assumption as $\phi_1(0) = 0$, we have $(0, \alpha) \in F_1$ for $\alpha \geq 0$. Therefore, from the inequality above, we have $\alpha^* \geq 0$. We claim that $\alpha^* \neq 0$. Suppose that $\alpha^* = 0$. Thus the above inequalities imply

$$\langle x^*, x_1\rangle \geq 0 \geq \langle x^*, x_2\rangle,\ \forall\ x_1 \in dom\ \phi_1,\ \forall\ x_2 \in dom\ \phi_2.$$

This implies that $dom\ \phi_1$ and $dom\ \phi_2$ can be separated, which contradicts the hypothesis that $ri\ dom\ \phi_1 \cap ri\ dom\ \phi_2 \neq \emptyset$. Hence, $\alpha^* > 0$ and can be normalized to one and thus

$$\begin{aligned}
\langle x^*, x\rangle + \alpha \geq 0,\ \forall\ (x, \alpha) \in F_1,\\
\langle x^*, x\rangle + \alpha \leq 0,\ \forall\ (x, \alpha) \in F_2.
\end{aligned}$$

In particular, for $(x, \phi_1(x)) \in F_1$ and $(x, -\phi_2(x)) \in F_2$, we have $-x^* \in \partial\phi_1(0)$ and $x^* \in \partial\phi_2(0)$, thereby leading to

$$0 \in \partial\phi_1(0) + \partial\phi_2(0),$$

thus establishing (2.23) and hence completing the proof. □

The necessity of the condition $ri\ dom\ \phi_1 \cap ri\ dom\ \phi_2 \neq \emptyset$ can be seen from the following example from Phelps [93]. Consider $\phi_1, \phi_2 : \mathbb{R}^2 \to \bar{\mathbb{R}}$ defined as

$$\begin{aligned}
\phi_1(x) &= \delta_{F_1}(x), \qquad & F_1 &= epi\ y^2,\ y \in \mathbb{R},\\
\phi_2(x) &= \delta_{F_2}(x), \qquad & F_2 &= \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = 0\}.
\end{aligned}$$

Here, $\partial(\phi_1 + \phi_2)(0) = \mathbb{R}^2$ whereas

$$\partial\phi_1(0) = \{(0,\xi) \in \mathbb{R}^2 : \xi \leq 0\} \quad \text{and} \quad \partial\phi_2(0) = \{(0,\xi) \in \mathbb{R}^2 : \xi \in \mathbb{R}\}.$$

Therefore, $\partial(\phi_1+\phi_2)(0) \neq \partial\phi_1(0)+\partial\phi_2(0)$. Observe that $dom\ \phi_1 \cap dom\ \phi_2 = F_1 \cap F_2 = \{(0,0)\}$ while $ri\ dom\ \phi_1 \cap ri\ dom\ \phi_2 = ri\ F_1 \cap ri\ F_2 = \emptyset$.

Now as an application of the Subdifferential Sum Rule, we prove the equality in Proposition 2.39 (i) under the assumption of $ri\ F_1 \cap ri\ F_2 \neq \emptyset$.

Proof of Proposition 2.39 (i). For convex sets $F_1, F_2 \subset \mathbb{R}^n$, define $\phi_1 = \delta_{F_1}$ and $\phi_2 = \delta_{F_2}$. Observe that $dom\ \phi_i = F_i$ for $i = 1,2$. If $ri\ F_1 \cap ri\ F_2 \neq \emptyset$, then $ri\ dom\ \phi_1 \cap ri\ dom\ \phi_2 \neq \emptyset$. Now applying the Sum Rule, Theorem 2.91,

$$\partial(\phi_1 + \phi_2)(\bar{x}) = \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}),\ \forall\ \bar{x} \in dom\ \phi_1 \cap dom\ \phi_2,$$

which along with the facts that $\delta_{F_1} + \delta_{F_2} = \delta_{F_1 \cap F_2}$ and $\partial\delta_F = N_F$ implies that

$$N_{F_1 \cap F_2}(\bar{x}) = N_{F_1}(\bar{x}) + N_{F_2}(\bar{x}),\ \forall\ \bar{x} \in F_1 \cap F_2,$$

hence completing the proof. □

Now if in Theorem 2.91, $\phi_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1,2$ are real-valued convex functions, then the Sum Rule can be derived using the directional derivative. We briefly discuss that approach from Hiriart-Urruty and Lemaréchal [63]. Using Theorem 2.79, the support of $\partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x})$ is $\phi_1'(\bar{x},.) + \phi_2'(\bar{x},.)$. Readers are advised to verify this fact using the definition of support. Also, the support of $\partial(\phi_1 + \phi_2)(\bar{x})$ is $(\phi_1 + \phi_2)'(\bar{x},.)$, which is same as that of $\partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x})$. Because the support functions are same for both sets,

$$\partial(\phi_1 + \phi_2)(\bar{x}) = \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}).$$

Observe that no additional assumption was required as here $ri\ dom\ \phi_1$ as well as $ri\ dom\ \phi_2$ is $\mathbb{R}^n$.

Other than the sum of convex functions being convex, from Proposition 2.53, we have that the composition of a nondecreasing convex function with a convex function is also convex. So before presenting the Chain Rule, we introduce the notion of increasing function defined over $\mathbb{R}^n$ and a result on the subdifferential of a nondecreasing function. Recall that in Proposition 2.53, the nondecreasing function ψ was defined over $\mathbb{R}$.

Definition 2.92 A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called *nondecreasing* if for $x, y \in \mathbb{R}^n$ with $x_i \geq y_i,\ i = 1,2,\ldots,n$, implies that $\phi(x) \geq \phi(y)$.

Theorem 2.93 *Consider a nondecreasing convex function $\phi : \mathbb{R}^n \to \mathbb{R}$. Then for every $x \in \mathbb{R}^n$, $\partial\phi(x) \subset \mathbb{R}^n_+$.*

Proof. Because ϕ is a nondecreasing convex function,

$$\phi(\bar{x}) \geq \phi(\bar{x} - e_i) \geq \phi(\bar{x}) + \langle \xi, -e_i \rangle,$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the i-th place and $\xi \in \partial\phi(\bar{x})$. This implies that

$$\phi(\bar{x}) \geq \phi(\bar{x}) - \xi_i,$$

that is, $\xi_i \geq 0$. Since i was arbitrary, $\xi_i \geq 0$, $i = 1, 2, \ldots, n$ and thus $\partial\phi(\bar{x}) \subset \mathbb{R}^n_+$. $\square$

We now present the subdifferential calculus rule of the composition of convex functions. The proof is from Hiriart-Urruty and Lemaréchal [63].

Theorem 2.94 *(Chain Rule) Consider a nondecreasing convex function $\phi : \mathbb{R}^m \to \mathbb{R}$ and a vector-valued function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ given by $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_m(x))$ where $\phi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$ be a convex function. Then*

$$\partial(\phi \circ \Phi)(\bar{x}) = \left\{ \sum_{i=1}^{m} \mu_i \xi_i : (\mu_1, \mu_2, \ldots, \mu_m) \in \partial\phi(\Phi(\bar{x})), \right.$$
$$\left. \xi_i \in \partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m \right\}.$$

Proof. Define

$$\mathcal{F} = \left\{ \sum_{i=1}^{m} \mu_i \xi_i : (\mu_1, \mu_2, \ldots, \mu_m) \in \partial\phi(\Phi(\bar{x})),\ \xi_i \in \partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m \right\}.$$

We will prove the result in the following steps:

1. We shall show that $\mathcal{F}$ is a convex compact set as $\partial(\phi \circ \Phi)$.
2. We shall calculate the support function of $\mathcal{F}$.
3. We shall calculate the support function of $\partial(\phi \circ \Phi)$ and establish that it is same as the support of $\mathcal{F}$.

The result is completed by the fact that two convex sets are equal if and only if their support functions are equal.

Step 1: Consider any $\xi \in \mathcal{F}$. Thus there exist $(\mu_1, \mu_2, \ldots, \mu_m) \in \partial\phi(\Phi(\bar{x}))$ and $\xi_i \in \partial\phi_i(\bar{x})$, $i = 1, 2, \ldots, m$, such that

$$\xi = \sum_{i=1}^{m} \mu_i \xi_i.$$

Therefore,

$$\|\xi\| \leq \sum_{i=1}^{m} |\mu_i| \, \|\xi_i\|.$$

By Proposition 2.83, $\partial\phi(\Phi(\bar{x}))$ as well $\partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m$, are bounded sets and hence ξ is bounded. Because $\xi \in \mathcal{F}$ was arbitrary, $\mathcal{F}$ is a bounded set. Moreover, $\partial\phi(\Phi(\bar{x}))$ and $\partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m$, are closed sets; thus $\mathcal{F}$ is also a closed set, thereby yielding the compactness of F.

Suppose that $\xi_1, \xi_2 \in \mathcal{F}$, which implies for $j = 1, 2$,

$$\xi_j = \sum_{i=1}^{m} \mu_i^j \xi_i^j,$$

where $(\mu_1^j, \mu_2^j, \ldots, \mu_m^j) \in \partial\phi(\Phi(\bar{x})$ and $\xi_i^j \in \partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m$, for $j = 1, 2$. Now for any $\lambda \in (0, 1)$, define

$$\xi_\lambda = (1 - \lambda)\xi_1 + \lambda\xi_2.$$

From Theorem 2.93, $\mu_i^j \geq 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2$. Define

$$\mu_i^\lambda = (1 - \lambda)\mu_i^1 + \lambda\mu_i^2,\ i = 1, 2, \ldots, m.$$

Note that $\mu_i^\lambda = 0$ only when $\mu_i^1 = \mu_i^2 = 0$ as $\lambda \in (0, 1)$. Therefore,

$$\xi = \sum_{i \in \bar{I}} \mu_i \left(\frac{(1 - \lambda)\mu_i^1}{\mu_i} \xi_i^1 + \frac{\lambda\mu_i^2}{\mu_i} \xi_i^2 \right),$$

where $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \mu_i > 0\}$. By Proposition 2.83, $\partial\phi(\Phi(\bar{x}))$ and $\partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m$, are convex sets and hence

$$(\mu_1, \mu_2, \ldots, \mu_m) \in \partial\phi(\Phi(\bar{x}))$$

and

$$\frac{(1 - \lambda)\mu_i^1}{\mu_i} \xi_i^1 + \frac{\lambda\mu_i^2}{\mu_i} \xi_i^2 \in \partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m,$$

thereby showing that $\mathcal{F}$ is convex.

Step 2: Denote

$$\Phi'(\bar{x}, d) = (\phi_1'(\bar{x}, d), \phi_2'(\bar{x}, d), \ldots, \phi_m'(\bar{x}, d)).$$

We will establish that

$$\sigma_{\mathcal{F}}(d) = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)).$$

Consider $\xi \in \mathcal{F}$, which implies that

$$\xi = \sum_{i=1}^{m} \mu_i \xi_i,$$

where $(\mu_1, \mu_2, \ldots, \mu_m) \in \partial\phi(\Phi(\bar{x}))$ and $\xi_i \in \partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m$. By Theorem 2.79,

$$\langle \xi_i, d \rangle \leq \phi_i'(\bar{x}, d),\ i = 1, 2, \ldots, m.$$

By Theorem 2.93, $\mu_i \geq 0,\ i = 1, 2, \ldots, m$, which along with the above inequality implies that

$$\langle \xi, d\rangle = \sum_{i=1}^{m} \mu_i \langle \xi_i, d\rangle \leq \sum_{i=1}^{m} \mu_i \phi_i'(\bar{x}, d).$$

As $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \partial\phi(\Phi(\bar{x}))$,

$$\sum_{i=1}^{m} \mu_i \phi_i'(\bar{x}, d) = \langle \mu, \Phi'(\bar{x}, d)\rangle \leq \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)).$$

We claim that there exists $\bar{\xi} \in \mathcal{F}$ such that

$$\langle \bar{\xi}, d\rangle = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)).$$

By Proposition 2.83, $\partial\phi(\Phi(\bar{x}))$ is compact and therefore, there exists $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_m) \in \partial\phi(\Phi(\bar{x}))$ such that

$$\sum_{i=1}^{m} \bar{\mu}_i \phi_i'(\bar{x}, d) = \langle \bar{\mu}, \Phi'(\bar{x}, d)\rangle = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)). \tag{2.24}$$

Also, for $i = 1, 2, \ldots, m$, $\partial\phi_i(\bar{x})$ is compact, which implies there exists $\bar{\xi}_i \in \partial\phi_i(\bar{x})$ such that

$$\langle \bar{\xi}_i, d\rangle = \phi_i'(\bar{x}, d),\ i = 1, 2, \ldots, m.$$

Therefore, the condition (2.24) becomes

$$\sum_{i=1}^{m} \bar{\mu}_i \langle \bar{\xi}_i, d\rangle = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)).$$

Denoting $\bar{\xi} = \sum_{i=1}^{m} \bar{\mu}_i \bar{\xi}_i \in \mathcal{F}$,

$$\langle \bar{\xi}, d\rangle = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)),$$

which implies that

$$\sigma_{\mathcal{F}}(d) = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)),\ \forall\ d \in \mathbb{R}^n.$$

Step 3: It is obvious that the support function of $\partial(\phi \circ \Phi)(\bar{x})$ is $(\phi \circ \Phi)'(\bar{x}, d)$. We claim that

$$(\phi \circ \Phi)'(\bar{x}, d) = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)).$$

For real-valued convex functions $\phi_i,\ i = 1, 2, \ldots, m$, from Definition 2.74 of directional derivative, it is obvious that

$$\phi_i(\bar{x} + \lambda d) = \phi_i(\bar{x}) + \lambda \phi_i'(\bar{x}, d) + o(\lambda),\ i = 1, 2, \ldots, m,$$

which implies that

$$\Phi(\bar{x} + \lambda d) = \Phi(\bar{x}) + \lambda \Phi'(\bar{x}, d) + o(\lambda).$$

By Theorem 2.69, ϕ is continuous on $ri\ dom\ \phi = \mathbb{R}^n$ which yields

$$\phi(\Phi(\bar{x} + \lambda d)) = \phi(\Phi(\bar{x}) + \lambda \Phi'(\bar{x}, d)) + o(\lambda),$$

which again by the definition of ϕ' leads to

$$\phi(\Phi(\bar{x} + \lambda d)) = \phi(\Phi(\bar{x})) + \lambda \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)) + o(\lambda).$$

Dividing throughout by $\lambda > 0$ and taking the limit as $\lambda \to 0$ reduces the above condition to

$$(\phi \circ \Phi)'(\bar{x}, d) = \phi'(\Phi(\bar{x}), \Phi'(\bar{x}, d)).$$

Because the support functions of both the sets are same, the sets $\partial(\phi \circ \Phi)$ and $\mathcal{F}$ coincide. □

As we will discuss in this book, one of the ways to derive the optimality conditions for (CP) is the max-function approach, thereby hinting at the use of subdifferential calculus for the max-function. Consider the convex functions $\phi_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$, and define the max-function

$$\phi(x) = \max\{\phi_1(x), \phi_2(x), \ldots, \phi_m(x)\}.$$

Observe that ϕ can be expressed as a composition of the functions

$$\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_m(x)) \quad \text{and} \quad \varphi(y) = \max\{y_1, y_2, \ldots, y_m\}$$

given by $\phi(x) = (\varphi \circ \Phi)(x)$. It is now natural to apply the Chain Rule presented above but along with that one needs to calculate $\partial\varphi$ or $\varphi'(x, d)$. So before moving on to establish the Max-Function Rule, we will present a result to derive $\varphi'(x, d)$. The proof is from Hiriart-Urruty [59].

Theorem 2.95 *Consider differentiable convex functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, m$. For $x \in \mathbb{R}^n$, define*

$$\varphi(x) = \max\{\varphi_1(x), \varphi_2(x), \ldots, \varphi_m(x)\}$$

and denote the active index set by $I(x)$ defined as

$$I(x) = \{i \in \{1, 2, \ldots, m\} : \varphi(x) = \varphi_i(x)\}.$$

Then

$$\varphi'(\bar{x}, d) = \max_{i \in I(\bar{x})} \{\langle \nabla \varphi_i(\bar{x}), d \rangle\}.$$

Proof. Without loss of generality, assume that $I(\bar{x}) = \{1, 2, \ldots, m\}$ because those φ_i where the maximum is not attained, do not affect $\varphi'(\bar{x}, d)$. By the definition of the max-function,

$$\varphi(\bar{x} + \lambda d) \geq \varphi_i(\bar{x} + \lambda d), \ \forall \ i = 1, 2, \ldots, m,$$

which implies that

$$\varphi(\bar{x} + \lambda d) - \varphi(\bar{x}) \geq \varphi_i(\bar{x} + \lambda d) - \varphi(\bar{x}), \ \forall \ i = 1, 2, \ldots, m.$$

As $\varphi(\bar{x}) = \varphi_i(\bar{x})$ for $i \in I(\bar{x})$,

$$\varphi(\bar{x} + \lambda d) - \varphi(\bar{x}) \geq \varphi_i(\bar{x} + \lambda d) - \varphi_i(\bar{x}), \ \forall \ i \in I(\bar{x}).$$

By Definition 2.74 of the directional derivative,

$$\varphi'(\bar{x}, d) \geq \lim_{\lambda \downarrow 0} \frac{\varphi_i(\bar{x} + \lambda d) - \varphi_i(\bar{x})}{\lambda}, \ \forall \ i \in I(\bar{x}).$$

Because $\phi_i, \ i \in I(\bar{x})$ are differentiable functions, which along with the above inequality yields

$$\varphi'(\bar{x}, d) \geq \max_{i \in I(\bar{x})} \langle \nabla\varphi_i(\bar{x}), d \rangle.$$

To establish the result, we will prove the reverse inequality, that is,

$$\varphi'(\bar{x}, d) \leq \max_{i \in I(\bar{x})} \langle \nabla\varphi_i(\bar{x}), d \rangle.$$

We claim that there exists a neighborhood $\mathcal{N}(\bar{x})$ such that $I(x) \subset I(\bar{x})$ for every $x \in \mathcal{N}(\bar{x})$. On the contrary, assume that there exists $\{x_k\} \subset \mathbb{R}^n$ with $x_k \to \bar{x}$ such that $I(x_k) \not\subset I(\bar{x})$. Therefore, we may choose $i_k \in I(x_k)$ but $i_k \notin I(\bar{x})$. As $\{i_k\} \subset \{1, 2, \ldots, m\}$ for every $k \in \mathbb{N}$, by the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, suppose that $i_k \to \bar{i}$. Because $I(x_k)$ is closed, $\bar{i} \in I(x_k)$, which implies $\varphi_{\bar{i}}(x_k) = \varphi(x_k)$. By Theorem 2.69, the functions are continuous on $\mathbb{R}^n$. Thus $\varphi_{\bar{i}}(\bar{x}) = \varphi(\bar{x})$, that is, $\bar{i} \in I(\bar{x})$. Because $i_k \notin I(\bar{x})$ for every $k \in \mathbb{N}$, which implies that $\bar{i} \notin I(\bar{x})$, which is a contradiction, thereby establishing the claim.

Now consider $\{\lambda_k\} \subset \mathbb{R}_+$ such that $\lambda_k \to 0$. Observe that

$$\varphi_{i_k}(\bar{x} + \lambda_k d) = \varphi(\bar{x} + \lambda_k d), \ \forall \ i_k \in I(\bar{x} + \lambda_k d).$$

For sufficiently large $k \in \mathbb{N}$, we may choose $i_k \in I(\bar{x})$. Because $I(\bar{x})$ is closed, which along with the Bolzano–Weierstrass Theorem implies that i_k has a convergent subsequence. Without loss of generality, assume that $\{i_k\}$ converges to $\bar{i} \in I(\bar{x})$. We may choose $i_k = \bar{i}$. Therefore,

$$\lim_{k \to \infty} \frac{\varphi(\bar{x} + \lambda_k d) - \varphi(\bar{x})}{\lambda_k} \leq \max_{i \in I(\bar{x})} \langle \nabla\varphi_i(\bar{x}), d \rangle.$$

By Theorem 2.76, the directional derivative of a convex function always exists and therefore,

$$\varphi'(\bar{x}, d) = \lim_{\lambda \to 0} \frac{\varphi(\bar{x} + \lambda d) - \varphi(\bar{x})}{\lambda} \leq \max_{i \in I(\bar{x})} \langle \nabla \varphi_i(\bar{x}), d \rangle,$$

hence completing the proof. □

We are now in a position to obtain the Subdifferential Max-Function Rule as an application of the Chain Rule, Theorem 2.94, and the result Theorem 2.95 established above.

Theorem 2.96 *(Max-Function Rule) Consider convex functions* $\phi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, *and let* $\phi(x) = \max\{\phi_1(x), \phi_2(x), \ldots, \phi_m(x)\}$. *Then*

$$\partial \phi(\bar{x}) = co \bigcup_{i \in I(\bar{x})}, \partial \phi_i(\bar{x}),$$

where $I(\bar{x})$ *denotes the active index set.*

Proof. In the discussion preceding Theorem 2.95, we observed that $\phi = \varphi \circ \Phi$, where

$$\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_m(x)) \quad \text{and} \quad \varphi(y) = \max\{y_1, y_2, \ldots, y_m\}$$

with $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. By Theorem 2.95,

$$\varphi'(y, d) = \max_{i \in I'(y)} \{\langle e_i, d \rangle\},$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^m$ with 1 at the i-th place and $I'(y) = \{i \in \{1, 2, \ldots, m\} : y_i = \varphi(y)\}$. It is obvious that $\varphi'(y, .)$ is a support function of $\{e_i \in \mathbb{R}^m : i \in I'(y)\}$ and by Proposition 2.61, it is also the support function of co $\{e_i \in \mathbb{R}^m : i \in I'(y)\}$. Therefore, by Theorem 2.79,

$$\partial \varphi(y) = co\ \{e_i \in \mathbb{R}^m : i \in I'(y)\},$$

that is,

$$\partial \varphi(y) = \{(\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m : \mu_i \geq 0,\ i \in I'(y),$$
$$\mu_i = 0,\ i \notin I'(y),\ \sum_{i=1}^{m} \mu_i = 1\}.$$

Thus,

$$\partial \varphi(\Phi(\bar{x})) = \{(\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m : \mu_i \geq 0,\ i \in I'(\Phi(\bar{x})),$$
$$\mu_i = 0,\ i \notin I'(\Phi(\bar{x})),\ \sum_{i=1}^{m} \mu_i = 1\}.$$

As $I'(\Phi(\bar{x})) = I(\bar{x})$, the above condition reduces to

$$\partial\varphi(\Phi(\bar{x})) = \{(\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m : \mu_i \geq 0,\ i \in I(\bar{x}),\\ \mu_i = 0,\ i \notin I(\bar{x}),\ \sum_{i=1}^{m} \mu_i = 1\}.$$

As φ is a nondecreasing convex function, applying Theorem 2.94 to $\phi = \varphi \circ \Phi$ yields

$$\begin{aligned}
\partial\phi(\bar{x}) &= \{\sum_{i=1}^{m} \mu_i \xi_i : (\mu_1, \mu_2, \ldots, \mu_m) \in \partial\phi(\Phi(\bar{x})),\\
&\qquad \xi_i \in \partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m\}\\
&= \{\sum_{i=1}^{m} \mu_i \xi_i : \mu_i \geq 0,\ i \in I(\bar{x}),\ \mu_i = 0,\ i \notin I(\bar{x}), \sum_{i=1}^{m} \mu_i = 1,\\
&\qquad \xi_i \in \partial\phi_i(\bar{x}),\ i = 1, 2, \ldots, m\},
\end{aligned}$$

which implies

$$\partial\phi(\bar{x}) = co \bigcup_{i \in I(\bar{x})} \partial\phi_i(\bar{x}),$$

thereby leading to the desired result. □

Observe that in the Max-Function Rule above, the maximum was over a finite index set. Now if the index set is a compact set, need not be finite, then what will the subdifferential for sup-function be? This aspect was looked into by Valadier [109] and thus is also referred to as the *Valadier Formula*. Below we present the Valadier Formula from Ruszczyński [102].

Theorem 2.97 *Consider a function*

$$\Phi(x) = \sup_{y \in Y} \phi(x, y),$$

where $\phi : \mathbb{R}^n \times Y \to \bar{\mathbb{R}}$. Let $\bar{x} \in dom\ \Phi$ such that
(i) $\phi(., y)$ is convex for every $y \in Y$,
(ii) $\phi(x, .)$ is usc for every $x \in \mathbb{R}^n$,
(iii) $Y \subset \mathbb{R}^m$ is compact.
Furthermore, if $\phi(., y)$ is continuous at $\bar{x}$ for every $y \in Y$, then

$$\partial\Phi(\bar{x}) = co \bigcup_{y \in \hat{Y}(\bar{x})} \partial_x \phi(\bar{x}, y),$$

where $\hat{Y}(\bar{x}) = \{y \in Y : \phi(\bar{x}, y) = \Phi(\bar{x})\}$ and $\partial_x \phi$ denotes the subdifferential with respect to x.

Proof. Observe that (ii) and (iii) ensure that $\hat{Y}(\bar{x})$ is nonempty and compact. Suppose that $\xi \in \partial_x \phi(\bar{x}, \bar{y})$ for some $\bar{y} \in \hat{Y}(\bar{x})$. By Definition 2.77 of the subdifferential,

$$\phi(x, \bar{y}) - \phi(\bar{x}, \bar{y}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\, x \in \mathbb{R}^n.$$

As $\bar{y} \in \hat{Y}(\bar{x})$, $\phi(\bar{x}, \bar{y}) = \Phi(\bar{x})$. Therefore, the above inequality leads to

$$\Phi(x) - \Phi(\bar{x}) = \sup_{y \in Y} \phi(x, y) - \phi(\bar{x}, \bar{y}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\, x \in \mathbb{R}^n,$$

thus implying that $\xi \in \partial \Phi(\bar{x})$. Because $\bar{y} \in \hat{Y}(\bar{x})$ and $\xi \in \partial_x \phi(\bar{x}, \bar{y})$ were arbitrary,

$$\partial \Phi(\bar{x}) \supset \partial_x \phi(\bar{x}, y), \ \forall\, y \in \hat{Y}(\bar{x}).$$

Because $\partial \Phi(\bar{x})$ is convex, the preceding inclusion yields

$$\partial \Phi(\bar{x}) \supset co \bigcup_{y \in \hat{Y}(\bar{x})} \partial_x \phi(\bar{x}, y).$$

To establish the converse, we will prove the reverse inclusion in the above relation. Because $\partial \Phi(\bar{x})$ is closed, we first show that $\bigcup_{y \in \hat{Y}(\bar{x})} \partial_x \phi(\bar{x}, y)$ is closed. Suppose that $\xi_k \in \partial_x \phi(\bar{x}, y_k)$, where $\{y_k\} \subset \hat{Y}(\bar{x})$ such that $\xi_k \to \bar{\xi}$. Because $\hat{Y}(\bar{x})$ is compact and hence closed, $\{y_k\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, suppose $y_k \to \bar{y}$, which by the closedness of $\hat{Y}(\bar{x})$ implies that $\bar{y} \in \hat{Y}(\bar{x})$. By the definition of subdifferential along with the facts that $\{y_k\} \subset \hat{Y}(\bar{x})$ and $\bar{y} \in \hat{Y}(\bar{x})$, that is, $\phi(\bar{x}, y_k) = \Phi(\bar{x}) = \phi(\bar{x}, \bar{y})$ imply that for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \phi(x, y_k) &\geq \phi(\bar{x}, y_k) + \langle \xi_k, x - \bar{x} \rangle \\ &= \phi(\bar{x}, \bar{y}) + \langle \xi_k, x - \bar{x} \rangle, \ \forall\, k \in \mathbb{N}. \end{aligned}$$

Taking the limit supremum as $k \to +\infty$, which by the upper semicontinuity of $\phi(x, .)$ over $\mathbb{R}^n$ leads to

$$\begin{aligned} \phi(x, \bar{y}) \geq \limsup_{k \to \infty} \phi(x, y_k) &\geq \phi(\bar{x}, \bar{y}) + \limsup_{k \to \infty} \langle \xi_k, x - \bar{x} \rangle \\ &= \phi(\bar{x}, \bar{y}) + \langle \bar{\xi}, x - \bar{x} \rangle, \ \forall\, x \in \mathbb{R}^n, \end{aligned}$$

thereby yielding that $\bar{\xi} \in \partial_x \phi(\bar{x}, \bar{y})$. Hence, $\bigcup_{y \in \hat{Y}(\bar{x})} \partial_x \phi(\bar{x}, y)$ is closed.

Now let us assume on the contrary that

$$\partial \Phi(\bar{x}) \not\subset co \bigcup_{y \in \hat{Y}(\bar{x})} \partial_x \phi(\bar{x}, y),$$

that is, there exists $\bar{\xi} \in \partial\Phi(\bar{x})$ such that

$$\bar{\xi} \notin co \bigcup_{y \in \hat{Y}(\bar{x})} \partial_x \phi(\bar{x}, y).$$

As $co \bigcup_{y \in \hat{Y}(\bar{x})} \partial_x \phi(\bar{x}, y)$ is a closed convex set, by the Strict Separation Theorem, Theorem 2.26 (iii), there exists $d \in \mathbb{R}^n$ with $d \neq 0$ such that

$$\langle \bar{\xi}, d \rangle > \langle \xi, d \rangle, \ \forall\, \xi \in \partial_x \phi(\bar{x}, y), \ \forall\, y \in \hat{Y}(\bar{x}). \tag{2.25}$$

Consider a sequence $\{\lambda_k\} \subset \mathbb{R}_+$ such that $\lambda_k \to 0$. As Φ is convex, by the definition of subdifferential,

$$\frac{\Phi(\bar{x} + \lambda_k d) - \Phi(\bar{x})}{\lambda_k} \geq \langle \bar{\xi}, d \rangle. \tag{2.26}$$

For $k \in \mathbb{N}$, define the set

$$Y_k = \left\{ y \in Y : \frac{\phi(\bar{x} + \lambda_k d, y) - \Phi(\bar{x})}{\lambda_k} \geq \langle \bar{\xi}, d \rangle \right\}.$$

We claim that Y_k is compact and nonempty. Consider $\{y_r\} \in Y_k$ such that $y_r \to \hat{y}$. Because $y_r \in Y_k$,

$$\frac{\phi(\bar{x} + \lambda_k d, y_r) - \Phi(\bar{x})}{\lambda_k} \geq \langle \bar{\xi}, d \rangle.$$

Taking the limit supremum as $r \to +\infty$, which along with the upper semicontinuity of $\phi(x, .)$ for every $x \in \mathbb{R}^n$ implies that

$$\frac{\phi(\bar{x} + \lambda_k d, \hat{y}) - \Phi(\bar{x})}{\lambda_k} \geq \langle \bar{\xi}, d \rangle.$$

Thus, $\hat{y} \in Y_k$ and hence Y_k is closed for every $k \in \mathbb{N}$. As $Y_k \subset Y$ and Y is compact, Y_k is closed and bounded and thus compact. Also by the upper semicontinuity of $\phi(x, .)$ for every $x \in \mathbb{R}^n$, $\hat{Y}(\bar{x} + \lambda_k d)$ is nonempty. From the inequality (2.26) and the definition of the set Y_k, $\hat{Y}(\bar{x} + \lambda_k d) \subset Y_k$ and hence Y_k is nonempty. For every $y \in Y$, consider the expression

$$\frac{\phi(\bar{x} + \lambda d, y) - \Phi(\bar{x})}{\lambda} = \frac{\phi(\bar{x} + \lambda d, y) - \phi(\bar{x}, y)}{\lambda} + \frac{\phi(\bar{x}, y) - \Phi(\bar{x})}{\lambda}. \tag{2.27}$$

From the discussion preceding Theorem 2.76 on directional derivatives, the first term on the right-hand side of the above expression is a nondecreasing function of λ, that is,

$$\frac{\phi(\bar{x} + \lambda_1 d, y) - \phi(\bar{x}, y)}{\lambda_1} \leq \frac{\phi(\bar{x} + \lambda_2 d, y) - \phi(\bar{x}, y)}{\lambda_2}, \ \forall\, 0 < \lambda_1 \leq \lambda_2. \tag{2.28}$$

Also, as $\Phi(\bar{x}) \geq \phi(\bar{x}, y)$ for every $y \in Y$,

$$\frac{\phi(\bar{x}, y) - \Phi(\bar{x})}{\lambda_1} \leq \frac{\phi(\bar{x}, y) - \Phi(\bar{x})}{\lambda_2}, \ \forall\ 0 < \lambda_1 \leq \lambda_2, \tag{2.29}$$

which implies that the second term is also nondecreasing in λ. Thus, combining the conditions (2.28) and (2.29), the expression

$$\frac{\phi(\bar{x} + \lambda_1 d, y) - \Phi(\bar{x})}{\lambda_1} \leq \frac{\phi(\bar{x} + \lambda_2 d, y) - \Phi(\bar{x})}{\lambda_2}, \ \forall\ 0 < \lambda_1 \leq \lambda_2,$$

that is, the expression (2.27) is nondecreasing in λ. From the above inequality, it is obvious that $Y_1 \subset Y_2$ for every $0 < \lambda_1 \leq \lambda_2$. As $\{\lambda_k\}$ is a decreasing sequence,

$$Y_1 \supset Y_2 \supset Y_3 \supset \ldots.$$

As for every $k \in \mathbb{N}$, Y_k is compact and nonempty, there exists $\tilde{y} \in Y_k$ for all $k \in \mathbb{N}$. Therefore,

$$\frac{\phi(\bar{x} + \lambda_k d, \tilde{y}) - \Phi(\bar{x})}{\lambda_k} \geq \langle \bar{\xi}, d \rangle, \ \forall\ k \in \mathbb{N},$$

which implies that the term on the left-hand side is bounded below for every $k \in \mathbb{N}$. By the continuity of $\phi(., y)$ at $\bar{x}$ for every $y \in Y$, $\phi(\bar{x} + \lambda_k d, \tilde{y}) \to \phi(\bar{x}, \tilde{y})$ which along with the lower boundedness yields that $\tilde{y} \in \hat{Y}(\bar{x})$, that is, $\Phi(\bar{x}) = \phi(\bar{x}, \tilde{y})$. Taking the limit as $k \to +\infty$ in the above inequality along with Definition 2.74 of the directional derivative implies that

$$\phi'((\bar{x}, \tilde{y}), d) \geq \langle \bar{\xi}, d \rangle.$$

As $\phi(., \tilde{y})$ is continuous at $\bar{x}$, any neighborhood of $\bar{x}$ is contained in *dom* $\phi(., \tilde{y})$. Thus, $\bar{x} \in$ *int dom* $\phi(., \tilde{y})$, which by Theorem 2.79 implies that $\phi'(\bar{x}, \tilde{y})$ is the support function of $\partial_x \phi(\bar{x}, \tilde{y})$. Also, by Proposition 2.82, $\partial_x \phi(\bar{x}, \tilde{y})$ is compact. Therefore, there exists $\xi \in \partial_x \phi(\bar{x}, \tilde{y})$ such that the above inequality becomes

$$\langle \xi, d \rangle \geq \langle \bar{\xi}, d \rangle,$$

thereby contradicting the inequality (2.25) as $\tilde{y} \in \hat{Y}(\bar{x})$, hence completing the proof. $\square$

From Proposition 2.56, another operation on the convex functions that leads to a convex function is the inf-convolution. We end this section of the subdifferential calculus rules by presenting the subdifferential rule for the inf-convolution for a particular case from Lucchetti [79].

Theorem 2.98 *Consider proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$. Let $\bar{x}, x_1, x_2 \in \mathbb{R}^n$ be such that*

$$x_1 + x_2 = \bar{x} \quad \text{ and } \quad (\phi_1 \,\square\, \phi_2)(\bar{x}) = \phi_1(x_1) + \phi_2(x_2).$$

Then

$$\partial(\phi_1 \,\square\, \phi_2)(\bar{x}) = \partial\phi_1(x_1) \cap \partial\phi_2(x_2).$$

Proof. Suppose that $\xi \in \partial\phi_1(x_1) \cap \partial\phi_2(x_2)$. By Definition 2.77 of the subdifferential, for $i = 1, 2$,

$$\phi_i(y_i) - \phi_i(x_i) \geq \langle \xi, y_i - x_i \rangle, \; \forall \; y_i \in \mathbb{R}^n.$$

Define $y_1 + y_2 = \bar{y}$. The above inequality along with the given hypothesis leads to

$$\phi_1(y_1) + \phi_2(y_2) \geq (\phi_1 \,\square\, \phi_2)(\bar{x}) + \langle \xi, \bar{y} - \bar{x} \rangle, \; \forall \; y_1, y_2 \in \mathbb{R}^n.$$

Taking the infimum over y_1 and y_2 satisfying $y_1 + y_2 = \bar{y}$ in the above inequality, which by Definition 2.54 of the inf-convolution yields

$$(\phi_1 \,\square\, \phi_2)(\bar{y}) \geq (\phi_1 \,\square\, \phi_2)(\bar{x}) + \langle \xi, \bar{y} - \bar{x} \rangle.$$

As $\bar{y} \in \mathbb{R}^n$ was arbitrary, the above inequality holds for every $\bar{y} \in \mathbb{R}^n$. Thus, $\xi \in \partial(\phi_1 \,\square\, \phi_2)(\bar{x})$. Because $\xi \in \partial\phi_1(x_1) \cap \partial\phi_2(x_2)$ was arbitrary, $\partial\phi_1(x_1) \cap \partial\phi_2(x_2) \subset \partial(\phi_1 \,\square\, \phi_2)(\bar{x})$.

Conversely, suppose that $\xi \in \partial(\phi_1 \,\square\, \phi_2)(\bar{x})$. Therefore,

$$\partial(\phi_1 \,\square\, \phi_2)(\bar{y}) \geq \phi_1(x_1) + \phi_2(x_2) + \langle \xi, \bar{y} - \bar{x} \rangle, \; \forall \; \bar{y} \in \mathbb{R}^n.$$

As the above inequality holds for any $\bar{y} \in \mathbb{R}^n$, then $\bar{y} = x + x_2$ for some $x \in \mathbb{R}^n$. Substituting in the above inequality along with the definition of the inf-convolution yields

$$\phi_1(x) + \phi_2(x_2) \geq \phi_1(x_1) + \phi_2(x_2) + \langle \xi, (x + x_2) - (x_1 + x_2) \rangle, \; \forall \; x \in \mathbb{R}^n,$$

which implies that

$$\phi_1(x) \geq \phi_1(x_1) + \langle \xi, x - x_1 \rangle, \; \forall \; x \in \mathbb{R}^n.$$

Therefore, $\xi \in \partial\phi_1(x_1)$. Similarly, it can be shown that $\xi \in \partial\phi_2(x_2)$ and hence $\xi \in \partial\phi_1(x_1) \cap \phi_2(x_2)$. Because $\xi \in \partial(\phi_1 \,\square\, \phi_2)(\bar{x})$ was arbitrary, $\partial(\phi_1 \,\square\, \phi_2)(\bar{x}) \subset \partial\phi_1(x_1) \cap \phi_2(x_2)$, thereby establishing the result. $\square$

2.5 Conjugate Functions

All this background on convexity, convex sets as well as convex functions, the subdifferentials and their calculus form a backbone for the study of convex optimization theory. Optimization problems appear not only in the specialized fields of engineering, management sciences, and finance, but also in some simple real-life problems. For instance, if the cost of manufacturing $x_1, x_2, \ldots, x_n$ quantities of n goods is given by $\phi(x)$ and the price of selling these goods is $\xi_1, \xi_2, \ldots, \xi_n$, respectively, then the manufacturer would like to choose the quantities $x_1, x_2, \ldots, x_n$ in such a way that it leads to maximum profit, where the profit function is given by the affine function $\{\langle \xi, x\rangle - \phi(x)\}$. Theoretically, this problem had been expressed using the *conjugate functions* of ϕ introduced by Fenchel [45], which forms a class of convex functions. As we will see in a short while, these conjugate functions are related to not only the subdifferential for a convex function by the Fenchel–Young inequality but also to the ε-subdifferential via its epigraph. For convex functions, the very idea of conjugacy seems to derive from the fact that a proper lsc convex function is a pointwise supremum of affine functions majorized by it. But before moving on to this result, we present the following lemma from Lucchetti [79].

Lemma 2.99 *Consider a proper lsc convex function $\phi : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$. Let $\bar{x} \in dom\ \phi$ and $\gamma \in \mathbb{R}$ such that $\phi(\bar{x}) > \gamma$. Then there exists $(a, b) \in \mathbb{R}^n \times \mathbb{R}$ such that the affine function $h(x) = \langle a, x\rangle + b$ satisfies*

$$\phi(x) \geq h(x),\ \forall\ x \in \mathbb{R}^n \quad \textit{and} \quad h(\bar{x}) > \gamma.$$

Proof. As ϕ is an lsc convex function, by Theorem 1.9 and Proposition 2.48, *epi* ϕ is a closed convex set in $\mathbb{R}^n \times \mathbb{R}$. From the given hypothesis, it is obvious that $(\bar{x}, \gamma) \notin epi\ \phi$. By the Strict Separation Theorem, Theorem 2.26 (iii), there exist $(a, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ with $(a, \lambda) \neq (0, 0)$ and $b \in \mathbb{R}$ such that

$$\langle a, x\rangle + \lambda\alpha \geq b > \langle a, \bar{x}\rangle + \lambda\gamma,\ \forall\ (x, \alpha) \in epi\ \phi. \tag{2.30}$$

In particular, taking $(\bar{x}, \phi(\bar{x})) \in epi\ \phi$, the above inequality reduces to

$$\lambda(\phi(\bar{x}) - \gamma) > 0.$$

As $\phi(\bar{x}) > \gamma$, the above strict inequality leads to $\lambda > 0$. Again, taking $(x, \phi(x)) \in epi\ \phi$ in the condition (2.30) yields

$$\phi(x) \geq h(x),\ \forall\ x \in dom\ \phi \quad \text{and} \quad h(\bar{x}) > \gamma,$$

where $h(x) = \langle \frac{-a}{\lambda}, x\rangle + \frac{b}{\lambda}$. Observe that for $x \notin dom\ \phi$, the first inequality holds trivially, that is,

$$\phi(x) \geq h(x),\ \forall\ x \in \mathbb{R}^n,$$

thereby establishing the result. □

Now we present the main result, the proof of which is from Lucchetti [79].

Theorem 2.100 *A proper lsc convex function $\phi : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ can be expressed as a pointwise supremum of the collection of all affine functions majorized by it, that is, for every $x \in \mathbb{R}^n$,*

$$\phi(x) = \sup\{h(x) : \phi(x) \geq h(x),\ h(x) = \langle a, x\rangle + b,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}\}.$$

Proof. Define the function $\Phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ as

$$\Phi(x) = \sup\{h(x) : \phi(x) \geq h(x),\ h(x) = \langle a, x\rangle + b,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}\}.$$

Because Φ is a pointwise supremum of affine functions, it is an lsc convex function. Also, as $\phi(x) \geq h(x)$, $\phi(x) \geq \Phi(x)$ for every $x \in \mathbb{R}^n$, which implies that *epi* ϕ is contained in the intersection of epigraph of the affine functions h, *epi* h, which are majorized by ϕ, that is, $\phi(x) \geq h(x)$ for every $x \in \mathbb{R}^n$. Therefore, to complete the proof, it is sufficient to prove that for $(\bar{x}, \gamma) \notin$ *epi* ϕ, there exists an affine function h such that $h(\bar{x}) > \gamma$. By Lemma 2.99, for $\bar{x} \in$ *dom* ϕ such an h exists.

Now suppose that $\bar{x} \notin$ *dom* ϕ. As $(\bar{x}, \gamma) \notin$ *epi* ϕ, working along the lines of the proof of Lemma 2.99, there exist $(a, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ with $(a, \lambda) \neq (0, 0)$ and $b \in \mathbb{R}$ such that

$$\langle a, x\rangle + \lambda\alpha \geq b > \langle a, \bar{x}\rangle + \lambda\gamma,\ \forall\ (x, \alpha) \in epi\ \phi.$$

If $\lambda \neq 0$, the affine function h exists as in the lemma. If $\lambda = 0$, the above inequality reduces to

$$\langle a, x\rangle \geq b > \langle a, \bar{x}\rangle,\ \forall\ x \in dom\ \phi.$$

From the above condition,

$$h(x) \leq 0,\ \forall\ x \in dom\ \phi \quad \text{and} \quad h(\bar{x}) > 0,$$

where $h(x) = \langle -a, x\rangle + b$. As a consequence of Lemma 2.99, it is obvious that a proper lsc convex function has at least one affine function majorized by it. Therefore, ϕ has an affine function, say $\bar{h}$, majorized by it, that is $\phi(x) \geq \bar{h}(x)$ for every $x \in \mathbb{R}^n$. Now for any $\mu > 0$,

$$\phi(x) \geq h(x) + \mu\bar{h}(x),\ \forall\ x \in dom\ \phi.$$

The above inequality holds trivially for $x \notin$ *dom* ϕ. Thus,

$$\phi(x) \geq (h + \mu\bar{h})(x),\ \forall\ x \in \mathbb{R}^n,$$

which implies the affine function $(h + \mu\bar{h})$ is majorized by ϕ. As $h(\bar{x}) > 0$, for μ sufficiently large, $(h + \mu\bar{h})(\bar{x}) > \gamma$, thereby establishing the result. $\square$

Denote the set of all affine functions by $\mathcal{H}$. Consider the *support set* of ϕ denoted by *supp*$(\phi, \mathcal{H})$, which is the collection of all affine functions majorized by ϕ, that is,

$$supp(\phi, \mathcal{H}) = \{h \in \mathcal{H} : h(x) \leq \phi(x),\ \forall\ x \in \mathbb{R}^n\}.$$

An affine function $h \in \mathcal{H}$ is the *affine support* of ϕ if

$$h(x) \leq \phi(x),\ \forall\ x \in \mathbb{R}^n \quad \text{and} \quad h(\bar{x}) = \phi(\bar{x}),\ \text{for some } \bar{x} \in \mathbb{R}^n.$$

Consider $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in dom\ \phi$ such that $\partial\phi(\bar{x})$ is nonempty. Then for any $\xi \in \partial\phi(\bar{x})$, by Definition 2.77,

$$\phi(x) \geq \langle \xi, x \rangle + (\phi(\bar{x}) - \langle \xi, \bar{x} \rangle), \ \forall\ x \in \mathbb{R}^n. \tag{2.31}$$

Define an affine function $h : \mathbb{R}^n \to \mathbb{R}$ given by

$$h(x) = \langle \xi, x \rangle + (\phi(\bar{x}) - \langle \xi, \bar{x} \rangle). \tag{2.32}$$

Combining (2.31) and (2.32),

$$h(x) \leq \phi(x), \ \forall\ x \in \mathbb{R}^n \quad \text{and} \quad h(\bar{x}) = \phi(\bar{x}),$$

thereby implying that $h \in \mathcal{H}$ is an affine support of ϕ. Therefore, if $\partial\phi$ is nonempty, then there exists an affine support to it.

Now consider a set $\Phi^* \subset \mathbb{R}^n$ defined as

$$\Phi^* = \{(\bar{\xi}, \bar{\alpha}) \in \mathbb{R}^n \times \mathbb{R} : h(x) = \langle \bar{\xi}, x \rangle - \bar{\alpha} \leq \phi(x)\},$$

which implies for every $x \in \mathbb{R}^n$, $h(x) \leq \phi(x)$. Therefore,

$$\bar{\alpha} \geq \sup_{x \in \mathbb{R}^n} \{\langle \bar{\xi}, x \rangle - \phi(x)\},$$

which implies Φ^* can be considered the epigraph of the function ϕ^*, which is the conjugate of ϕ. We formally introduce the notion of conjugate below.

Definition 2.101 Consider a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. The *conjugate* of ϕ, $\phi^* : \mathbb{R}^n \to \bar{\mathbb{R}}$, is defined as

$$\phi^*(\xi) = \sup_{x \in \mathbb{R}^n} \{\langle \xi, x \rangle - \phi(x)\}.$$

Observe that $\Phi^* = epi\ \phi^*$, as discussed above. The *biconjugate* of ϕ, ϕ^{**}, is the conjugate of ϕ^*, that is,

$$\phi^{**}(x) = \sup_{\xi \in \mathbb{R}^n} \{\langle \xi, x \rangle - \phi^*(\xi)\}.$$

Consider a set $F \subset \mathbb{R}^n$. The conjugate of the indicator function to the set F is

$$\delta_F^*(\xi) = \sup_{x \in \mathbb{R}^n} \{\langle \xi, x \rangle - \delta_F(x)\} = \sup_{x \in F} \langle \xi, x \rangle,$$

which is actually the support function to the set F. Therefore, $\delta_F^* = \sigma_F$ for any set F.

Observe that the definitions of conjugate and biconjugate functions are given for any arbitrary function. Below we present some properties of conjugate functions.

Proposition 2.102 *For any function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$, the conjugate function ϕ^* is always lsc convex. In addition, if ϕ is proper convex, then ϕ^* is also a proper convex function.*

Proof. Consider any $\xi_1, \xi_2 \in \mathbb{R}^n$. Then for every $\lambda \in [0,1]$,

$$\begin{aligned}\phi^*((1-\lambda)\xi_1 + \lambda\xi_2) &= \sup_{x\in\mathbb{R}^n} \{\langle((1-\lambda)\xi_1 + \lambda\xi_2), x\rangle - \phi(x)\} \\ &= \sup_{x\in\mathbb{R}^n} \{(1-\lambda)(\langle\xi_1, x\rangle - \phi(x)) + \lambda(\langle\xi_2, x\rangle - \phi(x))\},\end{aligned}$$

which by Proposition 1.7 leads to

$$\begin{aligned}\phi^*((1-\lambda)\xi_1 + \lambda\xi_2) &\le (1-\lambda) \sup_{x\in\mathbb{R}^n} \{\langle\xi_1, x\rangle - \phi(x)\} + \\ &\qquad\qquad \lambda \sup_{x\in\mathbb{R}^n} \{\langle\xi_2, x\rangle - \phi(x)\} \\ &= (1-\lambda)\phi^*(\xi_1) + \lambda\phi^*(\xi_2), \ \forall\ \lambda \in [0,1].\end{aligned}$$

Because ξ_1 and ξ_2 are arbitrary, from the above inequality ϕ^* is convex. Also, as ϕ^* is a pointwise supremum of affine functions $\langle x, .\rangle - \phi(x)$, it is lsc.

As ϕ is a proper convex function, *dom* ϕ is a nonempty convex set in $\mathbb{R}^n$, which by Proposition 2.14 (i) implies that *ri dom* ϕ is nonempty. Also, by Proposition 2.82, for any $\bar{x} \in$ *ri dom* ϕ, $\partial\phi(\bar{x})$ is nonempty. Suppose that $\xi \in \partial\phi(\bar{x})$, which by Definition 2.77 of the subdifferential implies that

$$\langle\xi, \bar{x}\rangle - \phi(\bar{x}) \ge \langle\xi, x\rangle - \phi(x), \ \forall\ x \in \mathbb{R}^n,$$

which along with the definition of conjugate ϕ^* implies that

$$\langle\xi, \bar{x}\rangle - \phi(\bar{x}) = \phi^*(\xi).$$

As $\phi(\bar{x})$ is finite, $\phi^*(\xi)$ is also finite, that is, $\xi \in$ *dom* ϕ^*. Also, by the properness of ϕ and the definition of ϕ^*, it is obvious that $\phi^*(\xi) > -\infty$ for every $x \in \mathbb{R}^n$, thereby showing that ϕ^* is proper convex function. □

Observe that ϕ^* is lsc convex irrespective of the nature of ϕ but for ϕ^* to be proper, we need ϕ to be a proper convex function. Simply assuming ϕ to be proper need not imply that ϕ^* is proper. For instance, consider $\phi(x) = -x^2$, which is a nonconvex proper function. Then $\phi^* \equiv +\infty$ and hence not proper. Next we state some conjugate rules that can be proved directly using the definition of conjugate functions.

Proposition 2.103 *Consider a function $\bar{\phi}$, $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$.*

(i) If $\bar{\phi} \le \phi$, then $\bar{\phi}^* \ge \phi^*$.

(ii) If $\bar{\phi}(x) = \phi(x) + c$, $\bar{\phi}^*(\xi) = \phi^*(\xi) - c$.

(iii) If $\bar{\phi}(x) = \lambda\phi(x)$ for $\lambda > 0$, $\bar{\phi}^*(\xi) = \lambda\phi^*(\xi/\lambda)$.

(iv) For every x and ξ in $\mathbb{R}^n$,

$$\phi^*(\xi) + \phi(x) \geq \langle \xi, x \rangle.$$

This is known as the Fenchel–Young Inequality*. Equivalently,*

$$\phi^{**}(x) \leq \phi(x), \ \forall \ x \in \mathbb{R}^n.$$

The readers are urged to verify these properties simply using Definition 2.101 of conjugate and biconjugate functions.

As we discussed in Theorem 2.100 that a convex function is pointwise supremum of affine functions, the biconjugate of the function plays an important role in this respect. Below we present a result that relates the biconjugate with the support set. The proof is along the lines of Hiriart-Urruty and Lemaréchal [63].

Theorem 2.104 *Consider a proper function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then ϕ^{**} is the pointwise supremum of all affine functions majorized by ϕ, that is,*

$$\phi^{**}(\bar{x}) = \sup_{h \in supp(\phi, \mathcal{H})} h(\bar{x}).$$

More precisely, $\phi^{**} = cl\ co\ \phi$.

Proof. An affine function h is majorized by ϕ, that is, $h(x) \leq \phi(x)$ for every $x \in \mathbb{R}^n$. Because an affine function is expressed as $h(x) = \langle \xi, x \rangle - \alpha$ for some $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$\langle \xi, x \rangle - \alpha \leq \phi(x), \ \forall \ x \in \mathbb{R}^n.$$

Therefore, by Definition 2.101 of the conjugate function, $\phi^*(\xi) \leq \alpha$, which implies $\xi \in dom\ \phi^*$. Then for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
\sup_{h \in supp(\phi, \mathcal{H})} h(x) &= \sup_{\xi \in dom\ \phi^*,\ \phi^*(\xi) \leq \alpha} \{\langle \xi, x \rangle - \alpha\} \\
&= \sup_{\xi \in dom\ \phi^*} \{\langle \xi, x \rangle - \phi^*(\xi)\} \\
&= \sup_{\xi \in \mathbb{R}^n} \{\langle \xi, x \rangle - \phi^*(\xi)\} = \phi^{**}(x),
\end{aligned}$$

thereby yielding the desired result. From Definition 2.57 of the closed convex function, $\phi^{**} = cl\ co\ \phi$, as desired. □

Combining Theorems 2.100 and 2.104 we have the following result for a proper lsc convex function.

Theorem 2.105 *Consider a proper lsc convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then $\phi^{**} = \phi$.*

Observe that the above theorem holds when the function is lsc. What if ϕ is only proper convex but not lsc, then how is one supposed to relate the function ϕ to its biconjugate ϕ^{**}? The next result from Attouch, Buttazzo, and Michaille [3] looks into this aspect.

Proposition 2.106 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Assume that ϕ admits a continuous affine minorant. Then $\phi^{**} = cl\ \phi$. Consequently, ϕ is lsc at $\bar{x} \in \mathbb{R}^n$ if and only if $\phi(\bar{x}) = \phi^{**}(\bar{x})$.*

Proof. By the Fenchel–Young inequality, Proposition 2.103 (iv),

$$\phi(x) \geq \langle \xi, x \rangle - \phi^*(\xi), \ \forall\, x \in \mathbb{R}^n,$$

which implies that $h(x) = \langle \xi, x \rangle - \phi^*(\xi)$ belongs to $supp(\phi, \mathcal{H})$. By Definition 2.101 of the biconjugate function,

$$\phi^{**}(x) = \sup_{\xi \in \mathbb{R}^n} \{\langle \xi, x \rangle - \phi^*(\xi)\},$$

which leads to ϕ^{**} being the upper envelope of the continuous affine minorants of ϕ. Applying Proposition 2.102 to ϕ^*, ϕ^{**} is a proper lsc convex function and thus,

$$\phi^{**} \leq cl\ \phi \leq \phi.$$

This inequality along with Proposition 2.103 (i) leads to

$$(\phi^{**})^{**} \leq (cl\ \phi)^{**} \leq \phi^{**}.$$

As ϕ^{**} and $cl\ \phi$ are both proper lsc convex functions, by Theorem 2.105,

$$(\phi^{**})^{**} = \phi^{**} \quad \text{ and } \quad (cl\ \phi)^{**} = cl\ \phi,$$

thereby reducing the preceding inequality to

$$\phi^{**} \leq cl\ \phi)^{**} \leq \phi^{**}.$$

Hence, $\phi^{**} = cl\ \phi$, thereby establishing the first part of the result.

From Chapter 1, we know that closure of a function ϕ is defined as

$$cl\ \phi(\bar{x}) = \liminf_{x \to \bar{x}} \phi(x),$$

which is the same as $\phi(\bar{x})$ if ϕ is lsc at $\bar{x}$ by Definition 1.4, thereby yielding $\phi(\bar{x}) = cl\ \phi(\bar{x})$. Consequently, by the first part, the lower semicontinuity of ϕ at $\bar{x}$ is equivalent to $\phi(\bar{x}) = \phi^{**}(\bar{x})$, thereby completing the proof. □

With all the preceding results, and discussions on the properties of the conjugates and biconjugates, we now move on to see how the conjugates of the function operations are defined. More precisely, if given some functions and we perform some operation on them, like the sum operation or the supremum operation, then how are their conjugates related to the conjugates of the given functions? In the next result from Hiriart-Urruty and Lemaréchal [63] and Rockafellar [97], we look into this aspect of conjugate functions.

Theorem 2.107 *(i) (Inf-Convolution Rule) Consider proper functions* $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$, *satisfying* $\bigcap_{i=1}^m \mathrm{dom}\ \phi_i^* \neq \emptyset$. *Then*

$$(\phi_1 \,\Box\, \phi_2 \,\Box\, \ldots \,\Box\, \phi_m)^* = \phi_1^* + \phi_2^* + \ldots + \phi_m^*.$$

(ii) (Sum Rule) Consider proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$, *satisfying* $\bigcap_{i=1}^m \mathrm{dom}\ \phi_i \neq \emptyset$. *Then*

$$(cl\ \phi_1 + cl\ \phi_2 + \ldots + cl\ \phi_m)^* = cl\ (\phi_1^* \,\Box\, \phi_2^* \,\Box\, \ldots \,\Box\, \phi_m^*).$$

If $\bigcap_{i=1}^m ri\ \mathrm{dom}\ \phi_i \neq \emptyset$, *then*

$$(\phi_1 + \phi_2 + \ldots + \phi_m)^* = \phi_1^* \,\Box\, \phi_2^* \,\Box\, \ldots \,\Box\, \phi_m^*$$

and for every $\xi \in \mathrm{dom}\ (\phi_1 + \phi_2 + \ldots + \phi_m)^*$, *the infimum of the problem*

$$\inf\{\phi_1^*(\xi_1) + \phi_2^*(\xi_2) + \ldots + \phi_m^*(\xi_m) : \xi_1 + \xi_2 + \ldots + \xi_m = \xi\}$$

is attained.

(iii) (Infimum Rule) Consider a family of proper functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i \in I$, *where* I *is an arbitrary index set, having a common affine minorant and satisfying* $\sup_{i\in I} \phi_i^*(\xi) < +\infty$ *for some* $\xi \in \mathbb{R}^n$. *Then*

$$(\inf_{i\in I} \phi_i)^* = \sup_{i\in I} \phi_i^*.$$

(iv) (Supremum Rule) Consider a family of proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i \in I$, *where* I *is an arbitrary index set. If* $\sup_{i\in I} \phi_i$ *is not indentically* $+\infty$, *then*

$$(\sup_{i\in I} \phi_i)^* = cl\ co(\inf_{i\in I} \phi_i^*).$$

Proof. (i) From Definition 2.101 of the conjugate function and Definition 2.54 of the inf-convolution along with Proposition 1.7,

$$\begin{aligned}
(\phi_1 \,\Box\, \ldots \,\Box\, \phi_m)^*(\xi) &= \sup_{x\in\mathbb{R}^n} \{\langle \xi, x\rangle - \inf_{x_1+\ldots+x_m=x} (\phi_1(x_1) + \ldots + \phi_m(x_m))\} \\
&= \sup_{x\in\mathbb{R}^n} \sup_{x_1+\ldots+x_m=x} \{\langle \xi, x\rangle - (\phi_1(x_1) + \ldots + \phi_m(x_m))\} \\
&= \sup_{x_1,\ldots,x_m\in\mathbb{R}^n} \{\langle \xi, x_1\rangle - \phi_1(x_1) + \ldots + \\
&\qquad\qquad \langle \xi, x_m\rangle - \phi_m(x_m)\} \\
&= \phi_1^*(\xi) + \ldots + \phi_m^*(x_m),
\end{aligned}$$

thereby establishing the desired result.

(ii) Replacing ϕ_i by ϕ_i^* for $i = 1, 2, \ldots, m$, in (i) along with Proposition 2.106 leads to

$$cl\ \phi_1^* + cl\ \phi_2 + \ldots + cl\ \phi_m = (\phi_1^* \,\Box\, \phi_2^* \,\Box\, \ldots \,\Box\, \phi_m^*)^*.$$

Taking the conjugate on both sides and again applying Proposition 2.106 yields the requisite condition,

$$(cl\ \phi_1^* + cl\ \phi_2 + \ldots + cl\ \phi_m)^* = cl\ (\phi_1^* \,\square\, \phi_2^* \,\square\, \ldots \,\square\, \phi_m^*).$$

If $\bigcap_{i=1}^m ri\ dom\ \phi_i$ is nonempty, then by Proposition 2.68,

$$cl\ \phi_1 + cl\ \phi_2 + \ldots + cl\ \phi_m = cl\ (\phi_1 + \phi_2 + \ldots + \phi_m).$$

Also, by the definition of conjugate functions,

$$\begin{aligned}(cl\ \phi_1 + cl\ \phi_2 + \ldots + cl\ \phi_m)^* &= (cl\ (\phi_1 + \phi_2 + \ldots + \phi_m))^* \\ &= (\phi_1 + \phi_2 + \ldots + \phi_m)^*.\end{aligned}$$

Now to establish the result, it is enough to prove that

$$\phi_1^* \,\square\, \phi_2^* \,\square\, \ldots \,\square\, \phi_m^*$$

is lsc. By Theorem 1.9, it is equivalent to showing that the lower-level set,

$$\mathcal{S}_\alpha = \{\xi \in \mathbb{R}^n : (\phi_1^* \,\square\, \ldots \,\square\, \phi_m^*)(\xi) \leq \alpha\},$$

is closed for every $\alpha \in \mathbb{R}$. Consider a bounded sequence $\{\xi_k\} \subset \mathcal{S}_\alpha$ such that $\xi_k \to \xi$. By Definition 2.54 of the inf-convolution, there exist $\xi_k^i \in \mathbb{R}^n$ with $\sum_{i=1}^m \xi_k^i = \xi_k$ such that

$$\phi_1^*(\xi_k^1) + \ldots + \phi_m^*(\xi_k^m) \leq \alpha + \frac{1}{k}, \ \forall\ k \in \mathbb{N}. \tag{2.33}$$

By assumption, suppose that $\hat{x} \in \bigcap_{i=1}^m ri\ dom\ \phi_i$. As $\phi_i,\ i = 1, 2, \ldots, m$, are convex, by Theorem 2.69, the functions are continuous at $\hat{x}$. Therefore, for some $\varepsilon > 0$ and $M_i \in \mathbb{R},\ i = 1, 2, \ldots, m$,

$$\phi_i(x) \leq M_i, \ \forall\ x \in \mathbb{B}_\varepsilon(\bar{x}),\ i = 1, 2, \ldots, m. \tag{2.34}$$

For any $d \in \mathbb{B}_\varepsilon(0)$, consider

$$\begin{aligned}\langle \xi_k^1, d \rangle &= \langle \xi_k^1, \hat{x} \rangle - \langle \xi_k^1, \hat{x} - d \rangle \\ &= \langle \xi_k^1, \hat{x} \rangle + \langle \xi_k^2, \hat{x} - d \rangle + \ldots + \langle \xi_k^m, \hat{x} - d \rangle - \langle \xi_k, \hat{x} - d \rangle,\end{aligned}$$

which by the Fenchel–Young inequality, Proposition 2.103 (iv), and the Cauchy–Schwarz inequality, Proposition 1.1, leads to

$$\begin{aligned}\langle \xi_k^1, d \rangle \leq\ & \phi_1^*(\xi_k^1) + \phi_1(\hat{x}) + \phi_2^*(\xi_k^2) + \phi_2(\hat{x} - d) + \ldots + \\ & \phi_m^*(\xi_k^m) + \phi_m(\hat{x} - d) + \|\xi_k\|\ \|\hat{x} - d\|.\end{aligned}$$

By the conditions (2.33) and (2.34), the above inequality reduces to

$$\langle \xi_k^1, d \rangle \leq \alpha + \frac{1}{k} + M_2 + \ldots + M_m + \|\xi_k\|\ \|\hat{x} - d\|,$$

which along with the boundedness of $\{\xi_k\}$ implies that $\{\xi_k^1\} \subset \mathbb{R}^n$ is a bounded sequence. Similarly, it can be shown that $\{\xi_k^i\}$, $i = 2, \ldots, m$, are bounded sequences. By the Bolzano–Weierstrass Theorem, Proposition 1.3, $\{\xi_k^i\}$, $i = 1, 2, \ldots, m$, have a convergent subsequence. Without loss of generality, assume that $\xi_k^i \to \xi_i$, $i = 1, 2, \ldots, m$. Because $\xi_k = \xi_k^1 + \xi_k^2 + \ldots + \xi_k^m$, as limit $k \to +\infty$,

$$\xi = \xi_1 + \xi_2 + \ldots + \xi_m.$$

By Proposition 2.102, ϕ_i^*, $i = 1, 2, \ldots, m$, are proper lsc convex functions, therefore taking the limit as $k \to +\infty$,

$$\phi_1^*(\xi_1) + \phi_2^*(\xi_2) + \ldots + \phi_m^*(\xi_m) \leq \alpha.$$

By the definition of inf-convolution, the above inequality leads to

$$(\phi_1^* \,\square\, \phi_2^* \,\square\, \ldots \,\square\, \phi_m^*)(\xi) \leq \alpha,$$

which implies that $\xi \in \mathcal{S}_\alpha$. Because $\alpha \in \mathbb{R}$ was arbitrary, the lower-level set is closed for every $\alpha \in \mathbb{R}$ and hence

$$\phi_1^* \,\square\, \phi_2^* \,\square\, \ldots \,\square\, \phi_m^*$$

is closed. Repeating the same arguments with

$$(\phi_1^* \,\square\, \phi_2^* \,\square\, \ldots \,\square\, \phi_m^*)(\xi) = \alpha \quad \text{and} \quad \xi_k = \xi$$

yields that the infimum is achieved, thereby completing the proof.

(iii) By Definition 2.101, for every $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
(\inf_{i \in I} \phi_i)^*(\xi) &= \sup_{x \in \mathbb{R}^n} \{\langle \xi, x\rangle - \inf_{i \in I} \phi_i(x)\} \\
&= \sup_{x \in \mathbb{R}^n} \sup_{i \in I} \{\langle \xi, x\rangle - \phi_i(x)\} \\
&= \sup_{i \in I} \sup_{x \in \mathbb{R}^n} \{\langle \xi, x\rangle - \phi_i(x)\} = \sup_{i \in I} \phi_i^*(\xi),
\end{aligned}$$

as desired.

(iv) Replacing ϕ_i by ϕ_i^* for $i = 1, 2, \ldots, m$ in (iii),

$$\sup_{i \in I} \phi_i^{**} = (\inf_{i \in I} \phi_i^*)^*.$$

As ϕ_i, $i = 1, 2, \ldots, m$, are lsc, the above condition reduces to

$$\sup_{i \in I} \phi_i = (\inf_{i \in I} \phi_i^*)^*.$$

Taking the conjugate on both sides leads to

$$(\sup_{i \in I} \phi_i)^* = (\inf_{i \in I} \phi_i^*)^{**},$$

which by Theorem 2.104 yields

$$(\sup_{i\in I}\phi_i)^* = cl\ co\ (\inf_{i\in I}\phi_i^*). \qquad \square$$

Next, using the Fenchel–Young inequality, we present an equivalent characterization of the subdifferential of a convex function.

Theorem 2.108 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then for any $x, \xi \in \mathbb{R}^n$*

$$\xi \in \partial\phi(x) \quad \Longleftrightarrow \quad \phi(x) + \phi^*(\xi) = \langle \xi, x\rangle.$$

In addition, if ϕ is also lsc, then for any x and ξ in $\mathbb{R}^n$

$$\xi \in \partial\phi(x) \quad \Longleftrightarrow \quad \phi(x) + \phi^*(\xi) = \langle \xi, x\rangle \quad \Longleftrightarrow \quad x \in \partial\phi^*(\xi).$$

Proof. Suppose that $\xi \in \partial\phi(\bar{x})$, which by Definition 2.77 of the subdifferential implies that

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x}\rangle,\ \forall\, x \in \mathbb{R}^n.$$

The above inequality leads to

$$\langle \xi, \bar{x}\rangle - \phi(\bar{x}) \geq \sup_{x\in\mathbb{R}^n}\{\langle \xi, x\rangle - \phi(x)\} = \phi^*(\xi),$$

that is,

$$\phi(\bar{x}) + \phi^*(\xi) \leq \langle \xi, \bar{x}\rangle$$

which along with the Fenchel–Young inequality, Proposition 2.103 (iv), reduces to the desired condition

$$\phi(\bar{x}) + \phi^*(\xi) = \langle \xi, \bar{x}\rangle.$$

Conversely, suppose that above condition is satisfied, which by Definition 2.101 of the conjugate function implies

$$\langle \xi, \bar{x}\rangle - \phi(\bar{x}) \geq \langle \xi, x\rangle - \phi(x),\ \forall\, x \in \mathbb{R}^n,$$

that is,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x}\rangle,\ \forall\, x \in \mathbb{R}^n.$$

Thus, $\xi \in \partial\phi(\bar{x})$, thereby establishing the equivalence.

Now if ϕ is lsc as well, then by Theorem 2.105, $\phi = \phi^{**}$. Then the equivalent condition can be expressed as $\bar{\xi} \in \partial\phi(\bar{x})$ if and only if

$$\phi^{**}(\bar{x}) + \phi^*(\bar{\xi}) = \langle \bar{\xi}, \bar{x}\rangle.$$

By Definition 2.101 of the biconjugate function, the above condition is equivalent to

$$\langle \bar{\xi}, \bar{x} \rangle - \phi^*(\bar{\xi}) \geq \langle \xi, \bar{x} \rangle - \phi^*(\xi), \ \forall\, \xi \in \mathbb{R}^n,$$

that is,

$$\phi^*(\xi) - \phi^*(\bar{\xi}) \geq \langle \xi - \bar{\xi}, \bar{x} \rangle, \ \forall\, \xi \in \mathbb{R}^n.$$

By the definition of subdifferential, $\bar{x} \in \partial \phi^*(\bar{\xi})$. The converse can be worked out along the lines of the previous part and thus establishing the desired relation. □

As an application of the above theorem, consider a closed convex cone $K \subset \mathbb{R}^n$. We claim that $\bar{\xi} \in \partial \delta_K(\bar{x})$ if and only if $\bar{x} \in \partial \delta_{K^\circ}(\bar{\xi})$. Suppose that $\bar{\xi} \in \partial \delta_K(\bar{x}) = N_K(\bar{x})$, which is equivalent to

$$\langle \bar{\xi}, x - \bar{x} \rangle \leq 0, \ \forall\, x \in K.$$

In particular, taking $x = 0$ and $x = 2\bar{x}$, respectively, implies that $\langle \bar{\xi}, \bar{x} \rangle = 0$. Therefore, the above inequality reduces to

$$\langle \bar{\xi}, x \rangle \leq 0, \ \forall\, x \in K,$$

which by Definition 2.30 implies that $\bar{\xi} \in K^\circ$. Thus, $\bar{\xi} \in N_K(\bar{x})$ is equivalent to

$$\bar{x} \in K, \qquad \bar{\xi} \in K^\circ, \qquad \langle \bar{\xi}, \bar{x} \rangle = 0.$$

For a closed convex cone K, by Proposition 2.31, $K^{\circ\circ} = K$. As $\bar{x} \in K = K^{\circ\circ}$,

$$\langle \xi, \bar{x} \rangle \leq 0, \ \forall\, \xi \in K^\circ.$$

Because $\langle \bar{\xi}, \bar{x} \rangle = 0$, the above condition is equivalent to

$$\langle \xi - \bar{\xi}, \bar{x} \rangle \leq 0, \ \forall\, \xi \in K^\circ,$$

which implies that $\bar{x} \in N_{K^\circ}(\bar{\xi})$, thereby proving our claim.

2.6 ε-Subdifferential

In Subsection 2.3.3 on differentiability properties of a convex function, from Proposition 2.82 and the examples preceding it, we noticed that $\partial \phi(x)$ may turn out to be empty, even though $x \in dom\ \phi$. To overcome this aspect of subdifferentials, the concept of the ε-subdifferential came into existence; it not only overcomes the drawback of subdifferentials but is also important from the optimization point of view. The idea can be found in the work of Brønsted and Rockafellar [19] but the theory of ε-subdifferential calculus was given by Hiriart-Urruty [58].

Definition 2.109 Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. For $\varepsilon > 0$, the *ε-subdifferential* of ϕ at $\bar{x} \in$ *dom* ϕ is given by

$$\partial_\varepsilon \phi(\bar{x}) = \{\xi \in \mathbb{R}^n : \phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x}\rangle - \varepsilon, \ \forall\, x \in \mathbb{R}^n\}.$$

For a zero function, $0 : \mathbb{R}^n \to \mathbb{R}$, defined as $0(x) = 0$ for every $x \in \mathbb{R}^n$, $\partial_\varepsilon 0(\bar{x}) = \{0\}$ for every $\varepsilon > 0$. Otherwise if there exists $\xi \in \partial_\varepsilon 0(\bar{x})$ with $\xi \neq 0$, by the above definition of ε-subdifferential,

$$\begin{aligned} 0 &\geq \langle \xi, x - \bar{x}\rangle - \varepsilon \\ &= \sum_{i=1}^{n} \xi_i (x_i - \bar{x}_i) - \varepsilon, \ \forall\, x \in \mathbb{R}^n. \end{aligned}$$

Because $\xi \neq 0$, there exists some $j \in \{1, 2, \ldots, n\}$ such that $\xi_j \neq 0$. In particular, taking $x_j = \bar{x}_j + \dfrac{2\varepsilon}{\xi_j}$ and $x_i = \bar{x}_i$, $i \neq j$, the above inequality yields $\varepsilon \leq 0$, which is a contradiction.

As shown in Section 2.4 that for a convex set $F \subset \mathbb{R}^n$, the subdifferential of the indicator function coincides with the normal cone, that is, $\partial \delta_F = N_F$. Along similar lines, we define the ε-normal set.

Definition 2.110 Consider a convex set $F \subset \mathbb{R}^n$. Then for $\varepsilon > 0$, the ε-subdifferential of the indicator function at $\bar{x} \in F$ is

$$\begin{aligned} \partial_\varepsilon \delta_F(\bar{x}) &= \{\xi \in \mathbb{R}^n : \delta_F(x) - \delta_F(\bar{x}) \geq \langle \xi, x - \bar{x}\rangle - \varepsilon, \ \forall\, x \in \mathbb{R}^n\} \\ &= \{\xi \in \mathbb{R}^n : \varepsilon \geq \langle \xi, x - \bar{x}\rangle, \ \forall\, x \in F\}, \end{aligned}$$

which is also called the *ε-normal set* and denoted as $N_{\varepsilon,F}(\bar{x})$. Note that $N_{\varepsilon,F}$ is not a cone unlike N_F, which is always a cone.

Recall the proper convex function $\phi : \mathbb{R} \to \bar{\mathbb{R}}$ given by

$$\phi(x) = \begin{cases} -\sqrt{x}, & 0 \leq x \leq 1, \\ +\infty, & \text{otherwise}, \end{cases}$$

considered in Subsection 2.3.3. As already mentioned, for $x = 0$, the subdifferential $\partial \phi(x)$ is empty. But for any $\varepsilon > 0$, the ε-subdifferential at $x = 0$ is $\partial_\varepsilon \phi(x) = \left(-\infty, \dfrac{-1}{2\varepsilon}\right]$ and hence nonempty.

In the proposition below we present some properties of the ε-subdifferential of the convex functions.

Proposition 2.111 *Consider a proper lsc convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and let $\varepsilon > 0$ be given. Then for every $\bar{x} \in$ dom ϕ, the ε-subdifferential $\partial_\varepsilon \phi(\bar{x})$ is a nonempty closed convex set and*

$$\partial \phi(\bar{x}) = \bigcap_{\varepsilon > 0} \partial_\varepsilon \phi(\bar{x}).$$

For $\varepsilon_1 \geq \varepsilon_2$, $\partial_{\varepsilon_2}(\bar{x}) \subset \partial_{\varepsilon_1}(\bar{x})$.

Proof. Observe that for $\bar{x} \in dom\ \phi$ and $\varepsilon > 0$, $\phi(\bar{x}) - \varepsilon < \phi(\bar{x})$, which implies $(\bar{x}, \phi(\bar{x}) - \varepsilon) \notin epi\ \phi$. Because ϕ is a lsc convex, by Theorem 1.9 and Proposition 2.48, $epi\ \phi$ is closed convex set in $\mathbb{R}^n \times \mathbb{R}$. Therefore, applying the Strict Separation Theorem, Theorem 2.26 (iii), there exists $(\xi, \gamma) \in \mathbb{R}^n \times \mathbb{R}$ with $(\xi, \gamma) \neq (0, 0)$ such that

$$\langle \xi, \bar{x} \rangle + \gamma(\phi(\bar{x}) - \varepsilon) < \langle \xi, x \rangle + \gamma\alpha, \ \forall\ (x, \alpha) \in epi\ \phi.$$

As $(x, \phi(x)) \in epi\ \phi$ for every $x \in dom\ \phi$, the above condition leads to

$$\langle \xi, \bar{x} \rangle + \gamma(\phi(\bar{x}) - \varepsilon) < \langle \xi, x \rangle + \gamma\phi(x), \ \forall\ x \in dom\ \phi. \tag{2.35}$$

In particular, taking $x = \bar{x}$ in the preceding inequality yields $\gamma > 0$. Now dividing (2.35) throughout by γ implies that

$$\langle -\frac{\xi}{\gamma}, x - \bar{x} \rangle - \varepsilon < \phi(x) - \phi(\bar{x}), \ \forall\ x \in dom\ \phi.$$

The above condition is also satisfied by $x \notin dom\ \phi$, which implies

$$\langle -\frac{\xi}{\gamma}, x - \bar{x} \rangle - \varepsilon < \phi(x) - \phi(\bar{x}), \ \forall\ x \in \mathbb{R}^n.$$

By Definition 2.109 of the ε-subdifferential, $-\dfrac{\xi}{\gamma} \in \partial_\varepsilon \phi(\bar{x})$. Thus, $\partial_\varepsilon \phi(\bar{x})$ is nonempty for every $\bar{x} \in dom\ \phi$.

Suppose that $\{\xi_k\} \subset \partial_\varepsilon \phi(\bar{x})$ such that $\xi_k \to \xi$. By the definition of ε-subdifferential,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi_k, x - \bar{x} \rangle - \varepsilon, \ \forall\ x \in \mathbb{R}^n.$$

Taking the limit as $k \to +\infty$, the above inequality leads to

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall\ x \in \mathbb{R}^n,$$

which implies that $\xi \in \partial_\varepsilon \phi(\bar{x})$, thereby yielding the closedness of $\partial_\varepsilon \phi(\bar{x})$.

Consider $\xi_1, \xi_2 \in \partial_\varepsilon \phi(\bar{x})$, which implies that for $i = 1, 2$,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle - \varepsilon, \ \forall\ x \in \mathbb{R}^n.$$

Therefore, for any $\lambda \in [0, 1]$,

$$\phi(x) - \phi(\bar{x}) \geq \langle (1 - \lambda)\xi_1 + \lambda\xi_2, x - \bar{x} \rangle - \varepsilon, \ \forall\ x \in \mathbb{R}^n,$$

which implies $(1 - \lambda)\xi_1 + \lambda\xi_2 \in \partial_\varepsilon \phi(\bar{x})$. Because ξ_1, ξ_2 were arbitrary, $\partial_\varepsilon \phi(\bar{x})$ is convex.

Now we will prove that

$$\partial\phi(\bar{x}) = \bigcap_{\varepsilon > 0} \partial_\varepsilon \phi(\bar{x}).$$

Suppose that $\xi \in \partial\phi(\bar{x})$, which by Definition 2.77 of the subdifferential implies that for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \phi(x) - \phi(\bar{x}) &\geq \langle \xi, x - \bar{x} \rangle \\ &\geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall\, \varepsilon > 0. \end{aligned}$$

Thus, by the definition of ε-subdifferential, $\xi \in \partial_\varepsilon \phi(\bar{x})$ for every $\varepsilon > 0$. Because $\xi \in \partial\phi(\bar{x})$ was arbitrary,

$$\partial\phi(\bar{x}) \subset \bigcap_{\varepsilon > 0} \partial_\varepsilon \phi(\bar{x}).$$

Conversely, consider $\xi \in \partial_\varepsilon \phi(\bar{x})$ for every $\varepsilon > 0$, which implies that for every $x \in \mathbb{R}^n$,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall\, \varepsilon > 0.$$

As the preceding inequality holds for every $\varepsilon > 0$, taking the limit as $\varepsilon \to 0$ leads to

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\, x \in \mathbb{R}^n,$$

thereby yielding $\xi \in \partial\phi(\bar{x})$. Because ξ was arbitrary, the reverse inclusion is satisfied, that is,

$$\partial\phi(\bar{x}) \supset \bigcap_{\varepsilon > 0} \partial_\varepsilon \phi(\bar{x}),$$

hence establishing the result.

The relation $\partial_{\varepsilon_2}(\bar{x}) \subset \partial_{\varepsilon_1}(\bar{x})$ for $\varepsilon_1 \geq \varepsilon_2$ can be easily worked out using the definition of ε-subdifferential. □

The proof of the nonemptiness of ε-subdifferential is from Lucchetti [79]. In the example, it is easy to observe that at $x = 0$, $\bigcap_{\varepsilon>0} \partial_\varepsilon \phi(x)$ is empty, as is $\partial\phi(x)$. Before moving any further, let us consider the absolute value function, $\phi(x) = |x|$. The subdifferential of ϕ is given by

$$\partial\phi(x) = \begin{cases} 1, & x > 0, \\ [-1, 1], & x = 0, \\ -1, & x < 0. \end{cases}$$

Now for $\varepsilon > 0$, the ε-subdifferential of ϕ is

$$\partial_\varepsilon \phi(x) = \begin{cases} [1 - \varepsilon/x, 1], & x > \varepsilon/2, \\ [-1, 1], & -\varepsilon/2 \leq x \leq \varepsilon/2, \\ [-1, -1 - \varepsilon/x] & x < -\varepsilon/2. \end{cases}$$

The graphs of $\partial\phi$ and $\partial_\varepsilon\phi$ for $\varepsilon = 1$ are shown in Figures 2.8 and 2.9. The graph of the subdifferential is a simple step function.

Similar to the characterization of the subdifferential in terms of the conjugate function, the following result provides a relation between the ε-subdifferential and the conjugate function.

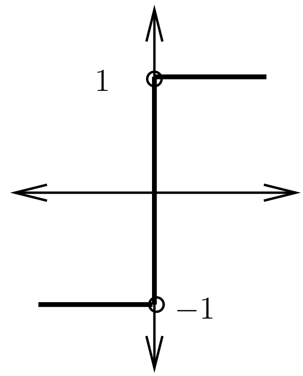

FIGURE 2.8: Graph of $\partial(|.|)$.

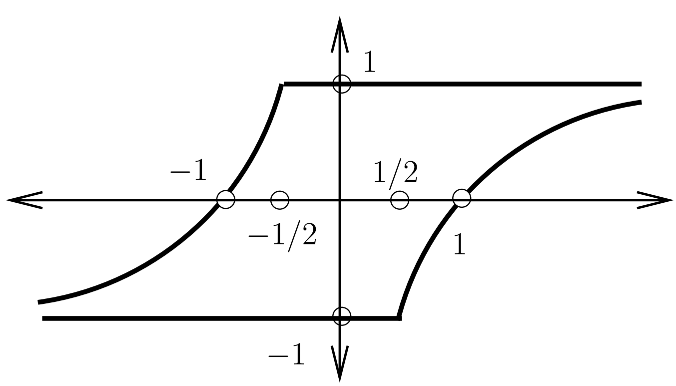

FIGURE 2.9: Graph of $\partial_1(|.|)$.

Theorem 2.112 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. Then for any $\varepsilon > 0$ and $x \in dom\ \phi$,*

$$\xi \in \partial_\varepsilon \phi(x) \quad \Longleftrightarrow \quad \phi(x) + \phi^*(\xi) - \langle \xi, x \rangle \leq \varepsilon.$$

Proof. Consider any $\xi \in \partial_\varepsilon \phi(\bar{x})$. By Definition 2.109 of ε-subdifferential,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall\ x \in \mathbb{R}^n,$$

which implies that

$$\langle \xi, x \rangle - \phi(x) + \phi(\bar{x}) - \langle \xi, \bar{x} \rangle \leq \varepsilon, \ \forall\ x \in \mathbb{R}^n.$$

By Definition 2.101 of the conjugate function,

$$\phi^*(\xi) + \phi(\bar{x}) - \langle \xi, \bar{x} \rangle \leq \varepsilon,$$

as desired.

Conversely, suppose that the inequality holds, which by the definition of conjugate function implies that

$$\langle \xi, x \rangle - \phi(x) + \phi(\bar{x}) - \langle \xi, \bar{x} \rangle \leq \varepsilon, \ \forall\ x \in \mathbb{R}^n,$$

which yields that $\xi \in \partial_\varepsilon \phi(\bar{x})$, thus establishing the equivalence. $\square$

As mentioned earlier, the notion of ε-subdifferential appears in the well-known work of Brønsted and Rockafellar [19] in which they estimated *how well $\partial_\varepsilon \phi$ "approximates" $\partial \phi$*. We present the modified version of that famous Brønsted–Rockafellar Theorem from Thibault [108] below. The proof involves the famous *Ekeland's Variational Principle* [41, 42, 43], which we state without proof before moving on with the result.

Theorem 2.113 *(Ekeland's Variational Principle) Consider a closed proper function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and for $\varepsilon > 0$, let $\bar{x} \in \mathbb{R}^n$ be such that*

$$\phi(\bar{x}) \leq \inf_{x \in \mathbb{R}^n} \phi(x) + \varepsilon.$$

Then for any $\lambda > 0$, there exists $x_\lambda \in \mathbb{R}^n$ such that

$$\|x_\lambda - \bar{x}\| \leq \frac{\varepsilon}{\lambda}, \qquad \phi(x_\lambda) \leq \phi(\bar{x}),$$

and x_λ is the unique minimizer of the unconstrained problem

$$\inf\ \phi(x) + \frac{\varepsilon}{\lambda}\|x - x_\lambda\| \quad \textit{subject to} \quad x \in \mathbb{R}^n.$$

Observe that the second condition in Ekeland's Variational Principle, $\phi(x_\lambda) \leq \phi(\bar{x})$, implies that

$$\phi(x_\lambda) - \phi(\bar{x}) \leq 0 \leq \varepsilon. \tag{2.36}$$

From the condition on $\bar{x}$,

$$\phi(\bar{x}) \leq \inf_{x \in \mathbb{R}^n} \phi(x) + \varepsilon \leq \phi(x_\lambda) + \varepsilon,$$

which implies that

$$\phi(\bar{x}) - \phi(x_\lambda) \leq \varepsilon.$$

The above condition along with (2.36) leads to

$$|\phi(x_\lambda) - \phi(\bar{x})| \leq \varepsilon.$$

Now to establish the modified version of Brønsted–Rockafellar Theorem, we can apply

$$|\phi(x_\lambda) - \phi(\bar{x})| \leq \varepsilon \quad \text{instead of} \quad \phi(x_\lambda) \leq \phi(\bar{x})$$

in the Ekeland's Variational Principle.

Theorem 2.114 *(A modified version of the Brønsted–Rockafellar Theorem) Consider a proper lsc convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\bar{x} \in dom\ \phi$. Then for any $\varepsilon > 0$ and for any $\xi \in \partial_\varepsilon \phi(\bar{x})$, there exist $x_\varepsilon \in \mathbb{R}^n$ and $\xi_\varepsilon \in \partial\phi(x_\varepsilon)$ such that*

$$\|x_\varepsilon - \bar{x}\| \leq \sqrt{\varepsilon}, \quad \|\xi_\varepsilon - \xi\| \leq \sqrt{\varepsilon} \quad \text{and} \quad |\phi(x_\varepsilon) - \langle \xi_\varepsilon, x_\varepsilon - \bar{x}\rangle - \phi(\bar{x})| \leq 2\varepsilon.$$

Proof. By Definition 2.109 of the ε-subdifferential, $\xi \in \partial_\varepsilon \phi(\bar{x})$ implies

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x}\rangle - \varepsilon, \ \forall\ x \in \mathbb{R}^n,$$

that is,

$$\phi(\bar{x}) - \langle \xi, \bar{x}\rangle \geq \phi(x) - \langle \xi, x\rangle + \varepsilon, \ \forall\ x \in \mathbb{R}^n.$$

By applying Ekeland's Variational Principle, Theorem 2.113, to $\phi - \langle \xi, .\rangle$ with $\lambda = \sqrt{\varepsilon}$, there exists $x_\varepsilon \in \mathbb{R}^n$ such that $\|x_\varepsilon - \bar{x}\| \leq \sqrt{\varepsilon}$,

$$|\phi(x_\varepsilon) - \langle \xi, x_\varepsilon\rangle - \phi(\bar{x}) + \langle \xi, \bar{x}\rangle| \leq \varepsilon \tag{2.37}$$

and

$$\phi(x_\varepsilon) - \langle \xi, x_\varepsilon\rangle \leq \phi(x) - \langle \xi, x\rangle + \sqrt{\varepsilon}\|x - x_\varepsilon\|, \ \forall\ x \in \mathbb{R}^n. \tag{2.38}$$

By the definition of subdifferential, the above condition (2.38) implies that

$$\xi \in \partial(\phi + \sqrt{\varepsilon}\|. - x_\varepsilon\|)(x_\varepsilon). \tag{2.39}$$

As *dom* $\|. - x_\varepsilon\| = \mathbb{R}^n$, by Theorem 2.69, $\|. - x_\varepsilon\|$ is continuous on $\mathbb{R}^n$. Therefore, by Theorem 2.91 along with the fact that $\partial(\|. - x_\varepsilon\|)(x_\varepsilon) = \mathbb{B}$, (2.39) becomes

$$\xi \in \partial\phi(x_\varepsilon) + \sqrt{\varepsilon}\mathbb{B}.$$

Thus, there exists $\xi_\varepsilon \in \partial\phi(x_\varepsilon)$ such that $\|\xi_\varepsilon - \xi\| \leq \sqrt{\varepsilon}$. From condition (2.37) along with the Cauchy–Schwarz inequality, Proposition 1.1,

$$\begin{aligned} |\phi(x_\varepsilon) - \langle \xi, x_\varepsilon - \bar{x}\rangle - \phi(\bar{x})| &\leq \varepsilon + |\langle \xi_\varepsilon - \xi, x_\varepsilon - \bar{x}\rangle| \\ &\leq \varepsilon + \|\xi_\varepsilon - \xi\|\|x_\varepsilon - \bar{x}\| = 2\varepsilon, \end{aligned}$$

thereby completing the proof. □

As in the study of optimality conditions, we need the subdifferential calculus rules; similarly, ε-subdifferentials also play a pivotal role in this respect. Below we present the ε-subdifferential Sum Rule, Max-Function and the Scalar Product Rules that we will need in our study of optimality conditions for the convex programming problem (CP). The proofs of the Sum and the Max-Function Rules are from Hiriart-Urruty and Lemaréchal [62].

Theorem 2.115 *(Sum Rule) Consider two proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$ such that ri dom $\phi_1 \cap$ ri dom $\phi_2 \neq \emptyset$. Then for $\varepsilon > 0$,*

$$\partial_\varepsilon(\phi_1 + \phi_2)(\bar{x}) = \bigcup_{\substack{\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0,\\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1}\phi_1(\bar{x}) + \partial_{\varepsilon_2}\phi_2(\bar{x}))$$

for every $\bar{x} \in$ dom $\phi_1 \cap$ dom ϕ_2.

Proof. Suppose that $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$. Consider $\xi_i \in \partial_{\varepsilon_i}\phi_i(\bar{x})$, $i = 1, 2$, which by Definition 2.109 of the ε-subdifferential implies that for every $x \in \mathbb{R}^n$,

$$\phi_i(x) - \phi_i(\bar{x}) \geq \langle \xi_i, x - \bar{x}\rangle - \varepsilon_i,\ i = 1, 2.$$

The above condition along with the assumption $\varepsilon_1 + \varepsilon_2 = \varepsilon$ leads to

$$\begin{aligned} (\phi_1 + \phi_2)(x) - (\phi_1 + \phi_2)(\bar{x}) &\geq \langle \xi_1 + \xi_2, x - \bar{x}\rangle - (\varepsilon_1 + \varepsilon_2) \\ &= \langle \xi_1 + \xi_2, x - \bar{x}\rangle - \varepsilon,\ \forall\, x \in \mathbb{R}^n, \end{aligned}$$

thereby yielding $\xi_1 + \xi_2 \in \partial_\varepsilon(\phi_1 + \phi_2)(\bar{x})$. Because $\varepsilon_i \geq 0$ and $\xi_i \in \partial_{\varepsilon_i}\phi_i(\bar{x})$ for $i = 1, 2$, were arbitrary,

$$\partial_\varepsilon(\phi_1 + \phi_2)(\bar{x}) \supset \bigcup_{\substack{\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0,\\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1}\phi_1(\bar{x}) + \partial_{\varepsilon_2}\phi_2(\bar{x}))$$

Conversely, suppose that $\xi \in \partial_\varepsilon(\phi_1 + \phi_2)(\bar{x})$, which by the definition of ε-subdifferential implies that

$$(\phi_1 + \phi_2)(x) - (\phi_1 + \phi_2)(\bar{x}) \geq \langle \xi, x - \bar{x}\rangle - \varepsilon,\ \forall\, x \in \mathbb{R}^n.$$

By Definition 2.101 of the conjugate function,

$$(\phi_1 + \phi_2)^*(\xi) + (\phi_1 + \phi_2)(\bar{x}) - \langle \xi, \bar{x}\rangle \leq \varepsilon. \tag{2.40}$$

By the Sum Rule of the conjugate function, Theorem 2.107 (ii), as the assumption $ri\ dom\ \phi_1 \cap ri\ dom\ \phi_2 \neq \emptyset$ holds,

$$(\phi_1 + \phi_2)^*(\xi) = (\phi_1^* \,\square\, \phi_2^*)(\xi),$$

and the infimum is attained, which implies there exist $\xi_i \in \mathbb{R}^n,\ i = 1, 2$, satisfying $\xi_1 + \xi_2 = \xi$ such that

$$(\phi_1 + \phi_2)^*(\xi) = \phi_1^*(\xi_1) + \phi_2^*(\xi_2).$$

Therefore, the inequality (2.40) becomes

$$(\phi_1^*(\xi_1) + \phi_1(\bar{x}) - \langle \xi_1, \bar{x}\rangle) + (\phi_2^*(\xi_2) + \phi_2(\bar{x}) - \langle \xi_2, \bar{x}\rangle) \leq \varepsilon.$$

Denote $\varepsilon_i = \phi_i^*(\xi_i) + \phi_i(\bar{x}) - \langle \xi_i, \bar{x}\rangle,\ i = 1, 2$, which by the Fenchel–Young inequality, Proposition 2.103 (iv), implies that $\varepsilon_i \geq 0,\ i = 1, 2$. Observe that $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$. Again, by the definition of conjugate function,

$$\begin{aligned} \phi_i(x) - \phi_i(\bar{x}) &\geq \langle \xi_i, x - \bar{x}\rangle - \varepsilon_i \\ &\geq \langle \xi_i, x - \bar{x}\rangle - \bar{\varepsilon}_i, \end{aligned}$$

where $\bar{\varepsilon}_i = \varepsilon_i + \dfrac{\varepsilon - \varepsilon_1 - \varepsilon_2}{2} \geq \varepsilon_i,\ i = 1, 2$. Therefore, for $i = 1, 2$,

$$\xi_i \in \partial_{\bar{\varepsilon}_i} \phi_i(\bar{x}),$$

with $\bar{\varepsilon}_1 + \bar{\varepsilon}_2 = \varepsilon$. Thus,

$$\xi = \xi_1 + \xi_2 \in \partial_{\bar{\varepsilon}_1} \phi_1(\bar{x}) + \partial_{\bar{\varepsilon}_2} \phi_2(\bar{x}).$$

Because $\xi \in \partial_\varepsilon(\phi_1 + \phi_2)(\bar{x})$ was arbitrary,

$$\partial_\varepsilon(\phi_1 + \phi_2)(\bar{x}) \subset \bigcup_{\substack{\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1} \phi_1(\bar{x}) + \partial_{\varepsilon_2} \phi_2(\bar{x})),$$

thereby completing the proof. □

Before proving the ε-subdifferential Max-Function Rule, we state a result from Hiriart-Urruty and Lemaréchal [62] without proof and present the Scalar Product Rule.

Proposition 2.116 *Consider proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, \ldots, m$. Let $\phi(x) = \max\{\phi_1(x), \phi_2(x), \ldots, \phi_m(x)\}$ and $p = \min\{m, n + 1\}$. For every $\xi \in dom\ \phi^* = co\ \bigcup_{i=1}^m dom\ \phi_i^*$, there exist $\xi_i \in dom\ \phi_i^*$ and $\lambda_i \geq 0$, $i = 1, 2, \ldots, p$, with $\sum_{i=1}^p \lambda_i = 1$ such that*

$$\phi^*(\xi) = \sum_{i=1}^p \lambda_i \phi_i^*(\xi_i) \quad \textit{and} \quad \xi = \sum_{i=1}^p \lambda_i \xi_i.$$

More precisely, (ξ_i, λ_i) *solve the problem*

$$\begin{array}{rl} \inf & \displaystyle\sum_{i=1}^{p} \lambda_i \phi_i^*(\xi_i) \\ \textit{subject to} & \xi = \displaystyle\sum_{i=1}^{p} \lambda_i \xi_i, \quad \sum_{i=1}^{p} \lambda_i = 1, \\ & \xi_i \in \textit{dom } \phi_i^*, \quad \lambda_i \geq 0, \ i = 1, 2, \ldots, p. \end{array} \tag{P}$$

For the ε-subdifferential Max-Function Rule, we will need the Scalar Product Rule that we present below.

Theorem 2.117 *(Scalar Product Rule) For a proper convex function* $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ *and any* $\varepsilon \geq 0$*,*

$$\partial_\varepsilon(\lambda g)(\bar{x}) = \lambda \partial_{\varepsilon/\lambda} g(\bar{x}), \ \forall \ \lambda > 0.$$

Proof. Suppose that $xi \in \partial_\varepsilon(\lambda\phi)(\bar{x})$, which by Definition 2.109 implies that

$$(\lambda\phi)(x) - (\lambda\phi)(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall \ x \in \mathbb{R}^n.$$

As $\lambda > 0$, dividing throughout by λ leads to

$$\phi(x) - \phi(\bar{x}) \geq \langle \frac{\xi}{\lambda}, x - \bar{x} \rangle - \frac{\varepsilon}{\lambda}, \ \forall \ x \in \mathbb{R}^n,$$

which implies $\dfrac{\xi}{\lambda} \in \partial_{\tilde{\varepsilon}}\phi(\bar{x})$, where $\tilde{\varepsilon} = \dfrac{\varepsilon}{\lambda}$, that is, $\xi \in \lambda\partial_{\tilde{\varepsilon}}\phi(\bar{x})$. Because $\xi \in \partial_\varepsilon(\lambda\phi)(\bar{x})$ was arbitrary,

$$\partial_\varepsilon(\lambda g)(\bar{x}) \subset \lambda \partial_{\tilde{\varepsilon}} g(\bar{x}).$$

Conversely, suppose that $\xi \in \lambda\partial_{\tilde{\varepsilon}}\phi(\bar{x})$ for $\lambda > 0$, which implies there exists $\tilde{\xi} \in \partial_{\tilde{\varepsilon}}\phi(\bar{x})$ such that $\xi = \lambda\tilde{\xi}$. By the definition of ε-subdifferential,

$$\phi(x) - \phi(\bar{x}) \geq \langle \tilde{\xi}, x - \bar{x} \rangle - \tilde{\varepsilon}, \ \forall \ x \in \mathbb{R}^n,$$

which implies

$$(\lambda\phi)(x) - (\lambda\phi)(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall \ x \in \mathbb{R}^n,$$

where $\varepsilon = \lambda\tilde{\varepsilon}$. Therefore, $\xi \in \partial_\varepsilon(\lambda\phi)(\bar{x})$. Because $\xi \in \lambda\partial_{\tilde{\varepsilon}}\phi(\bar{x})$ was arbitrary,

$$\partial_\varepsilon(\lambda g)(\bar{x}) \supset \lambda \partial_{\tilde{\varepsilon}} g(\bar{x}),$$

thereby yielding the desired result. □

Now we proceed with establishing the ε-subdifferential Max-Function Rule with the above results as the tool.

Theorem 2.118 *(Max-Function Rule) Consider proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$. Let $\phi(x) = \max\{\phi_1(x), \phi_2(x), \ldots, \phi_m(x)\}$ and $p = \min\{m, n+1\}$. Then $\xi \in \partial_\varepsilon \phi(\bar{x})$ if and only if there exist $\xi_i \in dom\ \phi_i^*$, $\varepsilon_i \geq 0$, and $\lambda_i \geq 0$, $i = 1, 2, \ldots, p$, with $\sum_{i=1}^{p} \lambda_i = 1$ such that*

$$\xi_i \in \partial_{\varepsilon_i/\lambda_i} \phi_i(\bar{x}) \quad \textit{for every} \quad i \quad \textit{satisfying} \quad \lambda_i > 0, \tag{2.41}$$

$$\xi = \sum_{i=1}^{p} \lambda_i \xi_i \quad \textit{and} \quad \sum_{i=1}^{p} \varepsilon_i + \phi(\bar{x}) - \sum_{i=1}^{p} \lambda_i \phi_i(\bar{x}) \leq \varepsilon. \tag{2.42}$$

Proof. By Proposition 2.116,

$$\phi^*(\xi) = \sum_{i=1}^{p} \lambda_i \phi_i^*(\xi_i),$$

where $p = \min\{m, n+1\}$ and $(\xi_i, \lambda_i) \in dom\ \phi_i^* \times \mathbb{R}_+$, $i = 1, 2, \ldots, p$, solves the problem (P), that is, satisfies

$$\xi = \sum_{i=1}^{p} \lambda_i \xi_i \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i = 1.$$

By the relation between the ε-subdifferential and the conjugate function, Theorem 2.112, as $\xi \in \partial_\varepsilon \phi(x)$,

$$\phi^*(\xi) + \phi(x) - \langle \xi, x \rangle \leq \varepsilon,$$

which by the conditions on (ξ_i, λ_i), $i = 1, 2, \ldots, p$, along with the definition of ϕ leads to

$$\sum_{i=1}^{p} \lambda_i \phi_i^*(\xi_i) + \phi(x) - \sum_{i=1}^{p} \lambda_i \langle \xi_i, x \rangle \leq \varepsilon. \tag{2.43}$$

The above condition can be rewritten as

$$\sum_{i=1}^{p} \varepsilon_i + \phi(x) - \sum_{i=1}^{p} \lambda_i \phi_i(x) \leq \varepsilon,$$

where $\varepsilon_i = \lambda_i(\phi_i^*(\xi_i) + \phi_i(x) - \langle \xi_i, x \rangle)$, $i = 1, 2, \ldots, p$, which by Theorem 2.112 yields that $\xi_i \in \partial_{\varepsilon_i/\lambda_i} \phi_i(x)$ provided $\lambda_i > 0$, thereby leading to the conditions (2.41) and (2.42) as desired.

Conversely, suppose that the conditions (2.41) and (2.42) hold. By Theorem 2.112, (2.41) implies that for $\lambda_i > 0$,

$$\lambda_i(\phi_i^*(\xi_i) + \phi_i(x) - \langle \xi_i, x \rangle) \leq \varepsilon_i,$$

which along with (2.42) lead to

$$\sum_{i=1}^{p} \lambda_i \phi_i^*(\xi_i) + \phi(x) - \sum_{i=1}^{p} \lambda_i \langle \xi_i, x \rangle \leq \varepsilon,$$

that is, the inequality (2.43). Invoking Proposition 2.116 yields

$$\phi^*(\xi) = \sum_{i=1}^{p} \lambda_i \phi_i^*(\xi_i),$$

which along with (2.43) and Theorem 2.112 implies that $\xi \in \partial_\varepsilon \phi(x)$, thereby completing the proof. □

Remark 2.119 In the above result, applying the Scalar Product Rule, Theorem 2.117, to the condition (2.41) implies that $\tilde{\xi}_i = \lambda_i \xi_i \in \partial_{\varepsilon_i}(\lambda_i \phi_i)(x)$ provided $\lambda_i > 0$. Therefore, $\xi \in \partial_\varepsilon \phi(x)$ is such that there exist $\tilde{\xi}_i \in \partial_{\varepsilon_i}(\lambda_i \phi_i)(x)$, $i = 1, 2, \ldots, p$, satisfying

$$\xi = \sum_{i=1}^{p} \tilde{\xi}_i \quad \text{and} \quad \sum_{i=1}^{p} \varepsilon_i + \phi(x) - \sum_{i=1}^{p} \lambda_i \phi_i(x) \leq \varepsilon.$$

As $p = \min\{m, n+1\}$, we consider two cases. If $p = m$, for some $j \in \{1, 2, \ldots, p\}$, define

$$\tilde{\varepsilon}_j = \varepsilon_j + (\varepsilon - \sum_{i=1}^{p} \varepsilon_i) \quad \text{and} \quad \tilde{\varepsilon}_i = \varepsilon_i, \ i \neq j$$

and the conditions become

$$\tilde{\xi}_i \in \partial_{\varepsilon_i}(\lambda_i \phi_i)(\bar{x}) \quad \text{for every} \quad i \quad \text{satisfying} \quad \lambda_i > 0, \tag{2.44}$$

$$\xi = \sum_{i=1}^{m} \tilde{\xi}_i \quad \text{and} \quad \sum_{i=1}^{m} \varepsilon_i + \phi(\bar{x}) - \sum_{i=1}^{m} \lambda_i \phi_i(\bar{x}) = \varepsilon. \tag{2.45}$$

If $p < m$, define $\lambda_i = 0$ and $\varepsilon_i > 0$ arbitrary for $i = p+1, p+2, \ldots, m$, such that $\sum_{i=p+1}^{m} \varepsilon_i = \varepsilon - \sum_{j=1}^{p} \varepsilon_j$. As already discussed, $\partial_{\varepsilon_i}(\lambda_i \phi_i)(x) = \{0\}$, $i = p+1, p+2, \ldots, m$ and hence yield the conditions (2.44) and (2.45). Thus, if $\xi \in \partial_\varepsilon \phi(x)$, then there exist $\xi_i \in dom\ \phi_i^*$, $\varepsilon_i \geq 0$ and $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, with $\sum_{i=1}^{m} \lambda_i = 1$ such that the coditions (2.44) and (2.45) hold.

In particular, for a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $\phi^+(x) = \max\{0, \phi(x)\}$,

$$\partial_\varepsilon(\phi^+)(\bar{x}) \subset \{\partial_\eta(\lambda\phi)(\bar{x}) : 0 \leq \lambda \leq 1, \ \eta \geq 0, \ \varepsilon = \eta + \phi^+(\bar{x}) - \lambda\phi(\bar{x})\}.$$

In the results stated above, the ε-subdifferential calculus rules were expressed in terms of the ε-subdifferential itself. Below we state a result by Hiriart-Urruty and Phelps [64] relating the Sum Rule of the subdifferentials and the ε-subdifferentials.

Theorem 2.120 *Consider two proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$. Then for any $\bar{x} \in dom\ \phi_1 \cap dom\ \phi_2$,*

$$\partial(\phi_1 + \phi_2)(\bar{x}) = \bigcap_{\varepsilon > 0} cl\ (\partial_\varepsilon \phi_1(\bar{x}) + \partial_\varepsilon \phi_2(\bar{x})).$$

Proof. Suppose that $\xi_i \in \partial_\varepsilon \phi_i(\bar{x}),\ i = 1, 2$, which implies

$$\phi_i(x) - \phi_i(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle - \varepsilon,\ \forall\, x \in \mathbb{R}^n,\ i = 1, 2.$$

Therefore,

$$(\phi_1 + \phi_2)(x) - (\phi_1 + \phi_2)(\bar{x}) \geq \langle \xi_1 + \xi_2, x - \bar{x} \rangle - 2\varepsilon,\ \forall\, x \in \mathbb{R}^n,$$

that is, $\xi_1 + \xi_2 \in \partial_{2\varepsilon}(\phi_1 + \phi_2)(\bar{x})$. Because $\xi_i \in \partial_\varepsilon \phi_i(\bar{x}),\ i = 1, 2$, are arbitrary, which along with the closedness of ε-subdifferential by Proposition 2.111 yields

$$\partial_{2\varepsilon}(\phi_1 + \phi_2)(\bar{x}) \supset cl\ (\partial_\varepsilon \phi_1(\bar{x}) + \partial_\varepsilon \phi_2(\bar{x})).$$

Further, applying Proposition 2.111 leads to

$$\partial(\phi_1 + \phi_2)(\bar{x}) \supset \bigcap_{\varepsilon > 0} cl\ (\partial_\varepsilon \phi_1(\bar{x}) + \partial_\varepsilon \phi_2(\bar{x})).$$

To establish the result, we shall prove the reverse containment in the above condition. Suppose that $\bar{\xi} \in \partial(\phi_1 + \phi_2)(\bar{x})$. By Theorem 2.108,

$$(\phi_1 + \phi_2)(\bar{x}) + (\phi_1 + \phi_2)^*(\bar{\xi}) = \langle \bar{\xi}, \bar{x} \rangle,$$

which along with the Fenchel–Young inequality, Proposition 2.103 (iv), implies that

$$(\phi_1 + \phi_2)(\bar{x}) + (\phi_1 + \phi_2)^*(\bar{\xi}) \leq \langle \bar{\xi}, \bar{x} \rangle.$$

Applying the Sum Rule of conjugate functions, Theorem 2.107 (ii), to proper lsc convex functions ϕ_1 and ϕ_2 leads to

$$(\phi_1 + \phi_2)^* = cl\ (\phi_1^* \,\square\, \phi_2^*).$$

Define

$$\phi(\xi) = (\phi_1^* \,\square\, \phi_2^*)(\xi) - \langle \xi, \bar{x} \rangle,$$

which implies $cl\ \phi = cl\ (\phi_1^* \,\square\, \phi_2^*) - \langle ., \bar{x} \rangle$. By the preceding conditions, denoting $\alpha = -(\phi_1 + \phi_2)(\bar{x})$ yields $\phi(\bar{\xi}) \leq \alpha$. It is easy to observe that

$$\{\xi \in \mathbb{R}^n : cl\ \phi(\xi) \leq \alpha\} = \bigcap_{\varepsilon > 0} cl\ \{\xi \in \mathbb{R}^n : \phi(\xi) \leq \alpha + \varepsilon/2\}.$$

Therefore, for every $\varepsilon > 0$,

$$\bar{\xi} \in cl\ \{\xi \in \mathbb{R}^n : \phi(\xi) \leq \alpha + \varepsilon/2\}.$$

If $\phi(\xi) \leq \alpha + \varepsilon/2$, then

$$\begin{aligned}
\phi(\xi) - \alpha &= \inf_{\xi = \xi_1 + \xi_2} \{\phi_1^*(\xi_1) + \phi_2^*(\xi_2) - \langle \xi_1, \bar{x} \rangle - \langle \xi_2, \bar{x} \rangle + \phi_1(\bar{x}) + \phi_2(\bar{x})\} \\
&= \inf_{\xi = \xi_1 + \xi_2} \{(\phi_1^*(\xi_1) - \langle \xi_1, \bar{x} \rangle + \phi_1(\bar{x})) + (\phi_2^*(\xi_2) - \langle \xi_2, \bar{x} \rangle + \phi_2(\bar{x}))\}.
\end{aligned}$$

Therefore, there exist ξ_1, ξ_2 such that $\xi = \xi_1 + \xi_2$ and

$$(\phi_1^*(\xi_1) - \langle \xi_1, \bar{x} \rangle + \phi_1(\bar{x})) + (\phi_2^*(\xi_2) - \langle \xi_2, \bar{x} \rangle + \phi_2(\bar{x}))\} < \varepsilon.$$

By the Fenchel–Young inequality,

$$\phi_i^*(\xi_i) - \langle \xi_i, \bar{x} \rangle + \phi_i(\bar{x}) \geq 0, \ i = 1, 2,$$

which along with Definition 2.101 of the conjugate and the preceding conditions imply that

$$\langle \xi_i, x - \bar{x} \rangle - \phi(x) + \phi_i(\bar{x}) \leq \varepsilon, \ \forall \ x \in \mathbb{R}^n, \ i = 1, 2,$$

that is, $\xi_i \in \partial_{\varepsilon_i} \phi_i(\bar{x})$ for $i = 1, 2$. Thus,

$$cl \ \{\xi \in \mathbb{R}^n : \phi(\xi) \leq \alpha + \varepsilon/2\} \subset cl \ (\partial_\varepsilon \phi_1(\bar{x}) + \partial_\varepsilon \phi_2(\bar{x})),$$

which implies $\bar{\xi} \in cl \ (\partial_\varepsilon \phi_1(\bar{x}) + \partial_\varepsilon \phi_2(\bar{x}))$ for every $\varepsilon > 0$. As $\bar{\xi} \in \partial(\phi_1 + \phi_2)(\bar{x})$ was arbitrary,

$$\partial(\phi_1 + \phi_2)(\bar{x}) \subset \bigcap_{\varepsilon > 0} cl \ (\partial_\varepsilon \phi_1(\bar{x}) + \partial_\varepsilon \phi_2(\bar{x})),$$

thus establishing the result. □

Now if one goes back to the optimality condition $0 \in \partial\phi(\bar{x})$ in Theorem 2.89, it gives an equivalent characterization to the point of minimizer $\bar{x}$ of the unconstrained problem (CP_u). So one will like to know then what the condition $0 \in \partial_\varepsilon \phi(\bar{x})$ implies. As it turns out, it leads to the concept of approximate optimality conditions, which we will deal with in one of the later chapters. For now we simply state the result on approximate optimality for the unconstrained convex programming problem (CP_u).

Theorem 2.121 *Consider a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and let $\varepsilon > 0$ be given. Then $0 \in \partial_\varepsilon \phi(\bar{x})$ if and only if*

$$\phi(\bar{x}) \leq \inf_{x \in \mathbb{R}^n} \phi(x) + \varepsilon.$$

The point $\bar{x}$ is called an ε-solution of (CP_u).

In the above theorem we mentioned only one of the notions of approximate solutions, namely the ε-solution. But there are other approximate solution concepts, as we shall see later in the book, some of which are motivated by the Ekeland's Variational Principle, Theorem 2.113.

2.7 Epigraphical Properties of Conjugate Functions

With the study of conjugate function and ε-subdifferential, we are now in a position to present the relation of the epigraph of conjugate functions with the ε-subdifferentials of a convex function from Jeyakumar, Lee, and Dinh [68]. This relation plays an important part in the study of sequential optimality conditions as we shall see in the chapter devoted to its study.

Theorem 2.122 *Consider a proper lsc convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and let $\bar{x} \in dom\ \phi$. Then*

$$epi\ \phi^* = \bigcup_{\varepsilon \geq 0} \{(\xi, \langle \xi, \bar{x} - \phi(\bar{x}) + \varepsilon) : \xi \in \partial_\varepsilon \phi(\bar{x})\}.$$

Proof. Denote

$$\mathcal{F} = \bigcup_{\varepsilon \geq 0} \{(\xi, \langle \xi, \bar{x} - \phi(\bar{x}) + \varepsilon) : \xi \in \partial_\varepsilon \phi(\bar{x})\}.$$

Suppose that $(\xi, \alpha) \in epi\ \phi^*$, which implies $\phi^*(\xi) \leq \alpha$. By Definition 2.101 of the conjugate function,

$$\langle \xi, x \rangle - \phi(x) \leq \alpha,\ \forall\ x \in \mathbb{R}^n.$$

Denoting $\varepsilon = \alpha - \langle \xi, \bar{x} \rangle + \phi(\bar{x})$, the above inequality becomes

$$\begin{aligned} \phi(x) - \phi(\bar{x}) &\geq \langle \xi, x \rangle - \phi(\bar{x}) - \alpha \\ &= \langle \xi, x - \bar{x} \rangle - \varepsilon,\ \forall\ x \in \mathbb{R}^n, \end{aligned}$$

which by Definition 2.109 of the ε-subdifferential implies that $\xi \in \partial_\varepsilon \phi(\bar{x})$. Therefore, $(\xi, \alpha) \in \mathcal{F}$. Because $(\xi, \alpha) \in epi\ \phi^*$ was arbitrary, $epi\ \phi^* \subset \mathcal{F}$.

Conversely, suppose that $(\xi, \alpha) \in \mathcal{F}$, which implies there exists $\varepsilon \geq 0$ and $\bar{x} \in dom\ \partial\phi$ with

$$\xi \in \partial_\varepsilon \phi(\bar{x}) \quad \text{and} \quad \alpha = \langle \xi, \bar{x} \rangle - \phi(\bar{x}) + \varepsilon.$$

As $\xi \in \partial_\varepsilon \phi(\bar{x})$, by the definition of ε-subdifferential,

$$\phi(x) - \phi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon,\ \forall\ x \in \mathbb{R}^n,$$

which by the definition of conjugate function leads to

$$\phi^*(\xi) \leq \langle \xi, \bar{x} \rangle - \phi(\bar{x}) + \varepsilon = \alpha.$$

Thus, $(\xi, \alpha) \in epi\ \phi^*$. Because $(\xi, \alpha) \in \mathcal{F}$ was arbitrary, $epi\ \phi^* \supset \mathcal{F}$, thereby establishing the result. $\square$

Next we discuss the epigraphical conditions for the operations of the conjugate functions.

Theorem 2.123 *(i) (Sum Rule) Consider proper lsc convex functions* $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2, \ldots, m$. *Then*

$$epi(\phi_1 + \phi_2 + \ldots + \phi_m)^* = cl\ (epi\ \phi_1^* + epi\ \phi_2^* + \ldots + epi\ \phi_m^*).$$

(ii) (Supremum Rule) Consider a family of proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i \in I$, *where* I *is an arbitrary index set. Then*

$$epi\ (\sup_{i \in I} \phi_i)^* = cl\ co\ \bigcup_{i \in I} epi\ \phi_i^*.$$

(iii) Consider proper lsc convex functions $\phi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$. *Define a vector-valued convex function* $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, *defined as* $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_m(x))$. *Then*

$$\bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\lambda\Phi)^* \quad \text{is a convex cone.}$$

(iv) Consider a proper lsc convex function $\phi : \mathbb{R}^n \to \mathbb{R}$. *Then for every* $\lambda > 0$, $epi\ (\lambda\phi)^* = \lambda\ epi\ \phi^*$.

Proof. (i) As $\phi_i,\ i = 1, 2, \ldots, m$ are proper lsc convex functions, the condition of Theorem 2.107 (ii) reduces to

$$(\phi_1 + \phi_2 + \ldots + \phi_m)^* = cl\ (\phi_1^* \,\square\, \phi_2^* \,\square \ldots \square\, \phi_m^*),$$

which implies

$$epi\ (\phi_1 + \phi_2 + \ldots + \phi_m)^* = epi\ cl\ (\phi_1^* \,\square\, \phi_2^* \,\square \ldots \square\, \phi_m^*).$$

By Definition 1.11 of the closure of a function, the above condition becomes

$$epi\ (\phi_1 + \phi_2 + \ldots + \phi_m)^* = cl\ epi\ (\phi_1^* \,\square\, \phi_2^* \,\square \ldots \square\, \phi_m^*),$$

which by Proposition 2.55 leads to the desired condition.

(ii) Theorem 2.107 (iv) along with Definition 2.57 of the closed convex hull of a function implies that

$$epi\ (\sup_{i \in I} \phi_i)^* = epi\ cl\ co\ (\inf_{i \in I} \phi_i^*) = cl\ co \bigcup_{i \in I} epi\ \phi_i^*,$$

thereby establishing the result.

(iii) Suppose that $(\xi, \alpha) \in \bigcup_{\lambda \in \mathbb{R}^m_+} (\lambda\Phi)^*$, which implies that there exists $\lambda' \in \mathbb{R}^m_+$ such that $(\xi, \alpha) \in epi\ (\lambda'\Phi)^*$. This along with Definition 2.101 of the conjugate function leads to

$$\langle \xi, x \rangle - (\lambda'\Phi)(x) \leq (\lambda'\Phi)^*(\xi) \leq \alpha,\ \forall\ x \in \mathbb{R}^n.$$

Multiplying throughout by any $\gamma > 0$,

$$\langle \gamma\xi, x\rangle - ((\gamma\lambda')\Phi)(x) \leq \gamma\alpha, \ \forall\, x \in \mathbb{R}^n,$$

where $\gamma\lambda' \in \mathbb{R}^m_+$. Again, by the definition of conjugate function, the above condition leads to

$$((\gamma\lambda')\Phi)^*(\gamma\xi) \leq \gamma\alpha,$$

which implies that

$$\gamma(\xi,\alpha) \in epi\ ((\gamma\lambda')\Phi)^* \subset \bigcup_{\lambda\in\mathbb{R}^m_+} epi\ (\lambda\Phi)^*, \ \forall\, \gamma > 0.$$

Hence, $\bigcup_{\lambda\in\mathbb{R}^m_+} epi\ (\lambda\Phi)^*$ is a cone.

Now consider $(\xi_i,\alpha_i) \in \bigcup_{\lambda\in\mathbb{R}^m_+} epi\ (\lambda\Phi)^*,\ i = 1,2$, which implies there exist $\lambda_i \in \mathbb{R}^m_+$ such that $(\xi_i,\alpha_i) \in epi\ (\lambda_i\Phi)^*$ for $i = 1,2$. Therefore, by the definition of conjugate function, for every $x \in \mathbb{R}^n$,

$$\langle \xi_i, x\rangle - (\lambda_i\Phi)(x) \leq \alpha_i,\ i = 1,2.$$

For any $\gamma \in [0,1]$, the above condition leads to

$$\langle (1-\gamma)\xi_1 + \gamma\xi_2, x\rangle - (\lambda'\Phi)(x) \leq (1-\gamma)\alpha_1 + \gamma\alpha_2,\ \forall\, x \in \mathbb{R}^n,$$

where $\lambda' = (1-\gamma)\lambda_1 + \gamma\lambda_2 \in \mathbb{R}^m_+$. Therefore,

$$(\lambda'\Phi)^*((1-\gamma)\xi_1 + \gamma\xi_2) \leq (1-\gamma)\alpha_1 + \gamma\alpha_2,$$

which implies that

$$(1-\gamma)(\xi_1,\alpha_1) + \gamma(\xi_2,\alpha_2) \in epi\ (\lambda'\Phi)^* \subset \bigcup_{\lambda\in\mathbb{R}^m_+} epi\ (\lambda\Phi)^*,\ \forall\, \gamma \in [0,1].$$

Because $(\xi_i,\alpha_i),\ i = 1,2$, were arbitrary, thus $\bigcup_{\lambda\in\mathbb{R}^m_+} epi\ (\lambda\Phi)^*$ is a convex set.

(iv) Suppose that $(\xi,\alpha) \in epi\ (\lambda\phi)^*$, which implies that $(\lambda\phi)^*(\xi) \leq \alpha$. As $\lambda > 0$, Proposition 2.103 (iii) leads to

$$\phi^*\left(\frac{\xi}{\lambda}\right) \leq \frac{\alpha}{\lambda},$$

which implies $\left(\frac{\xi}{\lambda},\frac{\alpha}{\lambda}\right) \in epi\ \phi^*$, that is, $(\xi,\alpha) \in \lambda\ epi\ \phi^*$. Because $(\xi,\alpha) \in epi\ (\lambda\phi)^*$ was arbitrary, $epi\ (\lambda\phi)^* \subset \lambda epi\ \phi^*$. The reverse inclusion can be obtained by following the steps backwards, thereby establishing the result. $\square$

From the above theorem, for two proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$,

$$epi(\phi_1 + \phi_2)^* = cl\ (epi\ \phi_1^* + epi\ \phi_2^*).$$

In general, $epi\ \phi_1^* + epi\ \phi_2^*$ need not be closed. But under certain additional conditions, it can be shown that $epi\ \phi_1^* + epi\ \phi_2^*$ is closed. We present below the result from Burachik and Jeyakumar [20] and Dinh, Goberna, López, and Son [32] to establish the same.

Proposition 2.124 *Consider two proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$, such that $dom\ \phi_1 \cap dom\ \phi_2 \neq \emptyset$. If $cone(dom\ \phi_1 - dom\ \phi_2)$ is a closed subspace or at least one of the functions is continuous at some point in $dom\ \phi_1 \cap dom\ \phi_2$, then $epi\ \phi_1^* + epi\ \phi_2^*$ is closed.*

Proof. As $cone(dom\ \phi_1 - dom\ \phi_2)$ is a closed subspace, by Theorem 1.1 of Attouch and Brézis [2] or Theorem 3.6 of Strömberg [107], the exact infimal convolution holds, that is,

$$(\phi_1 + \phi_2)^* = \phi_1^* \,\square\, \phi_2^*.$$

The above condition along with Theorem 2.123 (i) leads to

$$cl\ (epi\ \phi_1^* + epi\ \phi_2^*) = epi\ (\phi_1 + \phi_2)^* = epi\ (\phi_1^* \,\square\, \phi_2^*) = epi\ \phi_1^* + epi\ \phi_2^*,$$

thereby yielding the result that $epi\ \phi_1^* + epi\ \phi_2^*$ is closed.

Suppose that ϕ_1 is continuous at $\hat{x} \in dom\ \phi_1 \cap dom\ \phi_2$, which yields

$$0 \in core\ (dom\ \phi_1 - dom\ \phi_2).$$

This implies that $cone\ (dom\ \phi_1 - dom\ \phi_2)$ is a closed subspace and thus leads to the desired result. $\square$

Note that the result gives only sufficient condition for the closedness of $epi\ \phi_1^* + epi\ \phi_2^*$. The converse need not be true. For a better understanding, we consider the following example from Burachik and Jeyakumar [20]. Let $\phi_1 = \delta_{[0,\infty)}$ and $\phi_2 = \delta_{(-\infty,0]}$. Therefore,

$$epi\ \phi_1^* = epi\ \sigma_{[0,\infty)} = \mathbb{R}_- \times \mathbb{R}_+ \quad \text{and} \quad epi\ \phi_2^* = epi\ \sigma_{(-\infty,0]} = \mathbb{R}_+ \times \mathbb{R}_+,$$

which leads to $epi\ \phi_1^* + epi\ \phi_2^* = \mathbb{R} \times \mathbb{R}_+$, a closed convex cone. Observe that $cone(dom\ \phi_1 - dom\ \phi_2) = [0, \infty)$, which is not a subspace, and also neither ϕ_1 nor ϕ_2 are continuous at $dom\ \phi_1 \cap dom\ \phi_2 = \{0\}$. Thus, the condition, $epi\ \phi_1^* + epi\ \phi_2^*$ is closed, is a relaxed condition in comparison to the other assumptions.

Using this closedness assumption, Burachik and Jeyakumar [21] obtained an equivalence between the exact inf-convolution and ε-subdifferential Sum Rule, which we present next.

Theorem 2.125 *Consider two proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$, such that dom $\phi_1 \cap$ dom $\phi_2 \neq \emptyset$. Then the following are equivalent:*

(i) $(\phi_1 + \phi_2)^* = \phi_1^* \,\square\, \phi_2^*$ *with exact infimal convolution,*

(ii) For every $\varepsilon \geq 0$ and every $\bar{x} \in$ dom $\phi_1 \cap$ dom ϕ_2,

$$\partial_\varepsilon(\phi_1 + \phi_2)(\bar{x}) = \bigcup_{\substack{\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1}\phi_1(\bar{x}) + \partial_{\varepsilon_2}\phi_2(\bar{x})).$$

(iii) epi ϕ_1^* + epi ϕ_2^* is closed,

Proof. (i) $\Rightarrow$ (ii): The proof follows along the lines of Theorem 2.115.

(ii) $\Rightarrow$ (iii): Suppose that $(\xi, \gamma) \in$ *cl* (*epi* ϕ_1^* + *epi* ϕ_2^*). By Theorem 2.123 (i), $(\xi, \gamma) \in$ *epi* $(\phi_1 + \phi_2)^*$. Let $\bar{x} \in$ *dom* $\phi_1 \cap$ *dom* ϕ_2. By Theorem 2.122, there exists $\varepsilon \geq 0$ such that

$$\xi \in \partial_\varepsilon(\phi_1 + \phi_2)(\bar{x}) \quad \text{and} \quad \gamma = \langle \xi, \bar{x} \rangle - (\phi_1 + \phi_2)(\bar{x}) + \varepsilon.$$

By (ii), there exist $\varepsilon_i \geq 0$ and $\xi_i \in \partial_{\varepsilon_i}\phi_i(\bar{x}),\ i = 1, 2$, such that

$$\xi = \xi_1 + \xi_2 \quad \text{and} \quad \varepsilon = \varepsilon_1 + \varepsilon_2.$$

Define $\gamma_i = \langle \xi_i, \bar{x} \rangle - \phi_i(\bar{x}) + \varepsilon_i,\ i = 1, 2$. Then from Theorem 2.122, for $i = 1, 2$, $(\xi_i, \gamma_i) \in$ *epi* ϕ_i^*, which implies

$$(\xi, \gamma) = (\xi_1, \gamma_1) + (\xi_2, \gamma_2) \in epi\ \phi_1^* + epi\ \phi_2^*,$$

thereby leading to (iii).

(iii) $\Rightarrow$ (i): Suppose that there exists $\xi \in \mathbb{R}^n$ such that $\xi \in$ *dom* $(\phi_1 + \phi_2)^*(\xi)$. Otherwise (i) holds trivially. By (iii),

$$epi\ (\phi_1 + \phi_2)^* = cl\ (epi\ \phi_1^* + epi\ \phi_2^*) = epi\ \phi_1^* + epi\ \phi_2^*,$$

which implies $(\xi, (\phi_1 + \phi_2)^*(\xi)) \in$ *epi* ϕ_1^* + *epi* ϕ_2^*. Thus for $i = 1, 2$, there exist $(\xi_i, \gamma_i) \in$ *epi* ϕ_i^* such that

$$\xi = \xi_1 + \xi_2 \quad \text{and} \quad (\phi_1 + \phi_2)^* = \gamma_1 + \gamma_2,$$

which implies there exists $\bar{\xi} \in \mathbb{R}^n$ such that

$$\phi_1^*(\xi - \bar{\xi}) + \phi_2^*(\bar{\xi}) \leq (\phi_1 + \phi_2)^*(\xi).$$

Therefore,

$$(\phi_1^* \,\square\, \phi_2^*)(\xi) \leq \phi_1^*(\xi - \bar{\xi}) + \phi_2^*(\bar{\xi}) \leq (\phi_1 + \phi_2)^*(\xi).$$

By Theorem 2.107 and (iii),

$$(\phi_1 + \phi_2)^*(\xi) = cl\ (\phi_1^* \,\square\, \phi_2^*)(\xi) \leq (\phi_1^* \,\square\, \phi_2^*)(\xi),$$

which along with the preceding condition leads to the exact infimal convolution, thereby establishing (i). $\square$

Though it is obvious that under the closedness of $epi\ \phi_1^* + epi\ \phi_2^*$, one can obtain the subdifferential Sum Rule by choosing $\varepsilon = 0$ in (ii) of the above theorem, we present a detailed version of the result from Burachik and Jeyakumar [20]. Below is an alternative approach to the Sum Rule, Theorem 2.91.

Theorem 2.126 *Consider two proper lsc convex functions $\phi_i : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$, $i = 1, 2$, such that $dom\ \phi_1 \cap dom\ \phi_2 \neq \emptyset$. If $epi\ \phi_1^* + epi\ \phi_2^*$ is closed, then*

$$\partial(\phi_1 + \phi_2)(\bar{x}) = \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}),\ \forall\ \bar{x} \in dom\ \phi_1 \cap dom\ \phi_2.$$

Proof. Let $\bar{x} \in dom\ \phi_1 \cap dom\ \phi_2$. It is easy to observe that

$$\partial(\phi_1 + \phi_2)(\bar{x}) \supset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}).$$

To prove the result, we shall prove the converse inclusion. Suppose that $\xi \in \partial(\phi_1 + \phi_2)(\bar{x})$. By Theorem 2.108,

$$(\phi_1 + \phi_2)^*(\xi) + (\phi_1 + \phi_2)(\bar{x}) = \langle \xi, \bar{x} \rangle.$$

Therefore, the above condition along with the given hypothesis

$$(\xi, \langle \xi, \bar{x} \rangle - (\phi_1 + \phi_2)(\bar{x})) \in epi\ (\phi_1 + \phi_2)^* = epi\ \phi_1^* + epi\ \phi_2^*,$$

which implies that there exist $(\xi_i, \gamma_i) \in epi\ \phi_i^*,\ i = 1, 2$, such that

$$\xi = \xi_1 + \xi_2 \quad \text{and} \quad \langle \xi, \bar{x} \rangle - (\phi_1 + \phi_2)(\bar{x}) = \gamma_1 + \gamma_2.$$

Also, as $(\xi_i, \gamma_i) \in epi\ \phi_i^*,\ i = 1, 2$, which along with the above conditions lead to

$$\phi_1^*(\xi_1) + \phi_2^*(\xi_2) \leq \langle \xi, \bar{x} \rangle - (\phi_1 + \phi_2)(\bar{x}) = \langle \xi_1, \bar{x} \rangle + \langle \xi_2, \bar{x} \rangle - \phi_1(\bar{x}) - \phi_2(\bar{x}).$$

By the Fenchel–Young inequality, Proposition 2.103 (iv),

$$\phi_1^*(\xi_1) + \phi_2^*(\xi_2) \geq \langle \xi_1, \bar{x} \rangle + \langle \xi_2, \bar{x} \rangle - \phi_1(\bar{x}) - \phi_2(\bar{x}),$$

which together with the preceding inequality leads to

$$\phi_1^*(\xi_1) + \phi_2^*(\xi_2) = \langle \xi_1, \bar{x} \rangle + \langle \xi_2, \bar{x} \rangle - \phi_1(\bar{x}) - \phi_2(\bar{x}).$$

Again by the Fenchel–Young inequality and the above equation,

$$\phi_1^*(\xi_1) + \phi_1(\bar{x}) - \langle \xi_1, \bar{x} \rangle = \langle \xi_2, \bar{x} \rangle - \phi_2(\bar{x}) - \phi_2^*(\xi_2) \leq 0,$$

which by Theorem 2.108 yields $\xi_1 \in \partial\phi_1(\bar{x})$. Along similar lines it can be obtained that $\xi_2 \in \partial\phi_2(\bar{x})$. Thus,

$$\xi = \xi_1 + \xi_2 \in \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}),$$

which implies that

$$\partial(\phi_1 + \phi_2)(\bar{x}) \subset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}),$$

thereby leading to the desired result. □

We end this chapter with an application of Theorem 2.126 to provide an alternative assumption to establish equality in Proposition 2.39 (i).

Corollary 2.127 *Consider convex sets $F_1,\ F_2 \subset \mathbb{R}^n$ such that $F_1 \cap F_2 \neq \emptyset$. If epi σ_{F_1} + epi σ_{F_2} is closed, then*

$$N_{F_1 \cap F_2}(\bar{x}) = N_{F_1}(\bar{x}) + N_{F_2}(\bar{x}),\ \forall\ \bar{x} \in F_1 \cap F_2.$$

Proof. We know that for any convex set $F \subset \mathbb{R}^n$, $\delta_F^* = \sigma_F$. Thus the condition

$$\mathit{epi}\ \sigma_{F_1} + \mathit{epi}\ \sigma_{F_2} \quad \text{is closed}$$

is equivalent to

$$\mathit{epi}\ \delta_{F_1}^* + \mathit{epi}\ \delta_{F_2}^* \quad \text{is closed.}$$

As the condition for Theorem 2.126 to hold is satisfied,

$$\partial(\delta_{F_1} + \delta_{F_2})(\bar{x}) = \partial\delta_{F_1}(\bar{x}) + \partial\delta_{F_2}(\bar{x}),\ \bar{x} \in \mathit{dom}\ \delta_{F_1} \cap \mathit{dom}\ \delta_{F_2}.$$

Because $\delta_{F_1} + \delta_{F_2} = \delta_{F_1 \cap F_2}$, the above equality condition along with the fact that for any convex set $F \subset \mathbb{R}^n$, $\partial\delta_F = N_F$, yields the desired result, that is,

$$N_{F_1 \cap F_2}(\bar{x}) = N_{F_1}(\bar{x}) + N_{F_2}(\bar{x}),\ \forall\ \bar{x} \in F_1 \cap F_2. \qquad \square$$

Chapter 3

Basic Optimality Conditions Using the Normal Cone

3.1 Introduction

Recall the convex optimization problem presented in Chapter 1

$$\min\ f(x) \quad \text{subject to} \quad x \in C, \tag{CP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $C \subset \mathbb{R}^n$ is a convex set. It is natural to think that $f'(x, h)$ and $\partial f(x)$ will play a major role in the process of establishing the optimality conditions as these objects have been successful in overcoming the difficulty posed by the absence of a derivative. In this chapter we will not bother ourselves with extended-valued function but such a framework can be easily adapted into the current framework. But use of extended-valued convex functions might appear while doing some of the proofs, as one will need to use the calculus rules for subdifferentials like the Sum Rule or the Chain Rule. To begin our discussion more formally, we right away state the following basic result.

Theorem 3.1 *Consider the convex optimization problem (CP). Then $\bar{x}$ is a point of minimizer of (CP) if and only if either of two conditions hold:*

(i) $f'(\bar{x}, d) \geq 0,\ \forall\ d \in T_C(\bar{x})$ *or,*

(ii) $0 \in \partial f(\bar{x}) + N_C(\bar{x})$.

Proof. (i) As $\bar{x} \in C$ and C is a convex set,

$$\bar{x} + \lambda(x - \bar{x}) \in C, \forall\ x \in C,\ \forall\ \lambda \in [0, 1].$$

Also, as $\bar{x}$ is a point of minimizer of (CP), then for every $x \in C$

$$f(\bar{x} + \lambda(x - \bar{x})) \geq f(\bar{x}),\ \forall\ \lambda \in [0, 1].$$

Therefore,

$$\lim_{\lambda \downarrow 0} \frac{f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x})}{\lambda} \geq 0,$$

which implies

$$f'(\bar{x}, x - \bar{x}) \geq 0, \ \forall \ x \in C.$$

By Theorem 2.76, the directional derivative is sublinear in the direction and thus

$$f'(\bar{x}, d) \geq 0, \ \forall \ d \in cl \ cone(C - \bar{x}).$$

By Theorem 2.35, $T_C(\bar{x}) = cl \ cone(C - \bar{x})$ and therefore, the above inequality becomes

$$f'(\bar{x}, d) \geq 0, \ \forall \ d \in T_C(\bar{x}).$$

Conversely, suppose condition (i) holds. As $f : \mathbb{R}^n \to \mathbb{R}$, applying Proposition 2.83 and Theorem 2.79, for any $d \in \mathbb{R}^n$ there exists $\xi \in \partial f(\bar{x})$ such that

$$\langle \xi, d \rangle = f'(\bar{x}, d).$$

For every $x \in C$, $x - \bar{x} \in T_C(\bar{x})$. Therefore, the convexity of f along with the above condition and condition (i) implies that for every $x \in C$, there exists $\xi \in \partial f(\bar{x})$ such that

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle \geq 0, \ \forall \ x \in C,$$

thereby proving that $\bar{x}$ is the point of minimizer of f over C.

(ii) As $\bar{x}$ is a point of minimizer of f over C, we have that $\bar{x}$ also solves the problem

$$\min_{x \in \mathbb{R}^n} (f + \delta_C)(x).$$

Hence, by the optimality conditions for the unconstrained optimization problem, Theorem 2.89,

$$0 \in \partial(f + \delta_C)(\bar{x}).$$

Because $\bar{x} \in dom \ \delta_C$, by Proposition 2.14, $ri \ dom \ \delta_C = ri \ C$ is nonempty. Also $ri \ dom \ f = \mathbb{R}^n$ and hence $ri \ dom \ f \cap ri \ dom \ \delta_C \neq \emptyset$. Now using the Sum Rule, Theorem 2.91,

$$0 \in \partial f(\bar{x}) + \partial \delta_C(\bar{x}),$$

which by the fact that $\partial \delta_C(x) = N_C(x)$ leads to

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}).$$

Conversely, suppose that condition (ii) is satisfied, which means that there exists $\xi \in \partial f(\bar{x})$ such that $-\xi \in N_C(\bar{x})$, that is,

$$\langle \xi, x - \bar{x} \rangle \geq 0, \ \forall\, x \in C.$$

Therefore by the convexity of f, which along with the above inequality yields

$$f(x) \geq f(\bar{x}), \ \forall\, x \in C,$$

thereby leading to the desired result. □

By condition (ii) of the above theorem, there exists $\xi \in \partial f(\bar{x})$ such that

$$\langle -\xi, x \rangle \leq \langle -\xi, \bar{x} \rangle, \ \forall\, x \in C.$$

As $\bar{x} \in C$, the above condition yields that the support function to the set C at $-\xi$ is given by

$$\sigma_C(-\xi) = -\langle \xi, \bar{x} \rangle.$$

Thus, condition (ii) is equivalent to the above condition.

Again, by condition (ii) of Theorem 3.1, there exists $\xi \in \partial f(\bar{x})$ such that $-\xi \in N_C(\bar{x})$, which can be equivalently expressed as

$$\langle (\bar{x} - \alpha\xi) - \bar{x}, x - \bar{x} \rangle \leq 0, \ \forall\, x \in C, \ \forall\, \alpha \geq 0.$$

Therefore, by Proposition 2.52, condition (ii) is equivalent to

$$\bar{x} = proj_C(\bar{x} - \alpha\xi), \ \forall\, \alpha \geq 0.$$

We state the above discussion as the following result.

Theorem 3.2 *Consider the convex optimization problem (CP). Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exists $\xi \in \partial f(\bar{x})$ such that*

$$\textit{either} \quad \sigma_C(-\xi) = -\langle \xi, \bar{x} \rangle \quad \textit{or} \quad \bar{x} = proj_C(\bar{x} - \alpha\xi), \ \forall\, \alpha \geq 0.$$

3.2 Slater Constraint Qualification

Now consider the case where C is represented only through convex inequality constraints. Observe that the equality affine constraints of the form

$$h_j(x) = 0, \ j = 1, 2, \ldots, l,$$

where $h_j : \mathbb{R}^n \to \mathbb{R}, \ j = 1, 2, \ldots, l$, are affine functions can also be expressed in the convex inequality form as

$$h_j(x) \leq 0, \ j = 1, 2, \ldots, l,$$
$$-h_j(x) \leq 0, \ j = 1, 2, \ldots, l.$$

Thus, we define

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}, \tag{3.1}$$

where $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$, are convex functions. In practice, this is most often the case. In order to write an explicit optimality condition we need to compute $N_C(\bar{x})$ and express it in terms of the constraint functions $g_i,\ i = 1, 2, \ldots, m$. So how do we do that? In this respect, we present the following result.

Proposition 3.3 *Consider the set C as in (3.1). Assume that the active index set at $\bar{x}$, that is,*

$$I(\bar{x}) = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = 0\}$$

is nonempty. Let the Slater constraint qualification hold, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$, for $i = 1, 2, \ldots, m$. Then

$$N_C(\bar{x}) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i \xi_i \in \mathbb{R}^n : \xi_i \in \partial g_i(\bar{x}),\ \lambda_i \geq 0,\ i \in I(\bar{x}) \right\}.$$

In order to prove the above proposition, we need to do a bit of work, which we will do step by step. Denote the set on the right-hand side of the equality by

$$\widehat{S}(\bar{x}) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i \xi_i \in \mathbb{R}^n : \xi_i \in \partial g_i(\bar{x}),\ \lambda_i \geq 0,\ i \in I(\bar{x}) \right\}. \tag{3.2}$$

One might get curious as to what are these $\lambda_i,\ i = 1, 2, \ldots, m$, in the expression of the elements of $\widehat{S}(\bar{x})$. These are the *Lagrange multipliers*, vital stuff in optimization and that we need to discuss more in detail. In order to establish Proposition 3.3, that is, $N_C(\bar{x}) = \widehat{S}(\bar{x})$, we will prove that $\widehat{S}(\bar{x})$ is a closed convex cone for which we need the following lemma whose proof is as given in van Tiel [110].

Proposition 3.4 *Consider a nonempty compact set $A \subset \mathbb{R}^n$ with $0 \notin A$. Let K be the cone generated by A, that is,*

$$K = cone A = \{\lambda a \in \mathbb{R}^n : \lambda \geq 0,\ a \in A\}.$$

Then K is a closed set.

Proof. Consider a sequence $\{x_k\} \subset K$ such that $x_k \to \tilde{x}$. To prove the result, we need to show that $\tilde{x} \in K$. As $x_k \in K$, there exist $\lambda_k \geq 0$ and $a_k \in A$ such that $x_k = \lambda_k a_k$ for every k. Because A is compact, $\{a^\nu\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, $\{a_k\}$ has a convergent subsequence. Thus, without loss of generality, let $a_k \to \tilde{a}$ and as A is closed, $\tilde{a} \in A$. Because $0 \notin A$, it is simple to observe that there exists $\alpha > 0$ such that $\|a\| \geq \alpha$ for every $a \in A$. Hence,

$$|\lambda_k| = \frac{|\lambda_k|\|a_k\|}{\|a_k\|} \leq \frac{1}{\alpha}\|\lambda_k a_k\|.$$

As $\lambda_k a_k \to \tilde{x}$, $\|\lambda_k a_k\|$ is bounded, thereby implying that $\{\lambda_k\}$ is a bounded sequence, that by the Bolzano–Weierstrass Theorem, Proposition 1.3, has a convergent subsequence. Without loss of generality, assume that $\lambda_k \to \tilde{\lambda}$. This shows that

$$x_k = \lambda_k a_k \to \tilde{\lambda}\tilde{a} \quad \text{as} \quad k \to +\infty.$$

By the assumption $x_k \to \tilde{x}$ and as the limit is unique, $\tilde{\lambda}\tilde{a} = \tilde{x}$. Hence $\tilde{x} \in K$, thereby establishing the result. $\square$

We will now show that the set $\widehat{S}(\bar{x})$ is a closed convex cone. This fact will play a major role in the proof of Proposition 3.3.

Lemma 3.5 *Assume that $I(\bar{x})$ is nonempty and the Slater constraint qualification holds. Then the set $\widehat{S}(\bar{x})$ given by (3.2) is a closed convex cone.*

Proof. Observe that $\widehat{S}(\bar{x})$ is a cone. To prove the convexity of $\widehat{S}(\bar{x})$, let $v_1, v_2(\neq 0) \in \widehat{S}(\bar{x})$. Then $v_j = \sum_{i \in I(\bar{x})} \lambda_i^j \xi_i^j$ where $\lambda_i^j \geq 0$ and $\xi_i^j \in \partial g_i(\bar{x})$, $i \in I(\bar{x})$ for $j = 1, 2$. As $\widehat{S}(\bar{x})$ is a cone, to show that it is convex, by Theorem 2.20 we just have to show that $v_1 + v_2 \in \widehat{S}(\bar{x})$. Consider

$$\begin{aligned} v_1 + v_2 &= \sum_{i \in I(\bar{x})} \left(\lambda_i^1 \xi_i^1 + \lambda_i^2 \xi_i^2\right) \\ &= \sum_{i \in I(\bar{x})} (\lambda_i^1 + \lambda_i^2) \left\{ \frac{\lambda_i^1}{\lambda_i^1 + \lambda_i^2} \xi_i^1 + \frac{\lambda_i^2}{\lambda_i^1 + \lambda_i^2} \xi_i^2 \right\}. \end{aligned}$$

Because $\partial g_i(\bar{x})$ is a convex set,

$$\frac{\lambda_i^1}{\lambda_i^1 + \lambda_i^2} \xi_i^1 + \frac{\lambda_i^2}{\lambda_i^1 + \lambda_i^2} \xi_i^2 \in \partial g_i(\bar{x}).$$

Hence, $v_1 + v_2 \in \widehat{S}(\bar{x})$.

Finally, we have to show that $\widehat{S}(\bar{x})$ is closed. Consider the function

$$g(x) = \max\{g_1(x),\ g_2(x), \ldots, g_m(x)\}.$$

Moreover, as $I(\bar{x})$ is nonempty, $g(\bar{x}) = 0$ with $J(\bar{x}) = I(\bar{x})$, where

$$J(\bar{x}) = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = g(\bar{x})\}.$$

Further, from the Max-Function Rule, Theorem 2.96,

$$\partial g(\bar{x}) = co\left(\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})\right). \tag{3.3}$$

We claim that $\widehat{S}(\bar{x}) = cone(\partial g(\bar{x}))$, that is,

$$\widehat{S}(\bar{x}) = \{\lambda\xi \in \mathbb{R}^n : \lambda \geq 0,\ \xi \in \partial g(\bar{x})\}. \tag{3.4}$$

But before showing that $\widehat{S}(\bar{x})$ is given as above and applying Proposition 3.4 to conclude that $\widehat{S}(\bar{x})$ is closed, we first need to show that $0 \notin \partial g(\bar{x})$.

As the Slater constraint qualification holds, there exists $\hat{x}$ such that $g_i(\hat{x}) < 0$ for every $i = 1, 2, \ldots, m$. Hence $g(\hat{x}) < 0$. By the convexity of g,

$$\langle \xi, \hat{x} - \bar{x}\rangle \leq g(\hat{x}) - g(\bar{x}),\ \forall\, \xi \in \partial g(\bar{x}).$$

Because $J(\bar{x}) = I(\bar{x})$ is nonempty, for every $\xi \in \partial g(\bar{x})$,

$$\langle \xi, \hat{x} - \bar{x}\rangle < 0.$$

As $\hat{x} \neq \bar{x}$, it is clear that $0 \notin \partial g(\bar{x})$. Otherwise, if $0 \in \partial g(\bar{x})$, the above inequality will be violated. Hence, observe that $0 \notin \partial g_i(\bar{x})$ for every $i \in J(\bar{x}) = I(\bar{x})$.

Because $\widehat{S}(\bar{x})$ is a cone, $0 \in \widehat{S}(\bar{x})$. For $\lambda = 0$,

$$0 \in \{\lambda\xi \in \mathbb{R}^n : \lambda \geq 0,\ \xi \in \partial g(\bar{x})\}.$$

Consider $v \in \widehat{S}(\bar{x})$ with $v \neq 0$. We will show that

$$v \in \{\lambda\xi \in \mathbb{R}^n : \lambda \geq 0,\ \xi \in \partial g(\bar{x})\}.$$

As $v \in \widehat{S}(\bar{x})$, there exist $\lambda_i \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$, $i \in I(\bar{x})$ such that $v = \sum_{i \in I(\bar{x})} \lambda_i \xi_i$. Because $v \neq 0$ and $0 \notin \partial g_i(\bar{x})$ for all $i \in I(\bar{x})$, it is clear that all the $\lambda_i,\ i \in I(\bar{x})$ cannot be simultaneously zero and hence $\sum_{i \in I(\bar{x})} \lambda_i > 0$. Let $\alpha = \sum_{i \in I(\bar{x})} \lambda_i$ and thus $\sum_{i \in I(\bar{x})} \lambda_i/\alpha = 1$. Therefore,

$$\frac{1}{\alpha} v = \sum_{i \in I(\bar{x})} \frac{\lambda_i}{\alpha} \xi_i \in co\left(\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})\right),$$

which by (3.3) implies that $v \in \alpha\, \partial g(\bar{x})$. Hence,

$$\widehat{S}(\bar{x}) \subseteq \{\lambda\xi \in \mathbb{R}^n : \lambda \geq 0,\ \xi \in \partial g(\bar{x})\}.$$

Conversely, consider $v \in \{\lambda\xi \in \mathbb{R}^n : \lambda \geq 0,\ \xi \in \partial g(\bar{x})\}$ with $v \neq 0$.

Therefore, $v = \lambda\xi$ for some $\lambda \geq 0,\ \xi \in \partial g(\bar{x})$. The condition (3.3) yields that there exist $\mu_i \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$ for $i \in I(\bar{x})$ such that

$$\xi = \sum_{i \in I(\bar{x})} \mu_i \xi_i$$

with $\sum_{i \in I(\bar{x})} \mu_i = 1$. Therefore,

$$v = \sum_{i \in I(\bar{x})} \lambda \mu_i \xi_i = \sum_{i \in I(\bar{x})} \lambda'_i \xi_i,$$

where $\lambda'_i = \lambda\mu_i \geq 0$ for $i \in I(\bar{x})$. Hence, $v \in \widehat{S}(\bar{x})$. Because v was arbitrary, (3.4) holds, which along with the fact that $0 \notin \partial g(\bar{x})$ and Proposition 3.4 yields that $\widehat{S}(\bar{x})$ is closed. $\square$

Remark 3.6 It may be noted here that $\widehat{S}(\bar{x})$ is proved to be closed under the Slater constraint qualification, which is equivalent to

$$0 \notin co \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}).$$

This observation was made by Wolkowicz [112]. In the absence of such conditions, $\widehat{S}(\bar{x})$ need not be closed.

Now we turn to establish Proposition 3.3 according to which, if the Slater constraint qualification holds, then

$$N_C(\bar{x}) = \widehat{S}(\bar{x}).$$

Proof of Proposition 3.3. First we will prove that $\widehat{S}(\bar{x}) \subseteq N_C(\bar{x})$. Consider any $v \in \widehat{S}(\bar{x})$. Thus, there exist $\lambda_i \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$ for $i \in I(\bar{x})$ such that $v = \sum_{i \in I(\bar{x})} \lambda_i \xi_i$. Hence, for any $x \in C$,

$$\langle v, x - \bar{x} \rangle = \sum_{i \in I(\bar{x})} \lambda_i \langle \xi_i, x - \bar{x} \rangle.$$

By the convexity of $g_i,\ i \in I(\bar{x})$,

$$\langle \xi_i, x - \bar{x} \rangle \leq g_i(x) - g_i(\bar{x}) \leq 0,\ \forall\, x \in C.$$

Thus $\langle v, x - \bar{x} \rangle \leq 0$ for every $x \in C$, thereby showing that $v \in N_C(\bar{x})$.

Conversely, suppose that $v \in N_C(\bar{x})$. We have to show that $v \in \widehat{S}(\bar{x})$. On the contrary, assume that $v \notin \widehat{S}(\bar{x})$. As $\widehat{S}(\bar{x})$ is a closed convex cone, by the strict separation theorem, Theorem 2.26 (iii), there exists $w \in \mathbb{R}^n$ with $w \neq 0$ such that

$$\langle w, \xi \rangle \leq 0 < \langle w, v \rangle,\ \forall\, \xi \in \widehat{S}(\bar{x}).$$

As $\widehat{S}(\bar{x}) = cone\ co\left(\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})\right)$, for each $i \in I(\bar{x})$, $\langle w, \xi_i \rangle \leq 0$ for every $\xi_i \in \partial g_i(\bar{x})$, which along with Theorem 2.79 yields

$$g_i'(\bar{x}, w) \leq 0, \ \forall\ i \in I(\bar{x}). \tag{3.5}$$

Define

$$K = \{u \in \mathbb{R}^n : g_i'(\bar{x}, u) < 0, \ \forall\ i \in I(\bar{x})\}.$$

Our first step is to show that K is nonempty. By the Slater constraint qualification, there exists $\hat{x}$ such that $g_i(\hat{x}) < 0$ for every $i = 1, 2, \ldots, m$, and corresponding to that $\hat{x}$, set $u = \hat{x} - \bar{x}$. By the convexity of each g_i and Theorem 2.79,

$$g_i'(\bar{x}, \hat{x} - \bar{x}) \leq g_i(\hat{x}) - g_i(\bar{x}), \ \forall\ i \in I(\bar{x}),$$

which implies

$$g_i'(\bar{x}, \hat{x} - \bar{x}) < 0, \ \forall\ i \in I(\bar{x}).$$

Hence, $\hat{x} - \bar{x} \in K$, thereby showing that K is nonempty. Observe that for any $u \in K$, there exists $\lambda > 0$ sufficiently small such that $g_i(\bar{x} + \lambda u) < 0$ for all $i = 1, 2, \ldots, m$, which implies $\bar{x} + \lambda u \in C$. Therefore,

$$u \in \frac{1}{\lambda}(C - \bar{x}) \subseteq cone(C - \bar{x}) \subseteq cl\ cone(C - \bar{x}).$$

By Theorem 2.35, $u \in T_C(\bar{x})$. Because $T_C(\bar{x})$ is closed, $cl\ K \subseteq T_C(\bar{x})$. Also, as K is nonempty, it is simple to show that

$$cl\ K = \{u \in \mathbb{R}^n : g_i'(\bar{x}, u) \leq 0, \ \forall\ i \in I(\bar{x})\}.$$

By (3.5), $w \in cl\ K$ and hence, $w \in T_C(\bar{x})$. As $v \in N_C(\bar{x})$, $\langle v, w \rangle \leq 0$, thereby contradicting the fact that $\langle v, w \rangle > 0$ and thus establishing the result. $\square$

Recall condition (ii) from Theorem 3.1, that is,

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}).$$

By combining it with Proposition 3.3, we can conclude that under the Slater constraint qualification, $\bar{x}$ is a point of minimizer of the convex programming problem (CP) with C given by (3.1) if and only if there exists $\bar{\lambda} \in \mathbb{R}^m_+$ such that

$$0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i \partial g_i(\bar{x}).$$

Setting $\bar{\lambda}_i = 0$ for $i \notin I(\bar{x})$, the above expression can be rewritten as

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}) \quad \text{and} \quad \bar{\lambda}_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m.$$

The above two expressions form the celebrated *Karush–Kuhn–Tucker (KKT) optimality conditions* for the convex programming problem (CP) with C given by (3.1). The vector $\bar{\lambda} \in \mathbb{R}^m_+$ is called a *Lagrange multiplier* or a *Karush–Kuhn–Tucker (KKT) multiplier*. The second condition is known as the *complementary slackness condition*.

Now suppose that the KKT optimality conditions are satisfied. Then there exist $\xi_0 \in \partial f(\bar{x})$ and $\xi_i \in \partial g_i(\bar{x}),\ i = 1, 2, \ldots, m$, such that

$$0 = \xi_0 + \sum_{i=1}^{m} \lambda_i \xi_i. \tag{3.6}$$

Therefore, by the convexity of f and $g_i,\ i = 1, 2, \ldots, m$, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} f(x) - f(\bar{x}) &\geq \langle \xi_0, x - \bar{x} \rangle, \\ g_i(x) - g_i(\bar{x}) &\geq \langle \xi_i, x - \bar{x} \rangle,\ i = 1, 2, \ldots, m. \end{aligned}$$

The above inequalities along with (3.6) yields that for every $x \in \mathbb{R}^n$,

$$f(x) - f(\bar{x}) + \sum_{i=1}^{m} \lambda_i (g_i(x) - g_i(\bar{x})) \geq \langle \xi_0, x - \bar{x} \rangle + \sum_{i=1}^{m} \lambda_i \langle \xi_i, x - \bar{x} \rangle = 0. \tag{3.7}$$

The above inequality holds, in particular, for $x \in C \subset \mathbb{R}^n$. Invoking the complementary slackness condition along with the feasibility of $x \in C$, the condition (3.7) reduces to

$$f(x) \geq f(\bar{x}),\ \forall\ x \in C.$$

Thus, $\bar{x}$ is a point of minimizer of (CP).

This discussion can be summed up in the form of the following theorem.

Theorem 3.7 *Consider the convex programming problem (CP) with C given by (3.1). Assume that the Slater constraint qualification holds. Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exist $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}) \quad \textit{and} \quad \bar{\lambda}_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

It is obvious that the computation of the normal cone in Proposition 3.3 plays a major role in the derivation of the KKT optimality conditions. What is shown by the computation of the normal cone in Proposition 3.3 is that the Lagrange multipliers are not just auxiliary multipliers that help us convert a constrained problem into a unconstrained one but are related to the geometry of the feasible set.

Remark 3.8 In Proposition 3.3, we have seen how to compute the normal cone when the convex inequality constraints need not be smooth. Now if g_i, $i = 1, 2, \ldots, m$, are differentiable and the Slater constraint qualification holds, then from Proposition 3.3

$$N_C(\bar{x}) = \{v \in \mathbb{R}^n : \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}),\ \lambda_i \geq 0,\ \forall\, i \in I(\bar{x})\}. \tag{3.8}$$

This can be actually computed easily. Note that $v \in N_C(\bar{x})$ if $\bar{x}$ is a point of minimizer of the problem

$$\min\ -\langle v, x\rangle \quad \text{subject to} \quad g_i(x) \leq 0,\ i = 1, 2, \ldots, m.$$

As the Slater condition holds, by Theorem 3.7 there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, such that

$$-v + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0.$$

By the complementary slackness condition, $\lambda_i = 0,\ i \notin I(\bar{x})$; thus the above relation becomes

$$-v + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0.$$

Hence, any $v \in N_C(\bar{x})$ belongs to the set on the right-hand side. One can simply check that any element on the right-hand side is also an element in the normal cone. From (3.8), it is simple to see that $N_C(\bar{x})$ is a finitely generated cone with $\{\nabla g_i(\bar{x}) : i \in I(\bar{x})\}$ being the set of generators. Thus, $N_C(\bar{x})$ is polyhedral when the g_i, $i = 1, 2, \ldots, m$, are differentiable and the Slater constraint qualification holds.

Is the normal cone also polyhedral if the Slater constraint qualification holds but the constraint functions g_i, $i = 1, 2, \ldots, m$, are not be differentiable? What is seen from Proposition 3.3 is that in the case nondifferentiable constraints, $N_C(\bar{x})$ can be represented as

$$\begin{aligned} N_C(\bar{x}) &= \{\sum_{i \in I(\bar{x})} \lambda_i \xi_i \in \mathbb{R}^n : \lambda_i \geq 0,\ \xi_i \in \partial g_i(\bar{x}),\ i \in I(\bar{x})\} \\ &= \bigcup_{\xi_i \in \partial g_i(\bar{x})} \{\sum_{i \in I(\bar{x})} \lambda_i \xi_i \in \mathbb{R}^n : \lambda_i \geq 0,\ i \in I(\bar{x})\}, \end{aligned}$$

that is, the union of a family of polyhedral cones.

We will now show by an example that even though $N_C(\bar{x})$ is a union of a family of polyhedral cones, it itself need not be polyhedral. Consider the set $C \subseteq \mathbb{R}^3$ given as

$$C = \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} \leq -x_3,\ x_3 \leq 0\}.$$

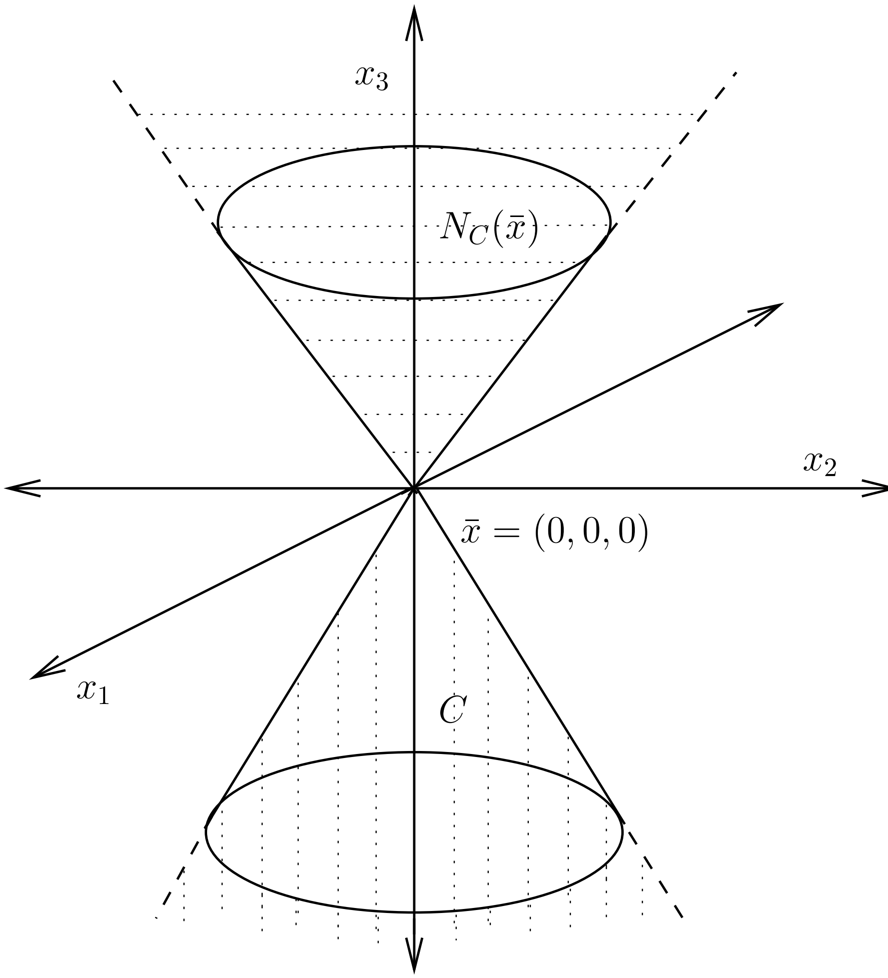

FIGURE 3.1: $N_C(\bar{x})$ is not polyhedral.

It is clear that C is described by the constraints

$$\begin{aligned} \sqrt{x_1^2 + x_2^2} + x_3 &\leq 0 \\ x_3 &\leq 0. \end{aligned}$$

Each of these are convex functions. It is simple to see that the Slater condition holds. Just take the point $\hat{x} = (0,0,-1)$. It is also simple to see that the first constraint is not differentiable at $\bar{x} = (0,0,0)$. However, from the geometry,

Figure 3.1, it is simple to observe that

$$N_C(\bar{x}) = \{v \in \mathbb{R}^3 : \sqrt{v_1^2 + v_2^2} \leq v_3,\ v_3 \geq 0\}.$$

It is easy to observe that this cone, which is also known as the second-order cone, is not polyhedral as it has an infinite number of generators and hence is not finitely generated.

3.3 Abadie Constraint Qualification

From the previous section it is obvious that to derive the KKT conditions an important feature is that the Slater constraint qualification is satisfied. But what happens if the Slater constraint qualification is not satisfied? Is there any other route to derive the KKT conditions? In this direction, we introduce what is known as the Abadie constraint qualification. Consider the problem (CP) with C given by (3.1), that is

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$$

where $g_i,\ i = 1, 2, \ldots, m$, are convex functions. Then the *Abadie constraint qualification* is said to hold at $\bar{x} \in C$ if

$$T_C(\bar{x}) = \{v \in \mathbb{R}^n : g_i'(\bar{x}, v) \leq 0,\ \forall\ i \in I(\bar{x})\}.$$

As C is convex, $(T_C(\bar{x}))^\circ = N_C(\bar{x})$ and the expression (ii) in Theorem 3.1 can be written as

$$0 \in \partial f(\bar{x}) + (T_C(\bar{x}))^\circ.$$

If the Abadie constraint qualification holds, we can compute the $N_C(\bar{x})$ as

$$N_C(\bar{x}) = (S(\bar{x}))^\circ,$$

where $(S(\bar{x}))^\circ$ denotes the polar cone of the cone

$$S(\bar{x}) = \{v \in \mathbb{R}^n : g_i'(\bar{x}, v) \leq 0,\ \forall\ i \in I(\bar{x})\}.$$

It can be easily verified that $S(\bar{x})$ is a closed convex cone. Also observe that

$$T_C(\bar{x}) \subset \{v \in \mathbb{R}^n : g_i'(\bar{x}, v) \leq 0,\ \forall\ i \in I(\bar{x})\}$$

is always satisfied. So one may simply consider the reverse inclusion as the Abadie constraint qualification. We will now compute $(S(\bar{x}))^\circ$. But before we do that, let us convince ourselves through an example that the Abadie constraint qualification can hold even if the Slater constraint qualification fails. Consider

$$C = \{x \in \mathbb{R} : |x| \leq 0,\ x \leq 0\}.$$

Here, $g_1(x) = |x|,\ g_2(x) = x$ and of course $C = \{0\}$. Let us set $\bar{x} = 0$. This shows that $T_C(\bar{x}) = \{0\}$. Further, because both constraints are active at $\bar{x}$,

$$\begin{aligned} S(\bar{x}) &= \{v \in \mathbb{R} : g_1'(\bar{x}, v) \leq 0,\ g_2'(\bar{x}, v) \leq 0\} \\ &= \{v \in \mathbb{R} : g_1'(\bar{x}, v) \leq 0,\ \langle \nabla g_2(\bar{x}), v \rangle \leq 0\} \\ &= \{v \in \mathbb{R} : |v| \leq 0,\ v \leq 0\} \\ &= \{0\}. \end{aligned}$$

Hence $T_C(\bar{x}) = S(\bar{x})$, showing that the Abadie constraint qualification holds while it is clear that the Slater constraint qualification does not hold.

Now we present the following result.

Proposition 3.9 $(S(\bar{x}))^\circ = cl\ \widehat{S}(\bar{x})$.

Proof. From the relation (3.2),

$$\widehat{S}(\bar{x}) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i \xi_i : \lambda_i \geq 0,\ \xi_i \in \partial g_i(\bar{x}),\ i \in I(\bar{x}) \right\}$$

is a convex cone from Lemma 3.5. Recall from the proof of Lemma 3.5 that $\widehat{S}(\bar{x})$ was shown to be closed under the Slater constraint qualification. In the absence of the Slater constraint qualification, $\widehat{S}(\bar{x})$ need not be closed. First we show that $cl\ \widehat{S}(\bar{x}) \subseteq (S(\bar{x}))^\circ$. Consider any $v \in \widehat{S}(\bar{x})$, which implies there exist $\lambda_i \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$ for $i \in I(\bar{x})$ such that $v = \sum_{i \in I(\bar{x})} \lambda_i \xi_i$. Consider any element $w \in S(\bar{x})$, that is, $g_i'(\bar{x}, w) \leq 0$ for $i \in I(\bar{x})$. Hence for every $i \in I(\bar{x})$, by Theorem 2.79, $\langle \xi_i, w \rangle \leq 0$ for every $\xi_i \in \partial g_i(\bar{x})$, which implies

$$\left\langle \sum_{i \in I(\bar{x})} \lambda_i \xi_i, w \right\rangle \leq 0,$$

that is, $\langle v, w \rangle \leq 0$. Because $w \in S(\bar{x})$ was arbitrarily chosen, $\widehat{S}(\bar{x}) \subseteq (S(\bar{x}))^\circ$, which by closedness of $(S(\bar{x}))^\circ$ leads to $cl\ \widehat{S}(\bar{x}) \subseteq (S(\bar{x}))^\circ$.

To complete the proof, we will establish the reverse inclusion, that is, $(S(\bar{x}))^\circ \subseteq cl\ \widehat{S}(\bar{x})$. On the contrary, assume that $(S(\bar{x}))^\circ \not\subseteq cl\ \widehat{S}(\bar{x})$, which implies there exists $w \in (S(\bar{x}))^\circ$ and $w \notin cl\ \widehat{S}(\bar{x})$. As $cl\ \widehat{S}(\bar{x})$ is a closed convex cone, by the strict separation theorem, Theorem 2.26 (iii), there exists $v \in \mathbb{R}^n$ with $v \neq 0$ such that

$$\sup_{\xi \in cl\ \widehat{S}(\bar{x})} \langle v, \xi \rangle < \langle v, w \rangle.$$

Because $0 \in cl\ \widehat{S}(\bar{x})$, $\langle v, w\rangle > 0$. We claim that $v \in (cl\ \widehat{S}(\bar{x}))^\circ$, that is, $\langle v, \xi\rangle \leq 0$ for every $\xi \in cl\ \widehat{S}(\bar{x})$. If $v \notin (cl\ \widehat{S}(\bar{x}))^\circ$, then there exists $\tilde{\xi} \in cl\ \widehat{S}(\bar{x})$ such that $\langle v, \tilde{\xi}\rangle > 0$. For every $\lambda > 0$, $\lambda\langle v, \tilde{\xi}\rangle = \langle v, \lambda\tilde{\xi}\rangle > 0$. Because $cl\ \widehat{S}(\bar{x})$ is a cone, $\lambda\tilde{\xi} \in cl\ \widehat{S}(\bar{x})$ for $\lambda > 0$, which means that as λ becomes sufficiently large, the inequality

$$\langle v, \lambda\tilde{\xi}\rangle < \langle v, w\rangle$$

will be violated. Thus, $v \in (cl\ \widehat{S}(\bar{x}))^\circ$. Further, observe that for $i \in I(\bar{x})$, $\xi_i \in \widehat{S}(\bar{x})$, where $\xi_i \in \partial g_i(\bar{x})$. Therefore, $\langle v, \xi_i\rangle \leq 0$ for every $\xi_i \in \partial g_i(\bar{x})$, $i \in I(\bar{x})$, which implies that $g_i'(\bar{x}, v) \leq 0$ for every $i \in I(\bar{x})$. This shows that $v \in S(\bar{x})$ and therefore, $\langle v, w\rangle \leq 0$ because $w \in (S(\bar{x}))^\circ$. This leads to a contradiction, thereby establishing the result. $\square$

The result below presents the KKT optimality conditions under the Abadie constraint qualification.

Theorem 3.10 *Consider the convex programming problem (CP) with C given by (3.1). Let $\bar{x}$ be a point of minimizer of (CP) and assume that the Abadie constraint qualification holds at $\bar{x}$. Then*

$$0 \in \partial f(\bar{x}) + cl\ \widehat{S}(\bar{x}). \tag{3.9}$$

Conversely, if (3.9) holds for some $\bar{x} \in \mathbb{R}^n$, then $\bar{x}$ is a point of minimizer of (CP). Moreover, the standard KKT optimality conditions hold at $\bar{x}$ if either $\widehat{S}(\bar{x})$ is closed or the functions g_i, $i \in I(\bar{x})$, are smooth functions.

Proof. If the Abadie constraint qualification holds at $\bar{x}$, then using Proposition 3.9, the relation (3.9) holds.

Conversely, suppose that (3.9) holds at $\bar{x}$. By the convexity of g_i, $i \in I(\bar{x})$, for every $\xi_i \in \partial g_i(\bar{x})$,

$$\langle \xi_i, x - \bar{x}\rangle \leq g_i(x) - g_i(\bar{x}) \leq 0,\ \forall\ x \in C.$$

For every $v \in \widehat{S}(\bar{x})$, there exist $\lambda_i \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$ for $i \in I(\bar{x})$ such that $v = \sum_{i\in I(\bar{x})} \lambda_i\xi_i$. Therefore, by the above inequality,

$$\langle v, x - \bar{x}\rangle = \sum_{i\in I(\bar{x})} \langle \lambda_i\xi_i, x - \bar{x}\rangle \leq 0,\ \forall\ x \in C,$$

which implies $v \in N_C(\bar{x})$. Thus $\widehat{S}(\bar{x}) \subseteq N_C(\bar{x})$, which along with the fact that $N_C(\bar{x})$ is closed implies that $cl\ \widehat{S}(\bar{x}) \subseteq N_C(\bar{x})$. Therefore, (3.9) yields

$$0 \in \partial f(\bar{x}) + N_C(\bar{x})$$

and hence, by Theorem 3.1 (ii), $\bar{x}$ is a point of minimizer of the convex programming problem (CP).

If $g_i,\ i \in I(\bar{x})$, are smooth,

$$\widehat{S}(\bar{x}) = \left\{ \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) \in \mathbb{R}^n : \lambda_i \geq 0,\ i \in I(\bar{x}) \right\}.$$

Thus, $\widehat{S}(\bar{x})$ is a finitely generated cone and hence is closed. Therefore, it is clear that when either $\widehat{S}(\bar{x})$ is closed or $g_i,\ i \in I(\bar{x})$ are smooth functions, then under the Abadie constraint qualification, the standard KKT conditions are satisfied. □

3.4 Convex Problems with Abstract Constraints

After studying the convex programming problem involving only inequality constraints, in this section we turn our attention to a slightly modified version of (CP), which we denote as $(CP1)$ given as

$$\begin{array}{rl} \min & f(x) \\ \text{subject to} & g_i(x) \leq 0,\ i = 1, 2, \ldots, m, \\ & x \in X, \end{array} \tag{$CP1$}$$

where we have the additional *abstract constraint* $x \in X$ with X as a closed convex subset of $\mathbb{R}^n$. The question is how to write down the KKT conditions for the problem $(CP1)$.

Theorem 3.11 *Let us consider the problem $(CP1)$. Assume the Slater-type constraint qualification, that is, there exists $\hat{x} \in ri\ X$ such that $g_i(\hat{x}) < 0$ for $i = 1, 2, \ldots, m$. Then the KKT optimality conditions are necessary as well as sufficient at a point of minimizer $\bar{x}$ of $(CP1)$ and are given as*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}) \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2 \ldots, m.$$

Proof. The problem $(CP1)$ can be written as

$$\min\ f(x) \quad \text{subject to} \quad x \in C \cap X,$$

where C is given by (3.1). Thus if $\bar{x}$ is a point of minimizer of $(CP1)$, then $\bar{x}$ solves the unconstrained problem

$$\min_{x \in \mathbb{R}^n} (f + \delta_{C \cap X})(x),$$

that is, $\bar{x}$ solves

$$\min_{x \in \mathbb{R}^n} (f + \delta_C + \delta_X)(x).$$

By the optimality condition for unconstrained problem, Theorem 2.89,

$$0 \in \partial(f + \delta_C + \delta_X)(\bar{x}).$$

The fact that $ri\ dom\ f = \mathbb{R}^n$ along with the Slater-type constraint qualification and Propositions 2.15 and 2.67 imply that $\hat{x} \in ri\ dom\ f \cap ri\ C \cap ri\ X$. Invoking the Sum Rule, Theorem 2.91, along with the facts that $\partial\delta_C(\bar{x}) = N_C(\bar{x})$ and $\partial\delta_X(\bar{x}) = N_X(\bar{x})$, the above relation leads to

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}) + N_X(\bar{x}).$$

The Slater-type constraint qualification implies the Slater constraint qualification which along with Proposition 3.3 yields

$$N_C(\bar{x}) = \{\sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) : \lambda_i \geq 0,\ i \in I(\bar{x})\}.$$

By choosing $\lambda_i = 0,\ i \notin I(\bar{x})$, the desired KKT optimality conditions are obtained.

Conversely, by the optimality condition, there exist $\xi_0 \in \partial f(\bar{x})$ and $\xi_i \in \partial g_i(\bar{x})$, $i = 1, 2 \ldots, m$, such that

$$-\xi_0 - \sum_{i=1}^{m} \lambda_i \xi_i \in N_X(\bar{x}),$$

that is,

$$\langle \xi_0, x - \bar{x}\rangle + \sum_{i=1}^{m} \langle \lambda_i \xi_i, x - \bar{x}\rangle \geq 0,\ \forall\ x \in X.$$

The convexity of f and g_i, $i = 1, 2, \ldots, m$, along with the above condition leads to

$$f(x) - f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(x) - \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \geq 0,\ \forall\ x \in X.$$

In particular, for any $x \in C$, the above inequality reduces to

$$f(x) \geq f(\bar{x}),\ \forall\ x \in X,$$

thereby establishing that $\bar{x}$ is a point of minimizer of $(CP1)$. □

Next consider the problem

$$\begin{array}{rl} \min & f(x) \\ \text{subject to} & x \in C = \{x \in \mathbb{R}^n : Ax = b\}, \end{array} \qquad (CP2)$$

where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$. It is clear that C is a polyhedron. Further, a point $\bar{x} \in C$ is a point of minimizer of f over C if and only if

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}).$$

If $v \in N_C(\bar{x})$, then $\bar{x}$ solves the following smooth problem

$$\begin{array}{rl} \min & -\langle v, x\rangle \\ \text{subject to} & Ax = b. \end{array}$$

As the constraints are affine, the KKT optimality conditions for this problem automatically hold, that is, there exists $\lambda \in \mathbb{R}^m$ such that

$$-v + A^T\lambda = 0,$$

that is, $v = A^T\lambda$. Therefore,

$$N_C(\bar{x}) = \left\{v \in \mathbb{R}^n : v = A^T\lambda,\ \lambda \in \mathbb{R}^m\right\}.$$

Hence, the optimality condition is that there exists $\lambda \in \mathbb{R}^m$ such that

$$-A^T\lambda \in \partial f(\bar{x}).$$

Using the convexity of f, the above relation implies that $\bar{x}$ is a point of minimizer of $(CP2)$. This discussion can be stated as the following theorem.

Theorem 3.12 *Consider the problem* $(CP2)$. *Then* $\bar{x}$ *is a point of minimizer of* $(CP2)$ *if and only if there exists* $\lambda \in \mathbb{R}^m$ *such that*

$$-A^T\lambda \in \partial f(\bar{x}).$$

3.5 Max-Function Approach

Until now the convex programming problems were tackled without modifying the constraint sets. But every convex programming problem can be expressed as nonsmooth convex programming with a lesser number of constraints. Consider the problem (CP) where C is given by (3.1), that is, convex inequality constraints. Assume that the objective function f and the constraints functions $g_i,\ i = 1, 2, \ldots, m$, are convex and smooth. Then (CP) can be equivalently posed as a problem with only one constraint, which is given as

$$\min\ f(x) \quad \text{subject to} \quad g(x) \leq 0, \tag{CP_{eq}}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$g(x) = \max\{g_1(x), g_2(x), \ldots, g_m(x)\}.$$

Hence g is intrinsically nonsmooth. We would like to invite the reader to deduce the optimality condition of the problem (CP) using (CP_{eq}). It is clear that one needs to use the Max-Function Rule, Theorem 2.96, for evaluating the subdifferential of the max-function. Thus, at a very fundamental level, every convex programming problem (smooth or nonsmooth) is a nonsmooth convex programming problem.

The Max-Function Rule is also in some sense very fundamental to convex programming problems as can be seen in the result below. In the following result, we derive the KKT optimality conditions for the convex programming problem (CP) with C given by (3.1) using the max-function approach.

Theorem 3.13 *Consider the convex programming problem (CP) with C given by (3.1). Assume that the Slater constraint qualification holds. Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exist $\lambda_i \geq 0,\ i = 1, 2, \ldots, m,$ such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}) \quad \text{and} \quad \bar{\lambda}_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

Proof. As $\bar{x}$ is a point of minimizer of (CP), it also solves the unconstrained problem

$$\min\ F(x) \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

where $F(x) = \max\{f(x) - f(\bar{x}), g_1(x), g_2(x), \ldots, g_m(x)\}$. Then by the unconstrained optimality condition, Theorem 2.89,

$$0 \in \partial F(\bar{x}).$$

Applying the Max-Function Rule, Theorem 2.96,

$$0 \in co\ \{\partial f(\bar{x}) \cup (\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}))\},$$

where $I(\bar{x})$ is the active index set at $\bar{x}$. Therefore, there exists $\lambda_i \geq 0$, $i \in \{0\} \cup I(\bar{x})$ satisfying $\sum_{i \in \{0\} \cup I(\bar{x})} \lambda_i = 1$ such that

$$0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}). \tag{3.10}$$

We claim that $\lambda_0 \neq 0$. On the contrary, assume that $\lambda_0 = 0$. Thus, the above inclusion reduces to

$$0 \in \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}),$$

that is, there exists $\xi_i \in \partial g_i(\bar{x})$ such that

$$0 = \sum_{i \in I(\bar{x})} \lambda_i \xi_i. \tag{3.11}$$

By the convexity of $g_i,\ i \in I(\bar{x})$,

$$g_i(x) = g_i(x) - g_i(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle,\ \forall\, x \in \mathbb{R}^n,\ i \in I(\bar{x}),$$

which along with (3.11) implies that

$$\sum_{i \in I(\bar{x})} \lambda_i g_i(x) \geq 0,\ \forall\, x \in \mathbb{R}^n.$$

As the Slater constraint qualification holds, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m$. Thus,

$$\sum_{i \in I(\bar{x})} \lambda_i g_i(\hat{x}) < 0,$$

which is a contradiction of the preceding inequality. Therefore, $\lambda_0 \neq 0$ and hence dividing (3.10) throughout by λ_0 yields

$$0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\lambda}_i \partial g_i(\bar{x}),$$

where $\bar{\lambda}_i = \dfrac{\lambda_i}{\lambda_0},\ i \in I(\bar{x})$. Taking $\bar{\lambda}_i = 0,\ i \notin I(\bar{x})$, the above condition becomes

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}),$$

that is, the KKT optimality condition. It is easy to observe that

$$\bar{\lambda}_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m,$$

thus yielding the desired conditions. The sufficiency part can be worked out using the convexity of the functions, as done in the previous KKT optimality theorems. □

3.6 Cone-Constrained Convex Programming

A convex optimization problem can be posed in a more general format. Consider a nonempty closed convex cone $S \subset \mathbb{R}^m$. Then consider the problem

$$\min\ f(x) \quad \text{subject to} \quad G(x) \in -S, \qquad (CCP)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $G : \mathbb{R}^n \to \mathbb{R}^m$ is a *S-convex function*; that is, for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$(1 - \lambda)G(x) + \lambda G(y) - G((1 - \lambda)x + \lambda y) \in S.$$

In particular, if $S = \mathbb{R}^m_+$, then the above problem reduces to (CP) with C given by (3.1). If $S = \mathbb{R}^s_+ \times \{0\}_{m-s}$, then (CCP) reduces to a convex problem with both inequality and equality constraints. If S is not these two cones, then (CCP) is called a *cone-constrained problem.* We will now derive optimality conditions for a slightly more general problem that has an added abstract constraint. Consider the problem

$$\min \; f(x) \quad \text{subject to} \quad G(x) \in -S, \; x \in X, \tag{$CCP1$}$$

where $X \subset \mathbb{R}^n$ is a nonempty closed convex set. There are many ways to approach this problem. Here we demonstrate one approach. Define

$$C = \{x \in \mathbb{R}^n : G(x) \in -S\}.$$

As S and X are nonempty convex sets, by Proposition 2.14, $ri\ S$ and $ri\ X$ are both nonempty. Assume that the Slater-type constraint qualification holds, that is, there exist $\hat{x} \in ri\ X$ such that $G(\hat{x}) \in -ri\ S$. The most natural approach is to observe that if $\bar{x}$ solves $(CCP1)$, then $\bar{x}$ is also a point of minimizer of the unconstrained problem

$$\min \; (f + \delta_{C \cap X})(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

This is equivalent to the problem

$$\min \; (f + \delta_C + \delta_X)(x).$$

As $dom\ f = \mathbb{R}^n$, which along with the Slater-type constraint qualification implies that $\hat{x} \in ri\ dom\ f \cap ri\ C \cap ri\ X$. Invoking the Sum Rule, Theorem 2.91,

$$0 \in \partial f(\bar{x}) + \partial \delta_C(\bar{x}) + \partial \delta_X(\bar{x}).$$

and thus,

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}) + N_X(\bar{x}).$$

So our main concern now is to explicitly compute $N_C(\bar{x})$. How does one do that? We have already observed that it is not so straightforward to compute the normal cone when the inequality constraints are not smooth. Let us now mention that in this case also we do not consider G to be differentiable. Thus we shall now introduce the notion of a subdifferential of a *cone convex function.* As G is a S-convex function, we call an $m \times n$ matrix A to be a *subgradient* of G at $x \in \mathbb{R}^n$

$$G(y) - G(x) - A(y - x) \in S, \; \forall\ y \in \mathbb{R}^n.$$

Then the *subdifferential* of the cone convex function G at x is given as

$$\partial G(x) = \{A \in \mathbb{R}^{m \times n} : G(y) - G(x) - A(y - x) \in S, \; \forall\ y \in \mathbb{R}^n\}.$$

The important question is whether the set $\partial G(x)$ is nonempty.

It was shown, for example, in Luc, Tan, and Tinh [78] that if G is an S-convex function, then G is continuous on $\mathbb{R}^n$. Further, G is also a locally Lipschitz function on $\mathbb{R}^n$; that is, for any $x_0 \in \mathbb{R}^n$, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that there exists $L_{x_0} > 0$ satisfying

$$\|G(y) - G(x)\| \leq L_{x_0}\|y - x\|, \ \forall\ x, y \in \mathcal{N}(x_0).$$

Observe that L_{x_0} depends on the chosen x_0 and is also called the *Lipschitz constant* at x_0. Also, note that a locally Lipschitz vector function need not be differentiable everywhere. For a locally Lipschitz function G, the *Clarke Jacobian* of G at x is given as follows,

$$\partial_C G(x) = co \ \left\{ A \in \mathbb{R}^{m\times n} : A = \lim_{k\to\infty} JG(x_k) \text{ where } x_k \to x, \ x_k \in \mathcal{D} \right\},$$

where $\mathcal{D}$ is the set of points on $\mathbb{R}^n$ at which G is differentiable and $JG(y)$ denotes the Jacobian of G at y. In fact, there is a famous theorem of Rademacher that says that $\mathbb{R}^n \backslash \mathcal{D}$ is a set of Lebesgue measure zero. The set $\partial_C G(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$ and is convex and compact. For more details on the Clarke Jacobian, see for example Clarke [27] or Demyanov and Rubinov [30]. The property that will be important to us is the Clarke Jacobian as a set-valued map is locally bounded and graph closed.

It was shown for example in Luc, Tan, and Tinh [78] that $\partial_C G(x) \subseteq \partial G(x)$, thereby proving that if G is an S-convex function, then $\partial G(x) \neq \emptyset$ for every $x \in \mathbb{R}^n$. Before we proceed to develop the optimality conditions for $(CCP1)$, let us look at a locally Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$.

Recall that a function $\phi : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at x_0 if there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 and $L_{x_0} > 0$ such that

$$|\phi(y) - \phi(x)| \leq L_{x_0}\|y - x\|, \ \forall\ x, y \in \mathcal{N}(x_0).$$

Naturally a locally Lipschitz scalar-valued function is not differentiable everywhere and the Rademacher Theorem tells us that the set of points where ϕ is not differentiable forms a set of measure zero. Therefore, at any $x \in \mathbb{R}^n$, the *Clarke generalized gradient* or *Clarke subdifferential* is given as

$$\partial^\circ \phi(x) = co \ \{\xi \in \mathbb{R}^n : \xi = \lim_{k\to\infty} \nabla\phi(x_k) \text{ where } x_k \to x, \ x_k \in \tilde{\mathcal{D}}\},$$

where $\tilde{\mathcal{D}}$ denotes the set of points at which ϕ is differentiable. One can observe that if $m = 1$, the Clarke Jacobian reduces to the Clarke subdifferential. The Clarke subdifferential is nonempty, convex, and compact. If $\bar{x}$ is a local minimum of ϕ over $\mathbb{R}^n$, then $0 \in \partial^\circ\phi(\bar{x})$. It is important to note that this condition is necessary but not sufficient. Now we state a calculus rule that will be useful in our computation of the normal cone. The Sum Rule is from Clarke [27].

Consider two locally Lipschitz functions $\phi_1, \phi_2 : \mathbb{R}^n \to \mathbb{R}$. Then

$$\partial^\circ(\phi_1 + \phi_2)(x) \subseteq \partial^\circ \phi_1(x) + \partial^\circ \phi_2(x).$$

If one of the functions is continuously differentiable, then equality holds.

The Chain Rule that we state is from Demyanov and Rubinov [30] (see also Dutta [36]).

Consider the function $\phi \circ \Phi$ where $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ and $\phi : \mathbb{R}^m \to \mathbb{R}$ are locally Lipschitz functions. Assume that ϕ is continuously differentiable. Then

$$\partial^\circ(\phi \circ \Phi)(x) = \{z^T \nabla \phi(\Phi(x)) \in \mathbb{R}^n : z \in \partial_C \Phi(x)\}.$$

Observe that $v \in N_C(\bar{x})$ (in the current context of $(CCP1)$) if and only if $\bar{x}$ is a point of minimizer of the problem

$$\min \; -\langle v, x\rangle \quad \text{subject to} \quad G(x) \in -S. \tag{NP}$$

For simplicity, assume that $C = \{x \in \mathbb{R}^n : G(x) \in -S\}$ is an n-dimensional convex set. The approach to derive the necessary and sufficient condition for optimality is due to Rockafellar [100] (see also Chapter 6 of Rockafellar and Wets [101]). As the above problem is a convex programming problem, $\bar{x}$ is a global point of minimizer of (NP). Further, without loss of generality, we can assume it to be a unique. Observe that if we define

$$f(x) = -\langle v, x\rangle \quad \text{and} \quad \tilde{f}(x) = -\langle v, x\rangle + \varepsilon\|x - \bar{x}\|^2,$$

then $\partial f(\bar{x}) = \partial \tilde{f}(\bar{x}) = \{-v\}$ and $\bar{x}$ is the unique minimizer of $\tilde{f}$ because it is a strictly convex function. Consider an n- dimension convex compact set $Y \subset \mathbb{R}^n$ such that $\bar{x} \in int\ Y$ and $C \cap Y \neq \emptyset$ (Figure 3.2). It is simple to see that $\bar{x}$ is also the unique minimizer of the problem

$$\min \; f(x) \quad \text{subject to} \quad G(x) \in -S, \; x \in Y. \tag{$NP1$}$$

Also observe that the normal cone $N_C(\bar{x}) = N_{C\cap Y}(\bar{x})$. (We would urge the readers to think why).

Our approach depends on the use of penalization, a method popular for designing algorithms for constrained optimization. Consider the problem

$$\min \; \hat{f}(x, u) = f(x) \quad \text{subject to} \quad G(x) - u = 0, \; (x, u) \in Y \times (-S). \tag{$\widehat{NP}$}$$

As $\bar{x}$ is the unique point of minimizer of $(NP1)$, we deduce that $(\bar{x}, \bar{u}) = (\bar{x}, G(\bar{x}))$ is the unique point of minimizer of $(\widehat{NP})$. For a sequence of $\varepsilon_k \downarrow 0$, consider the sequence of penalty approximations

$$\begin{aligned} &\min \; \hat{f}_k(x, u) = f(x) + \frac{1}{2\varepsilon_k}\|G(x) - u\|^2 \\ &\text{subject to} \;\; (x, u) \in Y \times (-S). \end{aligned} \tag{$\widehat{NP}_k$}$$

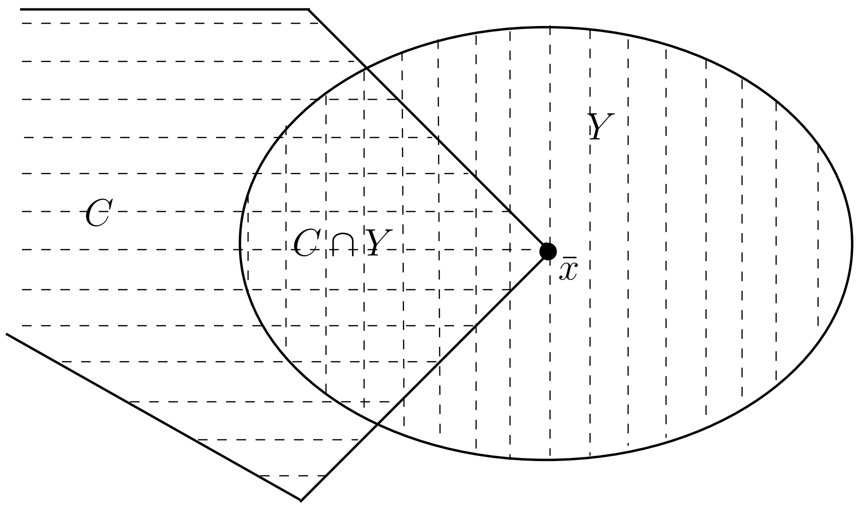

FIGURE 3.2: $C \cap Y$.

Consider the following closed set

$$S_k = \{(x, u) \in Y \times (-S) : \hat{f}(x, u) \le \hat{f}(\bar{x}, \bar{u}) = f(\bar{x})\}.$$

Note that $\bar{u} = G(\bar{x})$. Also, S_k is nonempty as $(\bar{x}, \bar{u}) \in S_k$ for each k. Because S_k is nonempty for each k, solving $(\widehat{NP}_k)$ is same as minimizing $\hat{f}_k$ over S_k. Denote the minimum of f over the compact set Y by μ. For any $(x, u) \in S_k$,

$$\hat{f}_k(x, u) \le \hat{f}_k(\bar{x}, \bar{u}) = f(\bar{x}),$$

which implies

$$f(x) + \frac{1}{2\varepsilon_k}\|G(x) - u\|^2 \le f(\bar{x}),$$

where $(x, u) \in S_k \subset Y \times (-S)$. As $f(x) \ge \mu$ for every $x \in Y$,

$$\mu + \frac{1}{2\varepsilon_k}\|G(x) - u\|^2 \le f(\bar{x}),$$

which leads to

$$\|G(x) - u\| \le \sqrt{2\varepsilon_k(f(\bar{x}) - \mu)}.$$

Thus for any given k,

$$S^k \subseteq \{(x, u) \in Y \times (-S) : \|G(x) - u\| \le \sqrt{2\varepsilon_k(f(\bar{x}) - \mu)}\}. \tag{3.12}$$

Also, for a fixed k,

$$\|u\| \leq \|G(x)\| + \sqrt{2\varepsilon_k(f(\bar{x}) - \mu)}.$$

As G is an S-convex function, it is also locally Lipschitz and hence $G(Y)$ is a compact set. This shows that the right-hand side of (3.12) is bounded for a fixed k. From this, along with the compactness of Y, we can deduce that S_k is compact and thus $\hat{f}_k$ achieves a minimum over S_k. Hence, $(\widehat{NP}_k)$ has a point of minimizer that naturally need not be unique. Denote a point of minimizer of $(\widehat{NP}_k)$ by (x_k, u_k) and thus, obtaining a bounded sequence $\{(x_k, u_k)\} \subset Y \times (-S)$, which satisfies

$$\|G(x_k) - u_k\| \leq \sqrt{2\varepsilon_k(f(\bar{x}) - \mu)},$$

and as $(x_k, u_k) \in S_k$,

$$f(x_k) \leq \hat{f}_k(x_k, u_k) \leq f(\bar{x}).$$

Because $\{(x_k, u_k)\}$ is bounded, by the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, assume that $x_k \to \tilde{x}$ and $u_k \to \tilde{u}$. Therefore, as $k \to \infty$, $\varepsilon_k \to 0$ and thus

$$\|G(\tilde{x}) - \tilde{u}\| = 0 \quad \text{and} \quad f(\tilde{x}) \leq f(\bar{x}).$$

Hence, $\tilde{u} = G(\tilde{x})$ and thus $(\tilde{x}, \tilde{u})$ is also a minimizer of $(\widehat{NP})$. But as $(\bar{x}, \bar{u})$ is the unique point of minimizer of $(\widehat{NP})$, we have $\tilde{x} = \bar{x}$ and $\tilde{u} = \bar{u}$.

Because (x_k, u_k) is a point of minimizer of $(\widehat{NP}_k)$, it is a simple exercise to see that x_k minimizes $\hat{f}_k(x, u_k)$ over Y and u_k minimizes $\hat{f}_k(x_k, u)$ over $-S$. Hence,

$$0 \in \partial_x^\circ \hat{f}_k(x_k, u_k) + N_Y(x_k), \tag{3.13}$$

$$0 \in \partial_u^\circ \hat{f}_k(x_k, u_k) + N_{-S}(u_k). \tag{3.14}$$

Now we analyze these conditions in more detail. Denote

$$y_k = \frac{1}{\varepsilon_k}(G(x_k) - u_k).$$

From condition (3.14),

$$-\nabla_u \hat{f}_k(x_k, u_k) \in N_{-S}(u_k).$$

One can easily compute $\nabla_u \hat{f}_k(x_k, u_k)$ to see that $y_k = -\nabla_u \hat{f}_k(x_k, u_k)$ and hence $y_k \in N_{-S}(u_k)$. Moreover, applying the Sum Rule and the Chain Rule for a locally Lipschitz to (3.13), then for each k,

$$0 \in -v + \partial_C G(x_k)^T y_k + N_Y(x_k). \tag{3.15}$$

Suppose that $\{y_k\}$ is bounded and thus by the Bolzano–Weierstrass Theorem has a convergent subsequence. Without loss of generality, suppose that $y_k \to \bar{y}$. Noting that the normal cone is graph closed as a set-valued map and $\partial_C G$ is locally bounded, taking the limit as $k \to \infty$ in (3.15) leads to

$$0 \in -v + \partial_C G(\bar{x})^T \bar{y} + N_Y(\bar{x}).$$

But as $\bar{x} \in int\ Y$, by Example 2.38 $N_Y(\bar{x}) = \{0\}$. As $\partial_C G(\bar{x}) \subset \partial G(\bar{x})$, $v \in \partial_C G(\bar{x})^T \bar{y} \subset \partial G(\bar{x})^T \bar{y}$. Thus, $v = z^T \bar{y}$ for some $z \in \partial G(\bar{x})$.

The important question is can $\{y_k\}$ be unbounded? We show that if $\{y_k\}$ is unbounded, then the Slater constraint qualification, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $G(\hat{x}) \in -ri\ S$ is violated.

On the contrary, assume that $\{y_k\}$ is unbounded and thus $\|y_k\| \to \infty$ as $k \to \infty$. Hence, noting that $\partial_C G(x_k) \subset \partial G(x_k)$, from (3.15) we have

$$0 \in \frac{1}{\|y_k\|}(-v) + \partial G(x_k)^T \frac{y_k}{\|y_k\|} + \frac{1}{\|y_k\|} N_Y(x_k),$$

which implies

$$0 \in \frac{1}{\|y_k\|}(-v) + \partial G(x_k)^T w_k + N_Y(x_k), \tag{3.16}$$

where $w_k = \dfrac{y_k}{\|y_k\|}$. Hence, $\{w_k\}$ is a bounded sequence and thus by the Bolzano–Weierstrass Theorem, Proposition 1.3, has a convergent subsequence. Without loss of generality, assume that $w_k \to \bar{w}$ with $\|\bar{w}\| = 1$. Hence from (3.16),

$$0 \in \partial G(\bar{x})^T \bar{w}. \tag{3.17}$$

As $y_k \in N_{-S}(u_k)$, we have $w_k \in N_{-S}(u_k)$. Again using the fact that the normal cone map has a closed graph, $\bar{w} \in N_{-S}(\bar{u})$. Hence,

$$\langle \bar{w}, z - G(\bar{x}) \rangle \leq 0, \ \ z \in -S.$$

Because S is a cone, $0 \in -S$, thus

$$\langle \bar{w}, -G(\bar{x}) \rangle \leq 0,$$

that is,

$$\langle \bar{w}, G(\bar{x}) \rangle \geq 0. \tag{3.18}$$

Consider $p \in -S$. As S is a convex cone, by Theorem 2.20, $G(\bar{x}) + p \in -S$. Hence,

$$\langle \bar{w}, G(\bar{x}) + p - G(\bar{x}) \rangle \leq 0,$$

which implies $\langle \bar{w}, p \rangle \leq 0$. Because p was arbitrary,

$$\langle \bar{w}, p \rangle \leq 0, \ \ \forall\, p \in -S.$$

Thus, $\bar{w} \in S^+$. Hence, $\langle \bar{x}, G(\bar{x}) \rangle \leq 0$, which together with (3.18) leads to $\langle \bar{w}, G(\bar{x}) \rangle = 0$. For any $y \in \mathbb{R}^n$,

$$G(y) - G(\bar{x}) - A(y - \bar{x}) \in S, \; \forall \; A \in \partial G(\bar{x}),$$

which implies

$$\langle \bar{w}, G(y) \rangle - \langle \bar{w}, G(\bar{x}) \rangle - \langle \bar{w}, A(y - \bar{x}) \rangle \geq 0, \; \forall \; A \in \partial G(\bar{x}).$$

From (3.17), if $\{y_k\}$ is unbounded, there exists $\bar{z} \in \partial G(\bar{x})$ such that $\bar{z}^T \bar{w} = 0$. Thus, from the above inequality we have

$$\langle \bar{w}, G(y) \rangle - \langle \bar{w}, G(\bar{x}) \rangle - \langle \bar{z}^T \bar{w}, (y - \bar{x}) \rangle \geq 0, \; \forall \; y \in \mathbb{R}^n,$$

which along with $\langle \bar{w}, G(\bar{x}) \rangle = 0$ and $\bar{z}^T \bar{w} = 0$ yields

$$\langle \bar{w}, G(y) \rangle \geq 0, \; \forall \; y \in \mathbb{R}^n.$$

If the Slater constraint qualification holds, there exists $\hat{x} \in \mathbb{R}^n$ such that $G(\hat{x}) \in -ri\ S$. As $\|\bar{w}\| = 1$ and $\bar{w} \in S^+$, $\langle \bar{w}, G(\hat{x}) \rangle < 0$, which contradicts the above inequality. Therefore, $\{y_k\}$ cannot be an unbounded sequence. Thus, we leave it to the reader to see that

$$v = z^T \bar{y}, \quad \text{where} \quad \bar{y} \in S^+,$$

and hence conclude that

$$N_C(\bar{x}) = \{v \in \mathbb{R}^n : \text{ there exists } \bar{y} \in S^+, \; z \in \partial G(\bar{x}) \text{ satisfying} \\ \langle \bar{y}, G(\bar{x}) \rangle = 0 \text{ such that } v = z^T \bar{y}\}.$$

The reader is now urged to write the necessary and sufficient optimality conditions for the problem $(CCP1)$, as the structure of the normal cone to C at $\bar{x}$ is now known.

Chapter 4

Saddle Points, Optimality, and Duality

4.1 Introduction

In the previous chapter, the KKT optimality conditions was studied using the normal cone as one of the main vehicles of expressing the optimality conditions. One of the central issues in the previous chapter was the computation of the normal cone at the point of the feasible set C where the set C was explicitly described by the inequality constraints. In this chapter our approach to the KKT optimality condition will take us deeper into convex optimization theory and also we can avoid the explicit computation of the normal cone. This approach uses the saddle point condition of the Lagrangian function associated with (CP). We motivate the issue using two-person-zero-sum games.

Consider a two-person-zero-sum game where we denote the players as Player 1 and Player 2 having strategy sets $X \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^m$, respectively, which we assume to be compact for simplicity. In each move of the game, the players reveal their choices simultaneously. For every choice $x \in X$ by Player 1 and $\lambda \in \Lambda$ by Player 2, an amount $\mathcal{L}(x, \lambda)$ is paid by Player 1 to Player 2. Now Player 1 behaves in the following way. For any given choice of strategy $x \in X$, he would like to know what the maximum amount he would have to give to Player 2. In effect, he computes the function

$$\phi(x) = \max_{\lambda \in \Lambda} \mathcal{L}(x, \lambda).$$

Further, it is natural that he would choose an $x \in X$ that minimizes $\phi(x)$, that is, Player 1 solves the problem

$$\min \ \phi(x) \quad \text{subject to} \quad x \in X,$$

which implies that in effect, he solves a minimax problem

$$\min_{x \in X} \max_{\lambda \in \Lambda} \mathcal{L}(x, \lambda).$$

Similarly, Player 2 would naturally want to know what the guaranteed amount he will receive once he makes a move $\lambda \in \Lambda$. This means he computes the function

$$\psi(\lambda) = \min_{x \in X} \mathcal{L}(x, \lambda).$$

Of course he would like to maximize the amount of money he gets and therefore solves the problem

$$\max\ \psi(\lambda) \quad \text{subject to} \quad \lambda \in \Lambda,$$

that is, he solves

$$\max_{\lambda\in\Lambda}\min_{x\in X}\mathcal{L}(x,\lambda).$$

Thus, in every game there are two associated optimization problems. The minimization problem for Player 1 and the maximization problem for Player 2. In the optimization literature, the problem associated with Player 1 is called the *primal problem* while that associated with Player 2 is called the *dual problem. Duality* is a deep issue in modern optimization theory. In this chapter, we will have quite a detailed discussion on duality in convex optimization. The game is said to have a *value* if

$$\min_{x\in X}\max_{\lambda\in\Lambda}\mathcal{L}(x,\lambda) = \max_{\lambda\in\Lambda}\min_{x\in X}\mathcal{L}(x,\lambda).$$

The above relation is the *minimax equality.*

For any given $\tilde{\lambda} \in \Lambda$,

$$\min_{x\in X}\mathcal{L}(x,\tilde{\lambda}) \leq \min_{x\in X}\max_{\lambda\in\Lambda}\mathcal{L}(x,\lambda).$$

Because $\tilde{\lambda} \in \Lambda$ is arbitrary, we obtain the *minimax inequality*, that is,

$$\max_{\lambda\in\Lambda}\min_{x\in X}\mathcal{L}(x,\lambda) \leq \min_{x\in X}\max_{\lambda\in\Lambda}\mathcal{L}(x,\lambda),$$

which always holds true.

Of course the minimax equality would hold true if a *saddle point* exists, that is, a pair $(\bar{x},\bar{\lambda}) \in X \times \Lambda$ exists that satisfies the following inequality,

$$\mathcal{L}(\bar{x},\lambda) \leq \mathcal{L}(\bar{x},\bar{\lambda}) \leq \mathcal{L}(x,\bar{\lambda}),\ \forall\ x \in X,\ \forall\ \lambda \in \Lambda.$$

The above relation is called the *saddle point condition.* It is easy to observe that $(\bar{x},\bar{\lambda}) \in X \times \Lambda$ is a saddle point if and only if

$$\max_{\lambda\in\Lambda}\mathcal{L}(\bar{x},\lambda) = \mathcal{L}(\bar{x},\bar{\lambda}) = \min_{x\in X}\mathcal{L}(x,\bar{\lambda}).$$

The above condition implies

$$\min_{x\in X}\max_{\lambda\in\Lambda}\mathcal{L}(x,\lambda) \leq \max_{\lambda\in\Lambda}\mathcal{L}(\bar{x},\lambda) = \min_{x\in X}\mathcal{L}(x,\bar{\lambda}) \leq \max_{\lambda\in\Lambda}\min_{x\in X}\mathcal{L}(x,\lambda),$$

which along with the minimax inequality yields the minimax equality.

Before moving on to study the optimality of the convex programming problem (CP) via the saddle point approach, we state the Saddle Point Theorem (Proposition 2.6.9, Bertsekas [12]) for which we will need the following notations.

For each $\lambda \in \Lambda$, define the proper function $\phi_\lambda : \mathbb{R}^n \to \bar{\mathbb{R}}$ as

$$\phi_\lambda(x) = \begin{cases} \mathcal{L}(x,\lambda), & x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

and for every $x \in X$, the proper function $\psi_x : \mathbb{R}^m \to \bar{\mathbb{R}}$ is given by

$$\psi_x(\lambda) = \begin{cases} -\mathcal{L}(x,\lambda), & \lambda \in \Lambda, \\ +\infty, & \text{otherwise.} \end{cases}$$

Proposition 4.1 *(Saddle Point Theorem) Assume that for every $\lambda \in \Lambda$, ϕ_λ and for every $x \in X$, ψ_x are lsc and convex. The set of saddle points of $\mathcal{L}$ is nonempty and compact under any one of the following conditions:*

(i) X and Λ are compact.

(ii) Λ is compact and there exists $\bar{\lambda} \in \Lambda$ and $\alpha \in \mathbb{R}$ such that the set

$$\{x \in X : \mathcal{L}(x,\bar{\lambda}) \le \alpha\}$$

is nonempty and compact.

(iii) X is compact and there exists $\bar{x} \in X$ and $\alpha \in \mathbb{R}$ such that the set

$$\{\lambda \in \Lambda : \mathcal{L}(\bar{x},\lambda) \ge \alpha\}$$

is nonempty and compact.

(iv) There exist $\bar{x} \in X$, $\bar{\lambda} \in \Lambda$, and $\alpha \in \mathbb{R}$ such that

$$\{x \in X : \mathcal{L}(x,\bar{\lambda}) \le \alpha\} \quad \textit{and} \quad \{\lambda \in \Lambda : \mathcal{L}(\bar{x},\lambda) \ge \alpha\}$$

are nonempty and compact.

This proposition will play a pivotal role in the study of enhanced optimality conditions in Chapter 5.

4.2 Basic Saddle Point Theorem

The saddle point condition can itself be taken as an optimality condition for the problem of Player 1, that is,

$$\min \phi(x) \quad \text{subject to} \quad x \in X.$$

Our question is, can we construct a function like $\mathcal{L}(x, \lambda)$ for the convex (CP) for which $f(x)$ can be represented in a way as $\phi(x)$ has been represented through $\mathcal{L}(x, \lambda)$? Note that if we remove the compactness from Λ, then $\phi(x)$ could take up $+\infty$ value for some x. It is quite surprising that for the objective function $f(x)$ of (CP), such a function can be obtained by considering the classical Lagrangian function from calculus.

For the problem (CP) with inequality constraints, we construct the *Lagrangian function* $L : \mathbb{R}^n \times \mathbb{R}^m_+ \to \mathbb{R}$ as

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m_+$. Observe that it is a simple matter to show that

$$\sup_{\lambda \in \mathbb{R}^m_+} L(x, \lambda) = \begin{cases} f(x), & x \text{ is feasible}, \\ +\infty, & \text{otherwise}. \end{cases}$$

Here, the Lagrangian function $L(x, \lambda)$ is playing the role of $\mathcal{L}(x, \lambda)$. So the next pertinent question is, if we can solve (CP) then does there exist a saddle point for it? Does the existence of a saddle point for $L(x, \lambda)$ guarantee that a solution to the original problem (CP) is obtained? The following theorem answers the above questions. Recall the convex programming problem

$$\min \ f(x) \quad \text{subject to} \quad x \in C \tag{CP}$$

with C given by

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, 2, \ldots, m\}, \tag{4.1}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are now assumed to be convex and non-affine functions.

Theorem 4.2 *Consider the convex programming problem (CP) with C given by (4.1). Assume that the Slater constraint qualification holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$. Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m) \in \mathbb{R}^m_+$ satisfying the complementary slackness condition, that is, $\bar{\lambda}_i g_i(\bar{x}) = 0$ for $i = 1, 2, \ldots, m$ and the saddle point condition*

$$L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}), \ \forall \ x \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^m_+.$$

Proof. As $\bar{x}$ is a point of minimizer of (CP) the following system

$$\left.\begin{array}{rcl} f(x) - f(\bar{x}) & < & 0 \\ g_i(x) & < & 0, \ i = 1, 2, \ldots, m \end{array}\right\},$$

has no solution. Define a set

$$\Lambda = \{(y_0, y) \in \mathbb{R} \times \mathbb{R}^m : \text{there exists } x \in \mathbb{R}^n \text{ such that} \\ f(x) - f(\bar{x}) < y_0,\ g_i(x) < y_i,\ i = 1, 2, \ldots, m\}.$$

We leave it the reader to prove that the set Λ is convex and open. It is clear that $(0, 0) \notin \Lambda$. Hence, by the Proper Separation Theorem, Theorem 2.26 (iv), there exists $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$ with $(\lambda_0, \lambda) \neq (0, 0)$ such that

$$\lambda_0 y_0 + \sum_{i=1}^{m} \lambda_i y_i \geq 0,\ \forall\ (y_0, y) \in \Lambda. \tag{4.2}$$

Corresponding to $\bar{x} \in \mathbb{R}^n$, for $y_i > 0,\ i = 0, 1, \ldots, m$, $(y_0, y) \in \Lambda$. Also, for any $\gamma > 0$, $(y_0 + \gamma, y) \in \Lambda$. Therefore, from condition (4.2),

$$\lambda_0 \geq -\frac{1}{\gamma}\{\lambda_0 y_0 + \sum_{i=1}^{m} \lambda_i y_i\},$$

which as the limit $\gamma \to \infty$ leads to $\lambda_0 \geq 0$. It is now left to the reader to prove in a similar fashion that $\lambda \in \mathbb{R}^m_+$.

For any $x \in \mathbb{R}^n$, consider a fixed $\alpha_i > 0, i = 0, 1, \ldots, m$. Then for any $\gamma_i > 0,\ i = 0, 1, \ldots, m$,

$$(f(x) - f(\bar{x}) + \gamma_0\alpha_0, g_1(x) + \gamma_1\alpha_1, \ldots, g_m(x) + \gamma_m\alpha_m) \in \Lambda.$$

Therefore, from (4.2),

$$\lambda_0(f(x) - f(\bar{x}) + \gamma_0\alpha_0) + \sum_{i=1}^{m} \lambda_i(g_i(x) + \gamma_i\alpha_i) \geq 0.$$

As $\gamma_i \to 0$, the above inequality yields

$$\lambda_0(f(x) - f(\bar{x})) + \sum_{i=1}^{m} \lambda_i g_i(x) \geq 0,\ \forall\ x \in \mathbb{R}^n. \tag{4.3}$$

We claim that $\lambda_0 \neq 0$. On the contrary, suppose that $\lambda_0 = 0$, thereby reducing (4.3) to

$$\sum_{i=1}^{m} \lambda_i g_i(x) \geq 0,\ \forall\ x \in \mathbb{R}^n.$$

This violates the Slater constraint qualification. Thus, $\lambda_0 > 0$. Therefore, denoting $\bar{\lambda}_i = \dfrac{\lambda_i}{\lambda_0}$, the condition (4.3) yields

$$f(x) - f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) \geq 0,\ \forall\ x \in \mathbb{R}^n.$$

In particular, $x = \bar{x}$ in the above inequality leads to $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) = 0$. Because the sum of negative numbers is zero only if each term is zero, the complementary slackness condition, that is, $\bar{\lambda}_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m$, holds. Therefore, the preceding inequality leads to

$$f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) \geq f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}),\ \forall\ x \in \mathbb{R}^n,$$

which implies

$$L(x, \bar{\lambda}) \geq L(\bar{x}, \bar{\lambda}),\ \forall\ x \in \mathbb{R}^n.$$

Further, for any $\lambda \in \mathbb{R}^m_+$, $\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \leq 0$. Thus,

$$f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \leq f(\bar{x}) = f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}),$$

that is,

$$L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}),\ \forall\ \lambda \in \mathbb{R}^m_+,$$

thereby establishing the saddle point condition.

Conversely, suppose that there exists $\bar{\lambda} \in \mathbb{R}^m_+$ such that the saddle point condition and the complementary slackness condition hold at $\bar{x}$. We first prove that $\bar{x}$ is feasible, that is, $-g(\bar{x}) = (-g_1(\bar{x}), -g_2(\bar{x}), \ldots, -g_m(\bar{x})) \in \mathbb{R}^m_+$. On the contrary, assume that $-g(\bar{x}) \notin \mathbb{R}^m_+$. As $\mathbb{R}^m_+$ is a closed convex cone, by the Strict Separation Theorem, Theorem 2.26 (iii), there exists $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$ such that

$$\langle \lambda, g(\bar{x}) \rangle = \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) > 0.$$

Therefore,

$$f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) > f(\bar{x}),$$

which implies $L(\bar{x}, \lambda) > L(\bar{x}, \bar{\lambda})$, thereby contradicting the saddle point condition. Hence, $\bar{x}$ is feasible to (CP).

Because $L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda})$ and the complementary slackness condition is satisfied,

$$f(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \leq f(x),\ \forall\ x \in C.$$

Thus, $\bar{x}$ is a point of minimizer of (CP). $\square$

The consequence of the saddle point criteria is simple. If $(\bar{x}, \bar{\lambda})$ is a saddle point associated with the Lagrangian function of (CP) where $\bar{x}$ is a point of minimizer of f over C, then

$$L(\bar{x}, \bar{\lambda}) = \min_{x \in \mathbb{R}^n} L(x, \bar{\lambda})$$

with $\bar{\lambda}_i g_i(\bar{x}) = 0$ for $i = 1, 2, \ldots, m$. Therefore, by the optimality condition for the unconstrained problem, Theorem 2.89,

$$0 \in \partial_x L(\bar{x}, \bar{\lambda}),$$

which under Slater constraint qualification yields

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}),$$

thus leading to the KKT optimality conditions for (CP).

4.3 Affine Inequalities and Equalities and Saddle Point Condition

Observe that in the previous section we had mentioned that the convex function are non-affine. This eventually has to do with the Slater constraint qualification. Consider the set

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 0, \ -x_1 \leq 0\}.$$

This set is described by affine inequalities. However, $C = \{(0, 0)\}$ and hence the Slater constraint qualification fails. The question is whether in such a situation the saddle point condition exists or not. What we show below is that the presence of affine inequalities does not affect the saddle point condition. In fact, we should only bother about the Slater constraint qualification for the convex non-affine inequalities. The presence of affine inequalities by itself is a constraint qualification. To the best of our knowledge, the first study in this respect was due to Jaffray and Pomerol [65]. We present their result establishing the saddle point criteria under a modified version of Slater constraint qualification using the separation theorem. For that we now consider the feasible set C of the convex programming problem (CP) defined by convex non-affine and affine inequalities as

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, 2, \ldots, m, \ h_j(x) \leq 0, \ j = 1, 2, \ldots, l\}, \quad (4.4)$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are convex functions while $h_j : \mathbb{R}^n \to \mathbb{R}$,

$j = 1, 2, \ldots, l$, are affine functions. Observe that C is a convex set. Corresponding to this convex programming problem (CP), the associated Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^l_+$ is defined as

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x).$$

Then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is the saddle point of (CP) with C given by (4.4) if

$$L(\bar{x}, \lambda, \mu) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu}).$$

We shall now present the proof of Jaffray and Pomerol in a more detailed and simplified manner.

Theorem 4.3 *Consider (CP) with C defined by (4.4). Assume that the modified Slater constraint qualification holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m$, and $h_j(\hat{x}) \leq 0,\ j = 1, 2, \ldots, l$. Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exist $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m_+ \times \mathbb{R}^l_+$ such that*

$$L(\bar{x}, \lambda, \mu) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu}),\ \forall\ x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^l_+$$

along with the complementary slackness conditions, that is,

$$\bar{\lambda}_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m \quad \text{and} \quad \bar{\mu}_j h_j(\bar{x}) = 0,\ j = 1, 2, \ldots, l.$$

Proof. Consider an index set J as the (possibly empty) maximal subset of $\{1, 2, \ldots, l\}$ such that there exists $\alpha_j > 0$ for $j \in J$ such that $\sum_{j \in J} \alpha_j h_j(x) = 0$ for every $x \in \mathbb{R}^n$. Observe that for every $x \in C$,

$$h_j(x) = 0,\ \forall\ j \in J.$$

Otherwise, if for some $x \in C$ and for some $j \in J$, $h_j(x) < 0$, the maximality of J is contradicted.

Define the *Lagrange covers* of (CP) as

$$\begin{aligned} \Lambda = \{(y_0, y, z) \in \mathbb{R}^{1+m+l} : &\text{ there exists } x \in \mathbb{R}^n \text{ such that} \\ & f(x) - f(\bar{x}) \leq y_0,\ g_i(x) \leq y_i,\ i = 1, 2, \ldots, m, \\ & h_j(x) \leq z_j,\ j \in J^c,\ h_j(x) = z_j,\ j \in J\}, \end{aligned}$$

where $J^c = \{j \in \{1, 2, \ldots, l\} : j \notin J\}$.

We claim that the set Λ is convex. Consider (y_0^1, y^1, z^1) and (y_0^2, y^2, z^2) in Λ with x_1 and x_2 the respective associated elements from $\mathbb{R}^n$. For any $\lambda \in [0, 1]$, $x = \lambda x_1 + (1 - \lambda) x_2 \in \mathbb{R}^n$. By the convexity of f and $g_i,\ i = 1, 2, \ldots, m$,

$$\begin{aligned} f(x) - f(\bar{x}) &\leq \lambda(f(x_1) - f(\bar{x})) + (1 - \lambda)(f(x_2) - f(\bar{x})) \\ &\leq \lambda y_0^1 + (1 - \lambda) y_0^2, \\ g_i(x) &\leq \lambda g_i(x_1) + (1 - \lambda) g_i(x_2) \leq \lambda y_i^1 + (1 - \lambda) y_i^2,\ i = 1, 2, \ldots, m, \end{aligned}$$

while the affineness of $h_j,\ j = 1, 2, \ldots, l$ leads to

$$h_j(x) = \lambda h_j(x_1) + (1-\lambda)h_j(x_2) \leq \lambda z_j^1 + (1-\lambda)z_j^2,\ j \in J^c,$$
$$h_j(x) = \lambda h_j(x_1) + (1-\lambda)h_j(x_2) = \lambda z_j^1 + (1-\lambda)z_j^2,\ j \in J.$$

Thus, for every $\lambda \in [0,1]$, $\lambda(y_0^1, y^1, z^1) + (1-\lambda)(y_0^2, y^2, z^2) \in \Lambda$ with $x \in \mathbb{R}^n$ as the associated element, thereby implying the convexity of Λ.

Observe that corresponding to the point of minimizer of (CP), $\bar{x} \in \mathbb{R}^n$, $(\bar{y}_0, 0, 0) \in \Lambda$ if and only if $\bar{y}_0 \geq 0$. Also, $(y_0, 0, 0)$ belongs to the affine hull of Λ for every $y_0 \in \mathbb{R}$, and hence, $(0,0,0)$ belongs to the relative boundary of Λ. Applying the Proper Separation Theorem, Theorem 2.26 (iv), to the Lagrange cover Λ and the relative boundary point $(0,0,0)$, there exists $(\lambda_0, \lambda, \mu) \in \mathbb{R}^{1+m+l}$ with $(\lambda_0, \lambda, \mu) \neq (0,0,0)$ such that

$$\lambda_0 y_0 + \sum_{i=1}^{m} \lambda_i y_i + \sum_{j=1}^{l} \mu_j z_j \geq 0,\ \forall\ (y_0, y, z) \in \Lambda \tag{4.5}$$

and for some $(y_0, y, z) \in \Lambda$,

$$\lambda_0 y_0' + \sum_{i=1}^{m} \lambda_i y_i' + \sum_{j=1}^{l} \mu_j z_j' > 0. \tag{4.6}$$

Consider $(y_0, y, z) \in \Lambda$. For any $\alpha_0 > 0$ and $\alpha \in int\ \mathbb{R}^m_+$, $(y_0+\alpha_0, y+\alpha, z) \in \Lambda$. Therefore, by (4.5), $i' = 0, 1, \ldots, m$,

$$\lambda_{i'} \geq -\frac{1}{\alpha_{i'}} \left\{ \lambda_0 y_0 + \sum_{i=1}^{m} \lambda_i y_i + \sum_{i=1}^{l} \mu_j z_j + \sum_{i=0}^{i'-1} \lambda_i \alpha_i + \sum_{i=i'+1}^{m} \lambda_i \alpha_i \right\},$$

which as the limit $\alpha_{i'} \to +\infty$ yields $\lambda_{i'} \geq 0,\ i' = 0, 1, \ldots, m$. Using the above technique, we can also show that $\mu_j \geq 0,\ j \in J^c$. The reader is advised to check this out. Observe that μ_j for $j \in J$ are unrestricted.

Let us proceed by assuming that J is nonempty. Therefore, there exist $\alpha_j > 0,\ j \in J$ such that $\sum_{j \in J} \alpha_j h_j(x) = 0$ for every $x \in \mathbb{R}^n$. Redefining $\lambda_i,\ i = 0, 1, \ldots, m$, and $\mu_j,\ j = 1, 2, \ldots, l$, as

$$\hat{\lambda}_i = \lambda_i,\ i = 0, 1, \ldots, m, \quad \hat{\mu}_j = \mu_j,\ j \in J^c \quad \text{and} \quad \hat{\mu}_j = \mu_j + \gamma\alpha_j,\ j \in J,$$

where $\gamma > 0$ is chosen such that $\hat{\mu}_j > 0$ for $j \in J$. Also, observe that

$$\sum_{j \in J} \hat{\mu}_j h_j(x) = \sum_{j \in J} \mu_j h_j(x) + \sum_{j \in J} \gamma\alpha_j h_j(x) = \sum_{j \in J} \mu_j h_j(x).$$

Thus, the conditions (4.5) and (4.6) hold for $(\hat{\lambda}_0, \hat{\lambda}, \hat{\mu})$ as well.

We claim that $\hat{\lambda}_0,\ \hat{\lambda}_i,\ i = 1, 2, \ldots, m$, and $\hat{\mu}_j,\ j \in J^c$, are not all simultaneously zero. On the contrary, assume that $\hat{\lambda}_0 = 0,\ \hat{\lambda}_i = 0,\ i = 1, 2, \ldots, m$, and $\hat{\mu}_j = 0,\ j \in J^c$. Therefore, from the construction of Λ along with (4.5) yields

$$\sum_{j \in J} \hat{\mu}_j h_j(x) \geq 0,\ \forall\, x \in \mathbb{R}^n.$$

As $\bar{x}$ is feasible for (CP), the above condition becomes

$$\sum_{j \in J} \hat{\mu}_j h_j(\bar{x}) = 0.$$

Therefore, the affine function $\sum_{j \in J} \hat{\mu}_j h_j(.)$ achieves its minimum over $\mathbb{R}^n$ at $\bar{x}$. Moreover, an affine function is unbounded over $\mathbb{R}^n$. This shows that

$$\sum_{j \in J} \hat{\mu}_j h_j(x) = 0,\ \forall\, x \in \mathbb{R}^n.$$

By condition (4.6), there exists $x' \in \mathbb{R}^n$ associated to $(y_0', y', z') \in \Lambda$ such that

$$\sum_{j \in J} \hat{\mu}_j h_j(x') > 0.$$

Hence, a contradiction is reached. Therefore, $\hat{\lambda}_0,\ \hat{\lambda}_i,\ i = 1, 2, \ldots, m$, and $\hat{\mu}_j,\ j \in J^c$, are not all simultaneously zero.

Next suppose that $\hat{\lambda}_0 = 0$ and $\hat{\lambda}_i = 0,\ i = 1, 2, \ldots, m$, and for some $j \in J^c$, $\hat{\mu}_j > 0$. Again working along the preceding lines, one obtains

$$\sum_{j \in J^c : \hat{\mu}_j > 0} \hat{\mu}_j h_j(x) + \sum_{j \in J} \hat{\mu}_j h_j(x) = 0,\ \forall\, x \in \mathbb{R}^n.$$

Observe that $\{j \in J^c : \hat{\mu}_j > 0\}$ is nonempty. Because the above condition holds for $j \in \{j \in J^c : \hat{\mu}_j > 0\} \cup \{j \in J : \hat{\mu}_j > 0\}$, thereby contradicting the maximality of the index set J. Hence $\hat{\lambda}_0$ and $\hat{\lambda}_i,\ i = 1, 2, \ldots, m$, are not simultaneously zero.

Assume that $\hat{\lambda}_0 = 0$. As the modified Slater constraint qualification holds, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m$, and $h_j(\hat{x}) \leq 0$, $j = 1, 2, \ldots, l$, corresponding to $\hat{x}$,

$$(f(\hat{x}) - f(\bar{x}), g_1(\hat{x}), \ldots, g_m(\hat{x}), h_1(\hat{x}), \ldots, h_l(\hat{x})) \in \Lambda,$$

which along with condition (4.5) and the modified Slater constraint qualification leads to

$$0 > \sum_{i=1}^{m} \hat{\lambda}_i g_i(\hat{x}) + \sum_{j=1}^{l} \hat{\mu}_j h_j(\hat{x}) \geq 0,$$

which is a contradiction. Hence $\hat{\lambda}_0 \neq 0$.

Now dividing (4.5) throughout by $\hat{\lambda}_0$ yields

$$y_0 + \sum_{i=1}^{m} \bar{\lambda}_i y_i + \sum_{j=1}^{l} \bar{\mu}_j z_j \geq 0, \ \forall \ (y_0, y, z) \in \Lambda, \tag{4.7}$$

where $\bar{\lambda}_i = \dfrac{\hat{\lambda}_i}{\hat{\lambda}_0}$, $i = 1, 2, \ldots, m$, and $\bar{\mu}_j = \dfrac{\hat{\mu}_j}{\hat{\lambda}_0}$, $j = 1, 2, \ldots, l$. Corresponding to every $x \in \mathbb{R}^n$,

$$(f(x) - f(\bar{x}), g_1(x), \ldots, g_m(x), h_1(x), \ldots, h_l(x)) \in \Lambda,$$

thereby reducing the inequality (4.7) to

$$f(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) + \sum_{j=1}^{l} \bar{\mu}_j h_j(x), \ \forall \ x \in \mathbb{R}^n. \tag{4.8}$$

By the feasibility of $\bar{x}$ for (CP) and the fact that $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m_+ \times \mathbb{R}^l_+$, condition (4.8) implies that

$$L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu}), \ \forall \ x \in \mathbb{R}^n.$$

In particular, taking $x = \bar{x}$ in (4.8), along with the feasibility of $\bar{x}$, leads to

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) + \sum_{j=1}^{l} \bar{\mu}_j h_j(\bar{x}) = 0.$$

This shows that

$$\bar{\lambda}_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m \quad \text{and} \quad \bar{\mu}_j h_j(\bar{x}) = 0, \ j = 1, 2, \ldots, l,$$

thereby establishing the complementary slackness condition. For any $(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^l_+$, again by the feasibility of $\bar{x}$,

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) + \sum_{j=1}^{l} \mu_j h_j(\bar{x}) \leq 0 = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) + \sum_{j=1}^{l} \bar{\mu}_j h_j(\bar{x}),$$

that is,

$$L(\bar{x}, \lambda, \mu) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}), \ \forall \ \lambda \in \mathbb{R}^m_+, \ \mu \in \mathbb{R}^l_+,$$

thereby leading to the desired result. The converse of the above the result can be obtained in a manner similar to Theorem 4.2. □

In the convex programming problem (CP) considered by Jaffray and Pomerol [65], the problem involved only convex non-affine and affine inequalities. Next we present a similar result from Florenzano and Van [47] to derive the saddle point criteria under a modified version of Slater constraint qualification but for a more general scenario involving additional affine equalities and abstract constraints in (4.4). Consider the feasible set C of the convex programming problem (CP) as

$$\begin{aligned} C = \{x \in X : g_i(x) \leq 0,\ i = 1, 2, \ldots, m,\ h_j(x) \leq 0,\ j = 1, 2, \ldots, s, \\ h_j(x) = 0,\ j = s+1, s+2, \ldots, l\}, \end{aligned} \quad (4.9)$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are convex non-affine functions; $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, l$, are affine functions; and $X \subset \mathbb{R}^n$ is a convex set. Corresponding to this problem, the associated Lagrangian function $L : X \times \mathbb{R}^m_+ \times \mathbb{R}^l \to \mathbb{R}$ is defined as

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x),$$

where $\mu = (\hat{\mu}, \tilde{\mu}) \in \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$. Then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is called the saddle point of the above problem if

$$L(\bar{x}, \lambda, \mu) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu}),\ \forall\ x \in X,\ \lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^l,$$

where $\mu = (\hat{\mu}, \tilde{\mu})$ and $\bar{\mu} = (\bar{\hat{\mu}}, \bar{\tilde{\mu}})$ are in $\mathbb{R}^s_+ \times \mathbb{R}^{l-s}$.

Theorem 4.4 *Consider the convex programming problem (CP) with C defined by (4.9). Let $\bar{x}$ be a point of minimizer of (CP). Assume that there exists $\hat{x} \in ri\ X$ such that*

$$\begin{aligned} h_j(\hat{x}) &\leq 0, \quad j = 1, 2, \ldots, s, \\ h_j(\hat{x}) &= 0, \quad j = s+1, s+2, \ldots, l. \end{aligned}$$

Then there exist $(\lambda_0, \lambda) \in \mathbb{R}_+ \times \mathbb{R}^m_+$ with $(\lambda_0, \lambda) \neq (0, 0)$, and $\mu = (\hat{\mu}, \tilde{\mu}) \in \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$ such that

$$\lambda_0 f(\bar{x}) \leq \lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x),\ \forall\ x \in X,$$

$$\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m, \qquad \textit{and} \qquad \hat{\mu}_j h_j(\bar{x}) = 0,\ j = 1, 2, \ldots, s.$$

Proof. Consider the set

$$\begin{aligned} \Lambda = \{(y_0, y, z) \in \mathbb{R}^{1+m+l} : \text{there exists } x \in X \text{ such that } f(x) - f(\bar{x}) < y_0, \\ g_i(x) < y_i,\ i = 1, 2, \ldots, m, \\ h_j(x) = z_j,\ j = 1, 2, \ldots, l\}. \end{aligned}$$

It can be easily shown as in the proof of Theorem 4.3 that Λ is a convex set.

Also, Λ is nonempty because corresponding to the point of minimizer $\bar{x}$ of (CP), one can define $(y_0, y, z) \in \Lambda$ as

$$y_0 > 0, \quad y_i > 0, \ i = 1, 2, \ldots, m, \quad \text{and} \quad z_j = h_j(\bar{x}), \ j = 1, 2, \ldots, l.$$

As Λ is a nonempty convex set, by Proposition 2.14, $ri\ \Lambda$ is also a nonempty convex set. Note that

$$\Lambda \cap (\mathbb{R}_-^{1+m+s} \times \{0_{\mathbb{R}^{l-s}}\}) = \emptyset.$$

Otherwise, there exists an element in Λ such that the associated $x \in X$ is feasible for (CP) satisfying $f(x) < f(\bar{x})$, which is a contradiction to the fact that $\bar{x}$ is a point of minimizer of (CP). Therefore, by Proposition 2.15,

$$ri\ \Lambda \cap ri\ (\mathbb{R}_-^{1+m+s} \times \{0_{\mathbb{R}^{l-s}}\}) = \emptyset.$$

Invoking the Proper Separation Theorem, Theorem 2.26 (iv), there exists $(\lambda_0, \lambda, \mu) \in \mathbb{R}^{1+m+l}$ with $(\lambda_0, \lambda, \mu) \neq (0, 0, 0)$ such that

$$\lambda_0 y_0 + \sum_{i=1}^{m} \lambda_i y_i + \sum_{j=1}^{l} \mu_j z_j \geq \lambda_0 w_0 + \sum_{i=1}^{m} \lambda_i w_i + \sum_{j=1}^{s} \mu_j v_j \tag{4.10}$$

for every $(y_0, y, z) \in \Lambda$ and $(w_0, w, v) \in \mathbb{R}^{1+m+s}$, and there exists $(y_0', y', z') \in \Lambda$ such that

$$\lambda_0 y_0' + \sum_{i=1}^{m} \lambda_i y_i' + \sum_{j=1}^{l} \mu_j z_j' > 0. \tag{4.11}$$

Let us partition $\mu = (\hat{\mu}, \tilde{\mu}) \in \mathbb{R}^s \times \mathbb{R}^{l-s}$. We claim that $\lambda_0 \geq 0$, $\lambda \in \mathbb{R}_+^m$ and $\hat{\mu} \in \mathbb{R}_+^s$. Corresponding to the point of minimizer $\bar{x}$, choose $y_0 > 0$, $y_i > 0$, $i = 1, 2, \ldots, m$, and $z_j = h_j(\bar{x})$, $j = 1, 2, \ldots, l$. From condition (4.10), for $i' = 0, 1, \ldots, m$,

$$\lambda_{i'} \geq \frac{1}{y_{i'}} \{ -\sum_{i=0}^{i'-1} \lambda_i y_i - \sum_{i=i'+1}^{m} \lambda_i y_i - \sum_{j=1}^{l} \mu_j z_j + \lambda_0 w_0 + \sum_{i=1}^{m} \lambda_i w_i + \sum_{j=1}^{s} \mu_j v_j \}.$$

Taking the limit as $y_{i'} \to \infty$ yields $\lambda_{i'} \geq 0$, $i' = 0, 1, \ldots, m$. Again from (4.10), for $j' = 1, 2, \ldots, s$,

$$\begin{aligned} \mu_{j'} \geq \frac{1}{v_{j'}} \{ \lambda_0 y_0 + \sum_{i=1}^{m} \lambda_i y_i + \sum_{j=1}^{l} \mu_j z_j - \lambda_0 w_0 - \sum_{i=1}^{m} \lambda_i w_i \\ - \sum_{j=1}^{j'-1} \mu_j v_j - \sum_{j=j'+1}^{s} \mu_j v_j \}. \end{aligned}$$

Taking the limit as $v_{j'} \to \infty$ leads to $\mu_{j'} \geq 0$, $j' = 1, 2, \ldots, s$.

Now consider any $x \in X$ and $\delta > 0$. Define

$$\begin{aligned} y_0 &= f(x) - f(\bar{x}) + \delta, \\ y_i &= g_i(x) + \delta, \quad i = 1, 2, \ldots, m, \\ z_j &= h_j(x), \quad j = 1, 2, \ldots, l. \end{aligned}$$

Therefore, $(y_0, y, z) \in \Lambda$ and for $(0, 0, 0) \in \mathbb{R}^{m+s}_{-} \times \{0_{\mathbb{R}^{l-s}}\}$, the condition (4.10) yields that for every $x \in X$ and every $\delta > 0$,

$$\lambda_0(f(x) - f(\bar{x})) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x) + \sum_{i=0}^{m} \lambda_i \delta \geq 0.$$

Because $\delta > 0$ was arbitrarily chosen, as $\delta \to 0$ the above condition reduces to

$$\lambda_0(f(x) - f(\bar{x})) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x) \geq 0, \ \forall\, x \in X. \tag{4.12}$$

In particular, for $x = \bar{x}$, condition (4.12) yields

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) + \sum_{j=1}^{l} \mu_j h_j(\bar{x}) \geq 0,$$

which along with the feasibility of $\bar{x}$ for (CP) leads to

$$\lambda_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m, \quad \text{and} \quad \hat{\mu}_j h_j(\bar{x}) = 0, \ j = 1, 2, \ldots, s,$$

as in the proof of Theorem 4.3.

We claim that $(\lambda_0, \lambda) \neq (0, 0)$. On the contrary, suppose that $(\lambda_0, \lambda) = (0, 0)$. Therefore, condition (4.12) leads to

$$\sum_{j=1}^{l} \mu_j h_j(x) \geq 0, \ \forall\, x \in X.$$

By the given hypothesis, for $\hat{x} \in ri\ X$ along with the above inequality implies that

$$\sum_{j=1}^{l} \mu_j h_j(\hat{x}) = 0,$$

that is, the affine function $\sum_{j=1}^{l} \mu_j h_j(.)$ achieves its minimum at a relative interior point. Because an affine function achieves its minimum at a boundary point, $\sum_{j=1}^{l} \mu_j h_j(.)$ has a constant value zero over X, that is,

$$\sum_{j=1}^{l} \mu_j h_j(x) = 0, \ \forall\, x \in X. \tag{4.13}$$

Corresponding to $(y_0', y', z') \in \Lambda$ satisfying (4.11) there exists $x' \in X$ such that

$$\sum_{j=1}^{l} \mu_j h_j(x') > 0,$$

which contradicts (4.13). Therefore, $\lambda_i,\ i = 0, 1, \ldots, m$, are not all simultaneously zero, which along with (4.12) leads to the desired result. $\square$

Theorem 4.5 *Consider the convex programming problem* (CP) *with* C *defined by (4.9). Assume that the modified Slater constraint qualification is satisfied, that is there exists* $\hat{x} \in ri\ X$ *such that*

$$\begin{aligned} g_i(\hat{x}) &< 0, \quad i = 1, 2, \ldots, m, \\ h_j(\hat{x}) &\leq 0, \quad j = 1, 2, \ldots, s, \\ h_j(\hat{x}) &= 0, \quad j = s+1, s+2, \ldots, l. \end{aligned}$$

Then $\bar{x}$ *is a point of minimizer of* (CP) *if and only if there exist* $\bar{\lambda} \in \mathbb{R}^m_+$, $\bar{\mu} = (\bar{\hat{\mu}}, \bar{\tilde{\mu}}) \in \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$ *such that*

$$L(\bar{x}, \lambda, \mu) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu}),\ \forall\ x \in X,\ \lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^l,$$

where $\mu = (\hat{\mu}, \tilde{\mu}) \in \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$ *along with*

$$\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m, \quad \text{and} \quad \hat{\mu}_j h_j(\bar{x}) = 0,\ j = 1, 2, \ldots, s.$$

Proof. Because the modified Slater constraint qualification is satisfied, the hypothesis of Theorem 4.4 also holds. Thus, if $\bar{x}$ is a point of minimizer of (CP), there exist $(\lambda_0, \lambda) \in \mathbb{R}_+ \times \mathbb{R}^m_+$ with $(\lambda_0, \lambda) \neq (0, 0)$ and $\mu = (\hat{\mu}, \tilde{\mu}) \in \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$ such that

$$\lambda_0 f(\bar{x}) \leq \lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x),\ \forall\ x \in X \tag{4.14}$$

and

$$\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m, \quad \text{and} \quad \mu_j h_j(\bar{x}) = 0,\ j = 1, 2, \ldots, s. \tag{4.15}$$

We claim that $\lambda_0 \neq 0$. On the contrary, suppose that $\lambda_0 = 0$. Because $(\lambda_0, \lambda) \neq (0, 0)$, $\lambda \neq 0$. Therefore, the optimality condition (4.14) becomes

$$\sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x) \geq 0,\ \forall\ x \in X.$$

In particular, for $x = \hat{x}$, the above condition along with the modified Slater constraint qualification leads to

$$0 > \sum_{i=1}^{m} \lambda_i g_i(\hat{x}) + \sum_{j=1}^{l} \mu_j h_j(\hat{x}) \geq 0,$$

which is a contradiction. Thus, $\lambda_0 > 0$ and hence dividing (4.14) throughout by λ_0 yields

$$f(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) + \sum_{j=1}^{l} \bar{\mu}_j h_j(x), \ \forall \ x \in X,$$

where $\bar{\lambda}_i = \dfrac{\lambda_i}{\lambda_0}$, $i = 1, 2, \ldots, m$, and $\bar{\mu}_j = \dfrac{\mu_j}{\lambda_0}$, $j = 1, 2, \ldots, l$. This inequality along with the condition (4.15) leads to

$$L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu}), \ \forall \ x \in X.$$

As $\bar{x}$ is feasible for (CP), $g(\bar{x}) \in -\mathbb{R}^m_+$, $-\hat{h}(\bar{x}) \in \mathbb{R}^s_+$ and $\tilde{h}(\bar{x}) = \{0\}_{\mathbb{R}^{l-s}}$. Therefore, for $\lambda \in \mathbb{R}^m_+$, $\mu = (\hat{\mu}, \tilde{\mu}) \in \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$,

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) + \sum_{j=1}^{l} \mu_j h_j(\bar{x}) \leq 0 = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) + \sum_{j=1}^{l} \bar{\mu}_j h_j(\bar{x}),$$

which leads to

$$L(\bar{x}, \lambda, \mu) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}),$$

thereby proving the desired saddle point result. The converse can be worked out as in Theorem 4.2. $\square$

Observe that the saddle point condition in the above theorem

$$L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu}), \ \forall \ x \in X$$

can be rewritten as

$$\begin{aligned} f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) + \sum_{j=1}^{s} \hat{\mu}_j h_j(\bar{x}) + \sum_{j=s+1}^{l} \tilde{\mu}_j h_j(\bar{x}) + \delta_X(\bar{x}) & \\ \leq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{s} \hat{\mu}_j h_j(x) + \sum_{j=s+1}^{l} \tilde{\mu}_j h_j(x) + \delta_X(x) & \end{aligned}$$

for every $x \in \mathbb{R}^n$. The above inequality implies that

$$0 \in \partial(f + \sum_{i=1}^{l} \lambda_i g_i + \sum_{j=1}^{s} \hat{\mu}_j h_j(x) + \sum_{j=s+1}^{l} \tilde{\mu}_j h_j(x) + \delta_X)(\bar{x}).$$

By the modified Slater constraint qualification $\hat{x} \in ri\ X$ and therefore, $ri\ dom\ f \cap \bigcap_{i=1}^{m} ri\ dom\ g_i \cap \bigcap_{j=1}^{l} ri\ dom\ h_j \cap ri\ dom\ \delta_X = ri\ X$ is nonempty. Applying the Sum Rule, Theorem 2.91 along with the fact that $\partial\delta_X(\bar{x}) = N_X(\bar{x})$ yields the KKT optimality condition

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + \sum_{j=1}^{s} \hat{\mu}_j \partial h_j(\bar{x}) + \partial(\sum_{j=s+1}^{l} \tilde{\mu}_j h_j)(\bar{x}) + N_X(\bar{x}).$$

By the affineness of $h_j,\ j = 1, 2, \ldots, l$, $\partial h_j(\bar{x}) = \{\nabla h_j(\bar{x})\}$, thereby reducing the above condition to the standard KKT optimality condition

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + \sum_{j=1}^{s} \hat{\mu}_j \nabla h_j(\bar{x}) + \sum_{j=s+1}^{l} \tilde{\mu}_j \nabla h_j(\bar{x}) + N_X(\bar{x}).$$

We state this discussion as the following result.

Theorem 4.6 *Consider the convex programming problem (CP) with C defined by (4.9). Assume that the modified Slater constraint qualification is satisfied. Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exist $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$; $\hat{\mu}_j \geq 0,\ j = 1, 2, \ldots, s$; and $\tilde{\mu}_j \in \mathbb{R},\ j = s+1, \ldots, l$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + \sum_{j=1}^{s} \hat{\mu}_j \nabla h_j(\bar{x}) + \sum_{j=s+1}^{l} \tilde{\mu}_j \nabla h_j(\bar{x}) + N_X(\bar{x})$$

along with

$$\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m, \quad \text{and} \quad \hat{\mu}_j h_j(\bar{x}) = 0,\ j = 1, 2, \ldots, s.$$

4.4 Lagrangian Duality

In the beginning of this chapter, we tried to motivate the notion of a saddle point using two-person-zero-sum games. We observed that two optimization problems were being simultaneously solved. Player 1 was solving a minimization problem while Player 2 was solving a maximization problem. The maximization problem is usually referred to as the *dual* of the minimization problem. Similarly, corresponding to the problem (CP), one can actually construct a dual problem following an approach quite similar to that of the two-person-zero-sum games. Consider the problem (CP) with the feasible set given by (4.9). Then if v_L denotes the optimal value of (CP), then observe that

$$v_L = \inf_{x \in C} \sup_{(\lambda, \hat{\mu}, \tilde{\mu}) \in \Omega} L(x, \lambda, \hat{\mu}, \tilde{\mu}),$$

where $\Omega = \mathbb{R}^m_+ \times \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$. Taking a clue from the two-person-zero-sum games, the dual problem to (CP) that we denote by (DP) can be stated as

$$\sup\ w(\lambda, \hat{\mu}, \tilde{\mu}) \quad \text{subject to} \quad (\lambda, \hat{\mu}, \tilde{\mu}) \in \Omega, \tag{DP}$$

where $w(\lambda, \hat{\mu}, \tilde{\mu}) = \min_{x \in X} L(x, \lambda, \hat{\mu}, \tilde{\mu})$. We denote the optimal value of (DP) by d_L. Our main aim here is to check if

$$d_L = \sup_{(\lambda,\hat{\mu},\tilde{\mu}) \in \Omega} w(\lambda, \hat{\mu}, \tilde{\mu}) = v_L, \tag{4.16}$$

that is,

$$\sup_{(\lambda,\hat{\mu},\tilde{\mu}) \in \Omega} \inf_{x \in X} L(x, \lambda, \hat{\mu}, \tilde{\mu}) = \inf_{x \in C} \sup_{(\lambda,\hat{\mu},\tilde{\mu}) \in \Omega} L(x, \lambda, \hat{\mu}, \tilde{\mu}).$$

The statement (4.16) is known as *strong duality*. We now present a result that shows when strong duality holds.

Theorem 4.7 *Consider the problem (CP) where the set C is defined by (4.9). Assume that (CP) has a lower bound, that is, it has an infimum value, v_L, that is finite. Also, assume that the modified Slater constraint qualification is satisfied. Then the dual problem (DP) has a supremum and the supremum is attained with*

$$v_L = d_L.$$

Proof. We always have $v_L \geq d_L$. This is absolutely straightforward and we urge the reader to establish this. This is called *weak duality*.

The problem (CP) has an infimum, v_L, that is,

$$v_L = \inf_{x \in C} f(x).$$

Working along the lines of the proof of Theorem 4.4, we conclude from (4.12) that there exists nonzero $(\bar{\lambda}_0, \bar{\lambda}, \bar{\hat{\mu}}, \bar{\tilde{\mu}}) \in \mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathbb{R}^s_+ \times \mathbb{R}^{l-s}$ such that

$$\bar{\lambda}_0(f(x) - v_L) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) + \sum_{j=1}^{s} \bar{\hat{\mu}}_j h_j(x) + \sum_{j=s+1}^{l} \bar{\tilde{\mu}}_j h_j(x) \geq 0, \ \forall\, x \in X.$$

As the modified Slater constraint qualification holds, by Theorem 4.5, it is simple to observe that $\bar{\lambda}_0 \neq 0$ and without loss of generality, assume $\bar{\lambda}_0 = 1$. Hence,

$$(f(x) - v_L) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) + \sum_{j=1}^{s} \bar{\hat{\mu}}_j h_j(x) + \sum_{j=s+1}^{l} \bar{\tilde{\mu}}_j h_j(x) \geq 0, \ \forall\, x \in X.$$

Therefore,

$$L(x, \bar{\lambda}, \bar{\hat{\mu}}, \bar{\tilde{\mu}}) \geq v_L, \ \forall\, x \in X,$$

that is, $w(\bar{\lambda}, \hat{\bar{\mu}}, \tilde{\bar{\mu}}) \geq v_L$. Hence,

$$\sup_{(\lambda,\hat{\mu},\tilde{\mu})\in\Omega} w(\lambda, \hat{\mu}, \tilde{\mu}) \geq w(\bar{\lambda}, \hat{\bar{\mu}}, \tilde{\bar{\mu}}) \geq v_L.$$

By the weak duality, $v_L \geq \sup_{(\lambda,\hat{\mu},\tilde{\mu})\in\Omega} w(\lambda, \hat{\mu}, \tilde{\mu})$. Thus,

$$\sup_{(\lambda,\hat{\mu},\tilde{\mu})\in\Omega} w(\lambda, \hat{\mu}, \tilde{\mu}) = v_L = \inf_{x\in C} f(x),$$

thereby establishing the strong duality between (CP) and (DP). $\square$

It is important to note that the assumption of the Slater constraint qualification is quite crucial as its absence can give a positive duality gap. We provide below the following famous example due to Duffin [35].

Example 4.8 Consider the primal problem

$$\inf \ e^{x_2} \quad \text{subject to} \quad \sqrt{x_1^2 + x_2^2} \leq x_1.$$

The Lagrangian dual problem is

$$\max w(\lambda) \quad \text{subject to} \quad \lambda \in \mathbb{R}^m_+,$$

where

$$w(\lambda) = \inf_{x\in\mathbb{R}^2} e^{x_2} + \lambda(\sqrt{x_1^2 + x_2^2} - x_1), \ \lambda \geq 0.$$

Observe that the only feasible point of the primal problem is $(x_1, x_2) = (0, 0)$ and hence $\inf e^{x_2} = e^0 = 1$. Thus, the minimum value or the infimum value of the primal problem is $v_L = 1$. Now let us evaluate the function $w(\lambda)$ for each $\lambda \geq 0$. Observe that for every fixed x_2, the term $(\sqrt{x_1^2 + x_2^2} - x_1) \to 0$ as $x_1 \to +\infty$. Thus, for each x_2, the value e^{x_2} dominates the expression

$$e^{x_2} + \lambda(\sqrt{x_1^2 + x_2^2} - x_1)$$

as $x_1 \to +\infty$. Hence, for a fixed x_2,

$$\inf_{x_1} e^{x_2} + \lambda(\sqrt{x_1^2 + x_2^2} - x_1) = e^{x_2}.$$

By letting $x_2 \to -\infty$,

$$w(\lambda) = 0, \ \forall \ \lambda \geq 0.$$

Therefore, the supremum value of the dual problem is $d_L = 0$. Hence, there is a positive duality gap. Observe that the Slater constraint qualification does not hold in the primal case.

We are now going to present some deeper properties of the dual variables (or Lagrange multipliers) for the problem (CP) with convex non-affine inequality, that is, the feasible set C is given by (4.1),

$$C = \{x \in X : g_i(x) \leq 0, \; i = 1, 2, \ldots, m\}.$$

The set of Lagrange multipliers at a given solution $\bar{x}$ of (CP) is given as

$$\mathcal{M}(\bar{x}) = \{\lambda \in \mathbb{R}^m_+ : 0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}), \; \lambda_i g_i(\bar{x}) = 0, \; i = 1, 2, \ldots, m\}.$$

It is quite natural to think that when we change $\bar{x}$, the set of multipliers will also change. We now show that for a convex programming problem, the set $\mathcal{M}(\bar{x})$ does not depend on the solution $\bar{x}$. Consider the set

$$\mathcal{M} = \{\lambda \in \mathbb{R}^m_+ : \inf_{x \in C} f(x) = \inf_{x \in \mathbb{R}^n} L(x, \lambda)\}. \tag{4.17}$$

In the following result we show that $\mathcal{M}(\bar{x}) = \mathcal{M}$ for any solution $\bar{x}$ of (CP). The proof of this fact is from Attouch, Buttazzo, and Michaille [3].

Theorem 4.9 *Consider the convex programming problem (CP) with C defined by (4.1). Let $\bar{x}$ be the point of minimizer of (CP). Then $\mathcal{M}(\bar{x}) = \mathcal{M}$.*

Proof. Suppose that $\lambda \in \mathcal{M}(\bar{x})$. Then

$$0 \in \partial_x L(\bar{x}, \lambda)$$

with $\lambda_i g_i(\bar{x}) = 0$, $i = 1, 2, \ldots, m$, where $\partial_x L$ denotes the subdifferential with respect to x. Hence, $\bar{x}$ solves the problem

$$\min L(x, \lambda) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

Therefore, for every $x \in \mathbb{R}^n$,

$$f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x),$$

which along with $\lambda_i g_i(\bar{x}) = 0$, $i = 1, 2, \ldots, m$, implies

$$f(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \; \forall \, x \in \mathbb{R}^n.$$

Thus,

$$f(\bar{x}) = \inf_{x \in \mathbb{R}^n} (f + \sum_{i=1}^{m} \lambda_i g_i)(x) = \inf_{x \in \mathbb{R}^n} L(x, \lambda).$$

Further, $f(\bar{x}) = \inf_{x \in C} f(x)$. Hence, $\lambda \in \mathcal{M}$.

Conversely, suppose that $\lambda \in \mathcal{M}$, which implies

$$f(\bar{x}) = \inf_{x \in \mathbb{R}^n} (f + \sum_{i=1}^{m} \lambda_i g_i)(x).$$

Therefore,

$$f(\bar{x}) \leq f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}),$$

thereby yielding

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \geq 0.$$

The above inequality along with the feasibility of $\bar{x}$ for (CP) and nonnegativity of λ_i, $i = 1, 2, \ldots, m$, leads to

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = 0.$$

This further yields

$$\lambda_i g_i(\bar{x}) = 0, \;\; i = 1, 2, \ldots, m.$$

Thus,

$$f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = \inf_{x \in \mathbb{R}^n} (f + \sum_{i=1}^{m} \lambda_i g_i)(x),$$

which implies that $\bar{x}$ solves the problem

$$\min L(x, \lambda) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

Therefore, $0 \in \partial_x L(\bar{x}, \lambda)$. As $dom\ f = dom\ g_i = \mathbb{R}^n$, $i = 1, 2, \ldots, m$, applying the Sum Rule, Theorem 2.91,

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}).$$

This combined with the fact that $\lambda_i g_i(\bar{x}) = 0$, $i = 1, 2, \ldots, m$, shows that $\lambda \in \mathcal{M}(\bar{x})$, thereby establishing that $\mathcal{M}(\bar{x}) = \mathcal{M}$. $\square$

Remark 4.10 In the above theorem, $\bar{x}$ was chosen to be any arbitrary solution of (CP). Thus, it is clear that $\mathcal{M}(\bar{x})$ is independent of the choice of $\bar{x}$ and hence $\mathcal{M}(\bar{x}) = \mathcal{M}$ for every solution $\bar{x}$ of (CP).

Note that the above result can be easily extended to the problem with feasible set C defined by (4.9), that is, convex non-affine and affine inequalities along with affine equalities. If we take a careful look at the set $\mathcal{M}$, we realize that for $\lambda \in \mathbb{R}^m_+$ it is not essential that (CP) has a solution; one merely needs (CP) to be bounded below. Thus Attouch, Buttazzo, and Michaille [3] call the set $\mathcal{M}$ to be the set of *generalized Lagrange multipliers*. Of course if (CP) has a solution, then $\mathcal{M}$ is the set of Lagrange multipliers. We now show how deeply the notion of Lagrange multipliers is associated with the perturbation of the constraints of the problem. From a numerical point of view, it is important to deal with constraint perturbations. Note that due to rounding off and other errors, often the iterates do not satisfy the constraints exactly but some perturbed version of it, that is, possibly in the form

$$g_i(x) \leq y_i, \ i = 1, 2, \ldots, m.$$

Thus, the function

$$v(y) = \inf\{f(x) : g_i(x) \leq y_i, \ i = 1, 2, \ldots, m\}$$

is called the *value function* or the *marginal function* associated with (CP). It is obvious that if $v(0) \in \mathbb{R}$, then $v(0)$ is the optimal value of (CP). We now establish that $v : \mathbb{R}^m \to \bar{\mathbb{R}}$ is a convex function. In order to show that, we need the following interesting and important lemma.

Lemma 4.11 *Consider* $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, *which is convex in both variables. Then the function*

$$\phi(v) = \inf_{u \in \mathbb{R}^n} \Phi(u, v)$$

is a convex function in v.

Proof. Consider $(v_i, \alpha_i) \in epi_s\ \phi$, $i = 1, 2$, that is,

$$\phi(v_i) < \alpha_i, \ i = 1, 2.$$

Therefore, there exist $\bar{u}_1, \bar{u}_2 \in \mathbb{R}^n$ such that by the definition of infimum,

$$\Phi(\bar{u}_i, v_i) < \alpha_i, \ i = 1, 2.$$

By the convexity of Φ, for every $\lambda \in [0, 1]$,

$$\begin{aligned} \Phi((1-\lambda)\bar{u}_1 + \lambda\bar{u}_2, (1-\lambda)v_1 + \lambda v_2) \quad &\leq \quad (1-\lambda)\Phi(\bar{u}_1, v_1) + \lambda\Phi(\bar{u}_2, v_2) \\ &< \quad (1-\lambda)\alpha_1 + \lambda\alpha_2, \end{aligned}$$

which implies

$$\phi((1-\lambda)v_1 + \lambda v_2) < (1-\lambda)\alpha_1 + \lambda\alpha_2, \ \forall\ \lambda \in [0, 1].$$

Thus

$$((1-\lambda)v_1 + \lambda v_2), (1-\lambda)\alpha_1 + \lambda\alpha_2) \in epi_s\ \phi,$$

which by Proposition 2.50 leads to the convexity of ϕ. $\square$

Observe that the value function can be expressed as

$$v(y) = \inf_{x \in \mathbb{R}^n} \{f(x) + \delta_{C(y)}(x)\}, \tag{4.18}$$

where $C(y) = \{x \in \mathbb{R}^n : g_i(x) \leq y_i,\ i = 1, 2, \ldots, m\}$. Now to prove the convexity of the value function, what one needs to show is that $f(x)+\delta_{C(y)}(x)$ is convex in both the variables x as well as y, and we leave it to the reader. Once that is done, we just have to use Lemma 4.11 to conclude that v is a convex function.

Through the following result given in Attaouch, Buttazzo, and Michaille [3], we show how the Lagrange multipliers (or the generalized Lagrange multipliers) are related to the value function.

Theorem 4.12 *(i) Let $v(0) \in \mathbb{R}$, then $\mathcal{M} = -\partial v(0)$. Further, if the Slater constraint qualification holds, then v is continuous at the origin and hence $\mathcal{M}$ is convex compact set in $\mathbb{R}^m_+$.*

(ii) Consider the problem

$$\sup \quad -v^*(-\lambda) \quad \textit{subject to} \quad \lambda \in \mathbb{R}^m_+. \tag{$DP1$}$$

The solutions of $(DP1)$ coincide with the set $\mathcal{M}$. Further, for every $\lambda \in \mathbb{R}^m_+$,

$$-v^*(-\lambda) = \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$$

Thus the problem $(DP1)$ coincides with the Lagrangian dual problem of (CP).

Proof. (i) We begin by proving $\mathcal{M} = -\partial v(0)$. Consider any $\lambda \in \mathcal{M}$. By the definition of the value function v (4.18) and $\mathcal{M}$ (4.17),

$$v(0) = \inf_{x \in \mathbb{R}^n} \{f + \delta_C\}(x) = \inf_{x \in \mathbb{R}^n} \{f + \sum_{i=1}^{m} \lambda_i g_i\}(x).$$

For any given $y \in \mathbb{R}^m$, consider the set

$$C(y) = \{x \in \mathbb{R}^n : g_i(x) \leq y_i,\ i = 1, 2, \ldots, m\}.$$

As $\lambda \in \mathbb{R}^m_+$, for any $x \in C(y)$,

$$\sum_{i=1}^{m} \lambda_i g_i(x) \leq \sum_{i=1}^{m} \lambda_i y_i,$$

which implies that

$$f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \leq f(x) + \sum_{i=1}^{m} \lambda_i y_i.$$

Therefore,

$$\inf_{x \in C(y)} \{f + \sum_{i=1}^{m} \lambda_i g_i\}(x) \leq \inf_{x \in C(y)} f(x) + \sum_{i=1}^{m} \lambda_i y_i. \tag{4.19}$$

Because $C(y) \subset \mathbb{R}^n$, by Proposition 1.7,

$$\inf_{x \in \mathbb{R}^n} \{f + \sum_{i=1}^{m} \lambda_i g_i\}(x) \leq \inf_{x \in C(y)} \{f + \sum_{i=1}^{m} \lambda_i g_i\}(x).$$

As $\lambda \in \mathcal{M}$, by (4.17) along with (4.19) leads to

$$v(0) \leq v(y) + \sum_{i=1}^{m} \lambda_i y_i,$$

that is,

$$v(y) \geq v(0) + \langle -\lambda, y - 0 \rangle, \ \forall\, y \in \mathbb{R}^m.$$

This yields that $-\lambda \in \partial v(0)$, thereby establishing that $\mathcal{M} \subset -\partial v(0)$.

Conversely, suppose that $\lambda \in -\partial v(0)$, that is, $-\lambda \in \partial v(0)$. We will prove that $\lambda \in \mathcal{M}$. Consider any $y \in \mathbb{R}^m_+$. Then it is easy to observe that $C \subset C(y)$. Again by Proposition 1.7,

$$\inf_{x \in C} f(x) \geq \inf_{x \in C(y)} f(x),$$

that is,

$$v(0) \geq v(y), \ \forall\, y \in \mathbb{R}^m_+.$$

As $-\lambda \in \partial v(0)$, which along with the above inequality leads to

$$\langle \lambda, y \rangle \geq v(0) - v(y) \geq 0.$$

Because $y \in \mathbb{R}^m_+$ was arbitrary, it is clear that $\lambda \in \mathbb{R}^m_+$. We now establish that $\lambda \in \mathcal{M}$ by proving that

$$\inf_{x \in C} f(x) = \inf_{x \in \mathbb{R}^n} \{f + \sum_{i=1}^{m} \lambda_i g_i\}(x),$$

that is,

$$v(0) = \inf_{x \in \mathbb{R}^n} \{f + \sum_{i=1}^{m} \lambda_i g_i\}(x).$$

Note that if $x \in C$, $g_i(x) \leq 0$, $i = 1, 2, \ldots, m$. Then $\sum_{i=1}^m \lambda_i g_i(x) \leq 0$ as $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$. Thus,

$$f(x) + \sum_{i=1}^m \lambda_i g_i(x) \leq f(x), \ \forall \ x \in C.$$

Therefore,

$$\inf_{x \in \mathbb{R}^n} \{f + \sum_{i=1}^m \lambda_i g_i\}(x) \leq \inf_{x \in C} \{f + \sum_{i=1}^m \lambda_i g_i\}(x) \leq \inf_{x \in C} f(x) = v(0). \tag{4.20}$$

The fact that $-\lambda \in \partial v(0)$ leads to

$$v(y) + \langle \lambda, y \rangle \geq v(0), \ \forall \ y \in \mathbb{R}^m,$$

that is, for every $y \in \mathbb{R}^m$,

$$v(y) + \sum_{i=1}^m \lambda_i y_i \geq v(0).$$

Consider any $\tilde{x} \in \mathbb{R}^n$ and set $\tilde{y} = g_i(\tilde{x})$, $i = 1, 2, \ldots, m$. Therefore, the above inequality leads to

$$v(\tilde{y}) + \sum_{i=1}^m \lambda_i g_i(\tilde{x}) \geq v(0).$$

By the definition (4.18) of value function $v(\tilde{y}) \leq f(\tilde{x})$, which along with the above inequality leads to

$$f(\tilde{x}) + \sum_{i=1}^m \lambda_i g_i(\tilde{x}) \geq v(0).$$

Because $\tilde{x}$ was arbitrary,

$$\inf_{x \in \mathbb{R}^n} \{f + \sum_{i=1}^m \lambda_i g_i\}(x) \geq v(0). \tag{4.21}$$

Combining (4.21) with (4.20),

$$v(0) = \inf_{x \in \mathbb{R}^n} \{f + \sum_{i=1}^m \lambda_i g_i\}(x).$$

Therefore, $\lambda \in \mathcal{M}$ and thus establishing that $\mathcal{M} = -\partial v(0)$.

Now assume that $v(0)$ is finite and the Slater constraint qualification holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$. Thus there exists $\delta > 0$ such that for every $y \in \mathbb{B}_\delta(0) = \delta\mathbb{B}$, $g_i(\hat{x}) < y_i,\ i = 1, 2, \ldots, m$, which implies that

$$v(y) \leq f(\hat{x}),\ \forall\ y \in \mathbb{B}_\delta(0). \tag{4.22}$$

As *dom* $f = \mathbb{R}^n$, $f(\hat{x}) < +\infty$, thereby establishing that v is bounded above on $\mathbb{B}_\delta(0)$. This fact shows that

$$\mathbb{B}_\delta(0) \times [f(\hat{x}), +\infty) \subset epi\ v.$$

We claim that $v(y) > -\infty$ for every $y \in \mathbb{R}^m$. On the contrary, assume that there exists $\hat{y} \in \mathbb{R}^m$ such that $v(\hat{y}) = -\infty$. Thus,

$$\{\hat{y}\} \times \mathbb{R} \subset epi\ v.$$

Consider $z = -\alpha\hat{y}$ such that $\alpha > 0$ and $\|z\| < \delta$. This is possible by choosing $\alpha = \dfrac{\delta}{2\|\hat{y}\|}$. Setting $\lambda = \dfrac{1}{1+\alpha}$, we have $\lambda \in (0,1)$ and $\alpha = \dfrac{1-\lambda}{\lambda}$. This implies that $z = \dfrac{-(1-\lambda)}{\lambda}\hat{y}$, that is,

$$\lambda z + (1-\lambda)\hat{y} = 0.$$

By choice, $z \in \mathbb{B}_\delta(0)$, which by (4.22) implies that $v(z) \leq f(\hat{x})$ and thus,

$$(z, f(\hat{x})) \in \mathbb{B}_\delta(0) \times [f(\hat{x}), +\infty) \subset epi\ v.$$

Further, for every $t \in \mathbb{R}$,

$$(\hat{y}, t) \in \{\hat{y}\} \times \mathbb{R} \subset epi\ v.$$

As v is convex, by Proposition 2.48, *epi* v is a convex set, which implies that

$$(\lambda z + (1-\lambda)\hat{y}, \lambda f(\hat{x}) + (1-\lambda)t) \in epi\ v,$$

that is,

$$(0, \lambda f(\hat{x}) + (1-\lambda)t) \in epi\ v.$$

Therefore,

$$v(0) \leq \lambda f(\hat{x}) + (1-\lambda)t,\ \forall\ t \in \mathbb{R}.$$

Taking the limit as $t \to -\infty$, $v(0) \leq -\infty$. But $v(0) \geq -\infty$ and hence $v(0) = -\infty$, which is a contradiction because $v(0) \in \mathbb{R}$. By Theorem 2.72, the function $v : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is majorized on a neighborhood of the origin and hence v is continuous at $y = 0$. Then by Proposition 2.82, $\partial v(0)$ is convex compact set, which implies so is $\mathcal{M}$.

(ii) We already know that $\lambda \in \mathcal{M}$ if and only $-\lambda \in \partial v(0)$. Therefore, from Theorem 2.108,

$$-\lambda \in \partial v(0) \qquad \Longleftrightarrow \qquad v(0) + v^*(-\lambda) = 0,$$

which implies $\lambda \in \mathcal{M}$ if and only if

$$v(0) + v^*(-\lambda) = 0.$$

From (i) we know that v is continuous at $y = 0$. Thus, by Proposition 2.106, $v(0) = v^{**}(0)$. By Definition 2.101 of the biconjugate,

$$v^{**}(0) = \sup_{\mu \in \mathbb{R}^m} \{-v^*(\mu)\} = \sup_{\mu \in \mathbb{R}^m} \{-v^*(-\mu)\}.$$

Thus $\lambda \in \mathcal{M}$ if and only if

$$-v^*(-\lambda) = v^{**}(0) = \sup_{\mu \in \mathbb{R}^m} \{-v^*(-\mu)\},$$

which is equivalent to the fact that λ solves the problem

$$\sup -v^*(\mu) \quad \text{subject to} \quad \mu \in \mathbb{R}^m.$$

Observe that

$$\begin{aligned}
v^*(\mu) &= \sup_{y \in \mathbb{R}^m} \{\langle \mu, y\rangle - v(y)\} \\
&= \sup_{y \in \mathbb{R}^m} \{\langle \mu, y\rangle - \inf_x \{f(x) : g_i(x) \leq y_i,\ i = 1, 2, \ldots, m\}\} \\
&= \sup_{y \in \mathbb{R}^m} \{\langle \mu, y\rangle + \sup_x \{-f(x) : g_i(x) \leq y_i,\ i = 1, 2, \ldots, m\}\} \\
&= \sup_{(y,x) \in \mathbb{R}^m \times \mathbb{R}^n} \{\langle \mu, y\rangle - f(x) : g_i(x) \leq y_i,\ i = 1, 2, \ldots, m\}.
\end{aligned}$$

If for some $i \in \{1, 2, \ldots, m\}$, $\mu_i > 0$, then $v^*(\mu) = +\infty$. So assume that $\mu \in -\mathbb{R}^m_+$. Then

$$v^*(\mu) = \sup_{x \in \mathbb{R}^n} \{-f(x) + \sup_{y_i \geq g_i(x)} \sum_{i=1}^m \mu_i y_i\}.$$

As $\sum_{i=1}^m \mu_i y_i = \langle \mu, y\rangle$ is a linear function,

$$\sup_{y_i \geq g_i(x)} \sum_{i=1}^m \mu_i y_i = \sum_{i=1}^m \mu_i g_i(x).$$

Hence, for $\mu \in -\mathbb{R}^m_+$,

$$v^*(\mu) = \sup_{x \in \mathbb{R}^n} \{\sum_{i=1}^m \mu_i g_i(x) - f(x)\}.$$

In particular, for $\mu = -\lambda$,

$$v^*(-\lambda) = \sup_{x \in \mathbb{R}^n} \{-(f(x) + \sum_{i=1}^m \lambda_i g_i(x))\},$$

which implies

$$-v^*(-\lambda) = \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$$

Thus, $-v^*(-\lambda) = w(\lambda)$, thereby showing that the dual problems (DP) and $(DP1)$ are the same. $\square$

4.5 Fenchel Duality

In the last section it was clear that the notion of conjugation is linked to the understanding of Lagrangian duality. In this section we explore this relation a bit more. We will focus on Fenchel duality where the dual problem is expressed explicitly in terms of the conjugate functions. Also we shall make a brief presentation of Rockafellar's perturbation approach to duality. Our approach to Fenchel duality will be that of Borwein and Lewis [17], which we present below.

Theorem 4.13 *Consider proper convex functions $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $g : \mathbb{R}^m \to \bar{\mathbb{R}}$ and a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. Let $v_F, d_F \in \bar{\mathbb{R}}$ be the optimal values of the primal and the dual problems given below:*

$$v_F = \inf_{x \in \mathbb{R}^n} \{f(x) + g(Ax)\} \quad \text{and} \quad d_F = \sup_{\phi \in \mathbb{R}^m} \{-f^*(A^T\lambda) - g^*(-\lambda)\},$$

where A^T denotes the conjugate of the linear map A or the transpose of the matrix represented by A. In fact, A can be viewed as an $m \times n$ matrix. Assume that the condition

$$0 \in core(dom\ g - A\ dom\ f)$$

holds. Then $v_F = d_F$ and the supremum in the dual problem is attained if the optimal value is finite. (Instead of the term core*, one can also use* interior *or* relative interior*.)*

Proof. We first prove that $v_F \geq d_F$, that is, the weak duality holds. By the definition of conjugate function, Definition 2.101,

$$f^*(A^T\lambda) = \sup_{x \in \mathbb{R}^n} \{\langle A^T\lambda, x\rangle - f(x)\} \geq \langle \lambda, Ax\rangle - f(x),\ \forall\ x \in \mathbb{R}^n,$$

which implies

$$f(x) \geq \langle \lambda, Ax\rangle - f^*(A^T\lambda),\ \forall\ x \in \mathbb{R}^n.$$

Similarly, we have

$$g(Ax) \geq -\langle \lambda, Ax \rangle - g^*(-\lambda),\ \forall\ x \in \mathbb{R}^n.$$

The above inequalities immediately show that for any $\lambda \in \mathbb{R}^m$ and any $x \in \mathbb{R}^n$,

$$f(x) + g(Ax) \geq -f^*(A^T\lambda) - g^*(-\lambda).$$

Thus, the above inequality it yields that $v_F \geq d_F$.

Next, to prove the equality under the given constraint qualification, define the function $h : \mathbb{R}^m \to \bar{\mathbb{R}}$ as

$$h(y) = \inf_{x \in \mathbb{R}^n} \{f(x) + g(Ax + y)\}.$$

In the parlance of optimization, h is referred to as the *optimal value function* or just a *value function*. Here the vector y acts as a parameter. See the previous section for more details. Using Lemma 4.11, it is easy to observe that h is convex. We urge the reader to reason out for himself / herself. Further, one must decide what *dom* h is. We claim that

$$dom\ h = dom\ g - A\ dom\ f.$$

Consider $y \in dom\ h$, that is, $h(y) < +\infty$. Hence there exists $x \in \mathbb{R}^n$ such that $x \in dom\ f$ and $Ax + y \in dom\ g$, which leads to

$$y \in dom\ g - A\ dom\ f.$$

This holds for every $y \in dom\ h$ and thus

$$dom\ h \subset dom\ g - A\ dom\ f.$$

Let $z \in dom\ g - A\ dom\ f$, which implies that there exists $u \in dom\ g$ and $\hat{x} \in dom\ f$ such that $z = u - A\hat{x}$. Hence $z + A\hat{x} \in dom\ g$, that is,

$$f(\hat{x}) + g(z + A\hat{x}) < +\infty.$$

Thus $h(z) < +\infty$, thereby showing that $z \in dom\ h$. This proves the assertion toward the domain of h.

Note that if $v_F = -\infty$, there is nothing to prove. Without loss of generality, we assume that v_F is finite. By assumption, $0 \in core(dom\ h)$ (or $0 \in int(dom\ h)$). By Proposition 2.82, $\partial h(0) \neq \emptyset$, which implies that there exists $-\xi \in \partial h(0)$. Thus, by Definition 2.77 of the subdifferential along with the definition of h,

$$h(0) \leq h(y) + \langle \xi, y \rangle \leq f(x) + g(Ax + y) + \langle \xi, y \rangle,\ \forall\ y \in \mathbb{R}^m.$$

Hence,

$$h(0) \leq \{f(x) - \langle A^*\xi, x \rangle\} + \{g(Ax + y) - \langle -\xi, Ax + y \rangle\}.$$

Taking the infimum first over y and then over x yields that

$$h(0) \leq -f^*(A^*\xi) - g^*(-\xi) \leq d_F \leq v_F \leq h(0),$$

thereby establishing that $v_F = d_F$. Observe that the dual value is obtained at $\lambda = \xi$. $\square$

It is important to mention that the above problem was also studied by Rockafellar [97]. In Rockafellar [97], the function g is taken to be a concave function, Definition 2.46, and the objective function of the primal problem and the dual problem are, respectively, given as

$$f(x) - g(Ax) \quad \text{and} \quad g_*(\lambda) - f^*(A^T\lambda).$$

Further, g_* denotes the *conjugate* of the concave function g, which is defined as

$$g_*(\lambda) = \inf_{y \in \mathbb{R}^m} \{\langle \lambda, y \rangle - g(y)\}.$$

From the historical point of view, we provide the statement of the classical Fenchel duality theorem as it appears in Rockafellar [97].

Theorem 4.14 *Consider a proper convex function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and a proper concave function $g : \mathbb{R}^n \to \bar{\mathbb{R}}$. Assume that one of the following conditions holds:*

(1) $ri(dom\ f) \cap ri(dom\ g) \neq \emptyset$,

(2) f and g are lsc and $ri(dom\ f^*) \cap ri(dom\ g_*) \neq \emptyset$.

Then

$$\inf_{x \in \mathbb{R}^n} \{f(x) - g(x)\} = \sup_{\lambda \in \mathbb{R}^n} \{g_*(\lambda) - f^*(\lambda)\}. \tag{4.23}$$

We request the readers to figure out how one will define the notion of a proper concave function. Of course, if we consider g to be a convex function, (4.23) can be written as

$$\inf_{x \in \mathbb{R}^n} \{f(x) + g(x)\} = \sup_{\lambda \in \mathbb{R}^n} \{-g^*(-\lambda) - f^*(\lambda)\}.$$

Note that this can be easily proved using Theorem 4.13 by taking A to be the identity mapping $I : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, $ri(dom\ f) \cap ri(dom\ g) \neq \emptyset$ shows that $0 \in int(dom\ g - dom\ f)$. Hence the result follows by invoking Theorem 4.13.

We now look into the perturbation-based approach. This approach is due to Rockafellar. Rockafellar's monogarph [99] entitled *Conjugate Duality and Optimization* makes a detailed study of this method in an infinite dimensional setting. We however discuss the whole issue from a finite dimensional viewpoint. In this approach, one considers the original problem being embedded in a family of problems. In fact, we begin by considering the *convexly parameterized family* of convex problems

$$\min\ F(x,y) \quad \text{subject to} \quad x \in \mathbb{R}^n, \tag{$CP(y)$}$$

where the vector y is called the *parameter* and the function $F : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ is assumed to be proper convex jointly in x and y. In fact, in such a situation, the optimal value function

$$v(y) = \inf_{x \in C} F(x,y)$$

is a convex function by Lemma 4.11. Of course, the function F is so chosen that $f_0(x) = F(x,0)$, where

$$f_0(x) = \begin{cases} f(x), & x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

In fact, (CP) can be viewed as

$$\min\ f_0(x) \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

thus embedding the original problem (CP) in $(CP(y))$.

Now we pose the standard convex optimization problem as $(CP(y))$. Consider the problem (CP) with C given by (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1,2,\ldots,m\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,2,\ldots,m$, are convex functions. Corresponding to (CP), introduce the family of parameterized problems $(CP(y))$ as follows

$$\min\ F(x,y) \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

where

$$F(x,y) = \begin{cases} f(x), & g_i(x) \leq y_i,\ i = 1,2,\ldots,m, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is clear that

$$F(x,0) = f_0(x) = \begin{cases} f(x), & g_i(x) \leq 0,\ i = 1,2,\ldots,m, \\ +\infty, & \text{otherwise.} \end{cases}$$

Recall that the Lagrangian function corresponding to (CP) is given by

$$L(x,\lambda) = \begin{cases} f(x) + \langle \lambda, g(x) \rangle, & \lambda \in \mathbb{R}^m_+, \\ +\infty, & \text{otherwise.} \end{cases}$$

Next we look at how to construct the dual problem for $(CP(y))$. Define the Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbb{R}}$ as

$$L(x,\lambda) = \inf_{y \in \mathbb{R}^m} \{F(x,y) + \langle y, \lambda \rangle\},$$

that is,

$$-L(x,\lambda) = \sup_{y\in\mathbb{R}^m} \{\langle y,\lambda\rangle - F(x,y)\}.$$

Observe that

$$\begin{aligned}
F^*(x^*,\lambda^*) &= \sup_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \{\langle x^*,x\rangle + \langle\lambda^*,y\rangle - F(x,y)\} \\
&= \sup_{x\in\mathbb{R}^n} \{\langle x^*,x\rangle + \sup_{y\in\mathbb{R}^m} (\langle\lambda^*,y\rangle - F(x,y))\} \\
&= \sup_{x\in\mathbb{R}^n} \{\langle x^*,x\rangle - L(x,\lambda^*)\}.
\end{aligned}$$

Thus,

$$-F^*(0,\lambda^*) = \inf_{x\in\mathbb{R}^n} L(x,\lambda^*).$$

Hence the Fenchel dual problem associated with (CP) is

$$\sup \ (-F^*(0,\lambda)) \quad \text{subject to} \quad \lambda \in \mathbb{R}^m. \qquad (DP_F)$$

With the given Lagrangian in a similar fashion as before, one can define a saddle point (x,λ) of the Lagrangian function $L(x,\lambda)$.

We now state without proof the following result. For proof, see for example Lucchetti [79] and Rockafellar [99].

Theorem 4.15 *Consider the problem (CP) and (DP_F) as given above. Then the following are equivalent:*

(i) $(\bar{x},\bar{\lambda})$ be a saddle point of L,

(ii) $\bar{x}$ is a solution for (CP) and $\bar{\lambda}$ is a solution for (DP_F) and there is no duality gap.

For more details on the perturbation-based approach, see Lucchetti [79] and Rockafellar [99].

4.6 Equivalence between Lagrangian and Fenchel Duality

In the previous sections, we studied two types of duality theories, namely the Lagrangian duality and the Fenchel duality. The obvious question that comes to mind is whether the two theories are equivalent or not. It was shown by Magnanti [81] that for a convex programming problem, both these forms of duality coincide. We end this chapter by taking a look at the equivalence between the two duality theories based on the approach of Magnanti [81].

Consider the following convex programming problems:

$$\begin{array}{lll} \text{Lagrange:} & \inf\ f(x) \quad \text{subject to} \quad x \in C, \\ \text{Fenchel:} & \inf\ (f_1(x) - f_2(x)) \quad \text{subject to} \quad x \in C_1 \cap C_2, \end{array}$$

where

$$C = \{x \in \mathbb{R}^n : g_i(x) \le 0,\ i = 1, 2, \ldots, m,\\ h_j(x) = 0,\ j = 1, 2, \ldots, l,\ x \in X\},$$

$f : X \to \mathbb{R}$, $f_1 : C_1 \to \mathbb{R}$ are convex functions; $f_2 : C_2 \to \mathbb{R}$ is a concave function; $g_i : X \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are convex non-affine functions; $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, l$, are affine functions; and C_1, C_2, X are convex subsets of $\mathbb{R}^n$. Denote the optimal values of the Lagrangian and the Fenchel convex programming problems as v_L and v_F, respectively. Observe that the Lagrangian problem is a particular case of (CP) with C given by (4.9). Corresponding to the two convex programming problem, we have the following dual problems:

$$\begin{array}{lll} \text{Lagrange:} & \sup \inf_{x \in X} L(x, \lambda, \mu) \quad \text{subject to} \quad (\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^l, \\ \text{Fenchel:} & \sup\ ((f_2)_*(\xi) - f_1^*(\xi)) \quad \text{subject to} \quad \xi \in \mathbb{R}^n, \end{array}$$

where the Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^l$ is defined as

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{l} \mu_j h_j(x).$$

As f_i are defined over C_i, that is, *dom* $f_i = C_i$ for $i = 1, 2$, the conjugate functions reduce to

$$\begin{aligned} f_1^*(\xi) &= \sup_{x \in C_1} \{\langle \xi, x\rangle - f_1(x)\}, \\ (f_2)_*(\xi) &= \inf_{x \in C_2} \{\langle \xi, x\rangle - f_2(x)\}. \end{aligned}$$

Denote the optimal values of the Lagrangian and the Fenchel dual problems as d_L and d_F, respectively. Note that $f_1^*(\xi) = +\infty$ for some $\xi \in \mathbb{R}^n$ is a possibility. Similarly, for the concave conjugate, $(f_2)_*(\xi) = -\infty$ for some $\xi \in \mathbb{R}^n$ is also a possibility. But these values play no role in the Fenchel dual problem and thus the problem may be considered as

$$\text{Fenchel:} \quad \sup\ ((f_2)_*(\xi) - f_1^*(\xi)) \quad \text{subject to} \quad \xi \in C_1^* \cap C_2^*,$$

where

$$C_1^* = \{\xi \in \mathbb{R}^n : f_1^*(\xi) < +\infty\} \quad \text{and} \quad C_2^* = \{\xi \in \mathbb{R}^n : (f_2)_*(\xi) > -\infty\}.$$

By Theorem 4.7 and Theorem 4.14, we have the strong duality results for the Lagrangian and the Fenchel problems, respectively, that is,

$$v_L = d_L \qquad \text{and} \qquad v_F = d_F.$$

Now we move on to show that the two strong dualities are equivalent. But before doing so we present a result from Magnanti [81] on relative interior.

Lemma 4.16 *Consider the set*

$$\begin{aligned} \Lambda = \{(y_0, y, z) \in \mathbb{R}^{1+m+l} : \ \textit{there exists } x \in X \textit{ such that } f(x) \leq y_0,& \\ g_i(x) \leq y_i,\ i = 1, 2, \ldots, m,& \\ h_j(x) = z_j,\ j = 1, 2, \ldots, l\}.& \end{aligned}$$

If $\hat{x} \in ri\ X$ *such that*

$$f(\hat{x}) < \hat{y}_0, \quad g_i(\hat{x}) < \hat{y}_i,\ i = 1, 2, \ldots, m, \quad \textit{and} \quad h_j(\hat{x}) = \hat{z}_j,\ j = 1, 2, \ldots, l,$$

then $(\hat{y}_0, \hat{y}, \hat{z}) \in ri\ \Lambda$.

Proof. By the convexity of the functions f, g_i, $i = 1, 2, \ldots, m$, and h_j, $j = 1, 2, \ldots, l$, and the set X, it is easy to observe that the set Λ is convex. It is left to the reader to verify this fact. To prove the result, we will invoke the Prolongation Principle, Proposition 2.14 (iii).

Consider $(y_0, y, z) \in \Lambda$, that is, there exists $x \in X$ such that

$$f(x) \leq y_0, \quad g_i(x) \leq y_i,\ i = 1, 2, \ldots, m, \quad \text{and} \quad h_j(x) = z_j,\ j = 1, 2, \ldots, l.$$

Because $X \subset \mathbb{R}^n$ is a nonempty convex set and $\hat{x} \in ri\ X$, by the Prolongation Principle, there exists $\gamma > 1$ such that

$$\gamma\hat{x} + (1 - \gamma)x \in X,$$

which by the convexity of X yields that

$$\alpha\hat{x} + (1 - \alpha)x \in X,\ \forall\ \alpha \in (1, \gamma]. \tag{4.24}$$

As $dom\ f = dom\ g_i = X$, $i = 1, 2, \ldots, m$ with $\hat{x} \in ri\ X$, for some $\alpha \in (1, \gamma]$,

$$f(\alpha\hat{x} + (1 - \alpha)x) \quad < \quad \alpha\hat{y}_0 + (1 - \alpha)y_0, \tag{4.25}$$

$$g_i(\alpha\hat{x} + (1 - \alpha)x) \quad < \quad \alpha\hat{y}_i + (1 - \alpha)y_i,\ i = 1, 2, \ldots, m. \tag{4.26}$$

By the affineness of h_j, $j = 1, 2, \ldots, l$,

$$h_j(\alpha\hat{x} + (1 - \alpha)x) < \alpha\hat{z}_j + (1 - \alpha)z_j,\ j = 1, 2, \ldots, l. \tag{4.27}$$

Combining the conditions (4.24) through (4.27) yields that for $\alpha > 1$,

$$\alpha(\hat{y}_0, \hat{y}, \hat{z}) + (1 - \alpha)(y_0, y, z) \in \Lambda.$$

Because $(y_0, y, z) \in \Lambda$ was arbitrary, by the Prolongation Principle, $(\hat{y}_0, \hat{y}, \hat{z}) \in ri\ \Lambda$ as desired. $\square$

Now we present the equivalence between the strong duality results.

Theorem 4.17 *Lagrangian strong duality is equivalent to Fenchel strong duality, that is, Theorem 4.7 is equivalent to Theorem 4.14.*

Proof. Suppose that the Lagrangian strong duality, Theorem 4.7, holds under the assumption of modified Slater constraint qualification, that is, there exists $\hat{x} \in ri\ X$ such that

$$g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m, \quad \text{and} \quad h_j(\hat{x}) = 0,\ j = 1, 2, \ldots, l.$$

Define $X = C_1 \times C_2 \times \mathbb{R}^n$ and $x = (x_1, x_2, x_3)$. The Fenchel convex programming problem can now be expressed as

$$v_F = \inf_{x \in C} (f_1(x_1) - f_2(x_2)),$$

where

$$C = \{x \in \mathbb{R}^{3n} : h_j^r(x) = (x_j - x_3)_r = 0,\ j = 1, 2,\ r = 1, 2, \ldots, n,\ x \in X\}.$$

Observe that here $h_j : \mathbb{R}^n \to \mathbb{R}^n$. Note that the reformulated Fenchel problem is nothing but the Lagrangian convex programming problem. The corresponding Lagrangian dual problem is as follows:

$$d_L = \sup_{(\mu_1, \mu_2) \in \mathbb{R}^{2n}} \inf_{x \in X} \{f_1(x_1) - f_2(x_2) + \sum_{r=1}^{n} \mu_1^r (x_1 - x_3)_r + \sum_{r=1}^{n} \mu_2^r (x_2 - x_3)_r\},$$

that is,

$$\begin{aligned} d_L = \sup_{(\mu_1, \mu_2) \in \mathbb{R}^{2n}} \inf_{x \in X} \{f_1(x_1) - f_2(x_2) &+ \langle \mu_1, x_1 \rangle + \langle \mu_2, x_2 \rangle \\ &- \langle \mu_1 + \mu_2, x_3 \rangle\}. \quad (4.28) \end{aligned}$$

From the assumption of Theorem 4.14,

$$ri(dom\ f_1) \cap ri(dom\ f_2) = ri\ C_1 \cap ri\ C_2 \neq \emptyset,$$

which implies there exists $\hat{x} \in \mathbb{R}^n$ such that $\hat{x} \in ri\ C_1 \cap ri\ C_2$. Therefore, $x = (\hat{x}, \hat{x}, \hat{x}) \in ri\ X$ such that $h_j^r(x) = 0,\ j = 1, 2,\ r = 1, 2, \ldots, n$; thereby implying that the modified Slater constraint qualification holds. Invoking the Lagrangian strong duality, Theorem 4.7,

$$v_F = d_L, \quad (4.29)$$

it is easy to note that if $\mu_1 \neq -\mu_2$, the infimum is $-\infty$ as $x_3 \in \mathbb{R}^n$. So taking the supremum over $\mu = -\mu_1 = \mu_2$ along with Proposition 1.7, the Lagrangian dual problem (4.28) leads to

$$\begin{aligned} d_L &= \sup_{\mu \in \mathbb{R}^n} \inf_{(x_1, x_2) \in C_1 \times C_2} \{f_1(x_1) - f_2(x_2) - \langle \mu, x_1 \rangle + \langle \mu, x_2 \rangle\} \\ &= \sup_{\mu \in \mathbb{R}^n} \{\inf_{x_2 \in C_2} (\langle \mu, x_2 \rangle - f_2(x_2)) + \inf_{x_1 \in C_1} (f_1(x_1) - \langle \mu, x_1 \rangle)\} \\ &= \sup_{\mu \in \mathbb{R}^n} \{(f_2)_*(\mu) - f_1^*(\mu)\}, \end{aligned}$$

thereby implying that $d_L = d_F$. This along with the relation (4.29) yields that $v_F = d_F$ and hence the Fenchel strong duality holds.

Conversely, suppose that the Fenchel strong duality holds under the assumption that $ri\ C_1 \cap ri\ C_2 \neq \emptyset$. Define

$$\begin{aligned} C_1 = \{(y_0, y, z) \in \mathbb{R}^{1+m+l} : \text{ there exists } x \in X \text{ such that } f(x) \leq y_0,\\ g_i(x) \leq y_i,\ i = 1, 2, \ldots, m,\\ h_j(x) = z_j,\ j = 1, 2, \ldots, l\} \end{aligned}$$

and

$$C_2 = \{(y_0, y, z) \in \mathbb{R}^{1+m+l} : y_i \leq 0,\ i = 1, 2, \ldots, m,\ z_j = 0,\ j = 1, 2, \ldots, l\}.$$

The Lagrange convex programming problem can now be expressed as

$$v_L = \inf\{y_0 : (y_0, y, z) \in C_1 \cap C_2\},$$

which is of the form of the Fenchel problem with $f_1(y_0, y, z) = y_0$ and $f_2(y_0, y, z) = 0$. The corresponding Fenchel dual problem is

$$\begin{aligned} d_F &= \sup\{((f_2)_*(\xi) - f_1^*(\xi)) : \xi = (\lambda_0, \lambda, \mu) \in \mathbb{R}^{1+m+l}\} \\ &= \sup_{(\lambda_0,\lambda,\mu)\in\mathbb{R}^{1+m+l}} \{ \inf_{(y_0,y,z)\in C_2} \{\lambda_0 y_0 + \langle \lambda, y\rangle + \langle \mu, z\rangle\} \\ &\qquad\qquad - \sup_{(y_0,y,z)\in C_1} \{\lambda_0 y_0 + \langle \lambda, y\rangle + \langle \mu, z\rangle - y_0\} \\ &= \sup_{(\lambda_0,\lambda,\mu)\in\mathbb{R}^{1+m+l}} \{ \inf_{(y_0,y,z)\in C_1} \{y_0 - \lambda_0 y_0 - \langle \lambda, y\rangle - \langle \mu, z\rangle\} \\ &\qquad\qquad + \inf_{(y_0,y,z)\in C_2} \{\lambda_0 y_0 + \langle \lambda, y\rangle + \langle \mu, z\rangle\}\}. \end{aligned} \tag{4.30}$$

By the assumption of Theorem 4.7, the modified Slater constraint qualification holds, which implies that there exists $\hat{x} \in ri\ X$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$, and $h_j(\hat{x}) = 0$, $j = 1, 2, \ldots, l$. As $dom\ g_i = X$, $i = 1, 2, \ldots, m$, by Theorem 2.69, g_i, $i = 1, 2, \ldots, m$, is continuous on $ri\ X$. Therefore, there exists $\hat{y}_i < 0$ such that $g_i(\hat{x}) < \hat{y}_i < 0$, $i = 1, 2, \ldots, m$. Also, as $dom\ f = X$ with $\hat{x} \in ri\ X$, one may choose $\hat{y}_0 \in \mathbb{R}$ such that $f(\hat{x}) < \hat{y}_0$. Thus, for $\hat{x} \in ri\ X$,

$$f(\hat{x}) < \hat{y}_0,\ g_i(\hat{x}) < \hat{y}_i,\ i = 1, 2, \ldots, m, \text{ and } h_j(\hat{x}) = \hat{z}_j,\ j = 1, 2, \ldots, l,$$

where $\hat{z}_j = 0$, $j = 1, 2, \ldots, l$. By Lemma 4.16, $(\hat{y}_0, \hat{y}, \hat{z}) \in ri\ C_1 \cap ri\ C_2$, which implies $ri\ C_1 \cap ri\ C_2 \neq \emptyset$. Thus, by Theorem 4.14,

$$v_L = d_F. \tag{4.31}$$

From the definition of C_2, the second infimum in (4.30) reduces to

$$\inf\{\lambda_0 y_0 + \sum_{i=1}^{m} \lambda_i y_i : y_0 \in \mathbb{R},\ y_i \leq 0,\ i = 1, 2, \ldots, m\}.$$

The infimum is $-\infty$ if either $\lambda_0 \neq 0$ or $\lambda_i > 0$ for some $i = 1, 2, \ldots, m$ and takes the value 0 otherwise. Therefore, the Fenchel dual problem becomes

$$
\begin{aligned}
d_F &= \sup_{(\lambda,\mu)\in\mathbb{R}^m_-\times\mathbb{R}^l} \inf_{(y_0,y,z)\in C_1} \{y_0 - \langle\lambda, y\rangle - \langle\mu, z\rangle\} \\
&= \sup_{(\lambda,\mu)\in\mathbb{R}^m_+\times\mathbb{R}^l} \inf_{(y_0,y,z)\in C_1} \{y_0 + \sum_{i=1}^m \lambda_i y_i + \sum_{j=1}^l \mu_j z_j\} \\
&= \sup_{(\lambda,\mu)\in\mathbb{R}^m_+\times\mathbb{R}^l} \inf_{x\in X} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^l \mu_j h_j(x)\},
\end{aligned}
$$

which yields that $d_F = d_L$. This along with (4.31) implies that $v_L = d_L$, thereby establishing the Lagrangian strong duality. $\square$

Chapter 5

Enhanced Fritz John Optimality Conditions

5.1 Introduction

Until now we have studied how to derive the necessary KKT optimality conditions for convex programming problems (CP) or its slight variations such as $(CP1)$, (CCP) or $(CCP1)$ via normal cone or saddle point approach. Observe that in the KKT optimality conditions, the multiplier associated with the subdifferential of the objective function is nonzero and thus normalized to one. As discussed in Chapters 3 and 4, some additional conditions known as the constraint qualifications are to be satisfied by the constraints to ensure that the multiplier is nonzero and hence the KKT optimality conditions hold. But in absence a of constraint qualification, one may not be able to derive KKT optimality conditions. For example, consider the problem

$$\min\ x \quad \text{subject to} \quad x^2 \leq 0.$$

In this example, $f(x) = x$ and $g(x) = x^2$ with $C = \{0\}$ at which none of the constraint qualifications is satisfied. Observe that the KKT optimality conditions is also not satisfied at the point of minimizer $\bar{x} = 0$, the only feasible point, as there do not exist $\lambda_0 = 1$ and $\lambda \geq 0$ such that

$$\lambda_0 \nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) = 0 \quad \text{and} \quad \lambda g(\bar{x}) = 0.$$

In this chapter, we will consider the convex programming problem

$$\min\ f(x) \quad \text{subject to } x \in C \tag{$CP1$}$$

where $(CP1)$ which involves not only inequality constraints but also additional abstract constraints, that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m,\ x \in X\}.$$

Here $f, g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$, are convex functions on $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$ is a closed convex set. Below we present the standard *Fritz John optimality conditions* for $(CP1)$.

Theorem 5.1 *Consider the convex programming problem* $(CP1)$ *and let* $\bar{x}$ *be a point of minimizer of* $(CP1)$. *Then there exist* $\lambda_i \geq 0$ *for* $i = 0, 1, \ldots, m$, *not all simultaneously zero, such that*

$$0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}) \quad \textit{and} \quad \lambda_i g_i(\bar{x}) = 0, \ \forall\ i = 1, 2, \ldots, m.$$

Proof. As $\bar{x}$ is a point of minimizer of $(CP1)$, it is a point of minimizer of the problem

$$\min\ F(x) \quad \text{subject to} \quad x \in X,$$

where $F(x) = \max\{f(x) - f(\bar{x}), g_1(x), g_2(x), \ldots, g_m(x)\}$ is a convex function. Therefore by the optimality condition (ii) of Theorem 3.1,

$$0 \in \partial F(\bar{x}) + N_X(\bar{x}).$$

Applying the Max-Function Rule, Theorem 2.96, there exist $\lambda_0 \geq 0$ and $\lambda_i \geq 0$, $i \in I(\bar{x})$, satisfying $\lambda_0 + \sum_{i \in I(\bar{x})} \lambda_i = 1$ such that

$$0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}),$$

where $I(\bar{x}) = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = 0\}$ is the active index set at $\bar{x}$. For $i \notin I(\bar{x})$, defining $\lambda_i = 0$ yields

$$0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x})$$

along with $\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m$, hence completing the proof. □

Note that in the example considered earlier, the Fritz John optimality condition holds if one takes $\lambda_0 = 0$ and $\lambda > 0$. Observe that the Fritz John optimality conditions are only necessary and not sufficient. To study the sufficiency optimality conditions, one needs KKT optimality conditions.

5.2 Enhanced Fritz John Conditions Using the Subdifferential

Recently, Bertsekas [11, 12] studied the Fritz John optimality conditions, which are more enhanced than those stated above and hence called them *enhanced Fritz John optimality conditions*. The proof of the enhanced Fritz John optimality condition involves the combination of the *quadratic penalty function* and *metric approximation* approaches. The penalty function approach is an important theoretical as well as algorithmic method in the study of constrained programming problems. Corresponding to the given problem, a sequence of the unconstrained penalized problem is formulated and in the limiting scenario, the sequence of point of minimizers of the penalized problem converges to the point of minimizer of the original constrained problem. The approach of metric approximations was introduced by Mordukhovich [84, 85]. This approach involves approximating the objective function and the constraint functions by smooth functions and reducing the constrained into an unconstrained problem. The work of Bertsekas [11, 12] was based mostly on the work of Hestenes [55], which was in turn motivated by the penalty function approach of McShane [83] to establish the Fritz John optimality conditions. It was the work of Hestenes [55] in which the complementary slackness was strengthened to obtain a somewhat weaker condition than the complementary violation condition, which we will discuss in the subsequent derivation of enhanced Fritz John optimality condition. In their works, McShane [83] and Hestenes [55] considered $X = \mathbb{R}^n$ while Bertsekas extended the study when $X \neq \mathbb{R}^n$. Below we discuss the above approach to establish the enhanced Fritz John optimality conditions for the convex programming problem $(CP1)$.

Theorem 5.2 *Consider the convex programming problem $(CP1)$ and let $\bar{x}$ be the point of minimizer of $(CP1)$. Then there exist $\lambda_i \geq 0$ for $i = 0, 1, \ldots, m$, not all simultaneously zero, such that*

(i) $0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x})$.

(ii) Consider the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$. If $\bar{I} \neq \emptyset$, then there exists a sequence $\{x_k\} \subset X$ that converges to $\bar{x}$ and is such that for all k sufficiently large,

$$f(x_k) < f(\bar{x}) \quad \text{and} \quad \lambda_i g_i(x_k) > 0, \ \forall\, i \in \bar{I}.$$

Proof. For $k = 1, 2, \ldots$, consider the penalized problem

$$\min F_k(x) \quad \text{subject to } x \in X \cap cl\, \mathbb{B}_\varepsilon(\bar{x}), \tag{P_k}$$

where $\varepsilon > 0$ is such that $f(\bar{x}) \leq f(x)$ for every $x \in cl\, \mathbb{B}_\varepsilon(\bar{x})$ feasible to $(CP1)$. The function $F_k : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$F_k(x) = f(x) + \frac{k}{2} \sum_{i=1}^{m} (g^+(x))^2 + \frac{1}{2} \|x - \bar{x}\|^2,$$

where $g^+(x) = \max\{0, g(x)\}$. By the convexity of the functions f and g_i, $i = 1, 2, \ldots, m$, F_k is a real-valued convex on $\mathbb{R}^n$. As *dom* $F_k = \mathbb{R}^n$, by Theorem 2.69, F_k is continuous on $\mathbb{R}^n$. Also, as X is a closed convex set and $cl\ \mathbb{B}_\varepsilon(\bar{x})$ is a compact convex set, $X \cap cl\ \mathbb{B}_\varepsilon(\bar{x})$ is a compact convex subset of $\mathbb{R}^n$. By the Weierstrass Theorem, Theorem 1.14, there exists a point of minimizer x_k for the problem (P_k). Therefore,

$$F_k(x_k) \leq F_k(\bar{x}),\ \forall\ k \in \mathbb{N},$$

which implies

$$f(x_k) + \frac{k}{2}\sum_{i=1}^{m}(g_i^+(x_k))^2 + \frac{1}{2}\|x_k - \bar{x}\|^2 \leq f(\bar{x}),\ \forall\ k \in \mathbb{N}. \tag{5.1}$$

Because $dom\ f = \mathbb{R}^n$, again by Theorem 2.69, f is continuous on $\mathbb{R}^n$. Hence, it is continuous on $X \cap cl\ \mathbb{B}_\varepsilon(\bar{x})$ and thus bounded over $X \cap cl\ \mathbb{B}_\varepsilon(\bar{x})$. By the boundedness of $f(x_k)$ over $X \cap cl\ \mathbb{B}_\varepsilon(\bar{x})$ and the relation (5.1), we have

$$\lim_{k\to\infty} g_i^+(x_k) = 0,\ i = 1, 2, \ldots, m. \tag{5.2}$$

Otherwise as $k \to +\infty$, the left-hand side of (5.1) also tends to infinity, which is a contradiction.

As $\{x_k\}$ is a bounded sequence, by the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, assume that $\{x_k\}$ converge to $\tilde{x} \in X \cap cl\ \mathbb{B}_\varepsilon(\bar{x})$. By the condition (5.2),

$$g_i(\tilde{x}) \leq 0,\ i = 1, 2, \ldots, m,$$

and hence $\tilde{x}$ is feasible for $(CP1)$. Taking the limit as $k \to +\infty$ in the condition (5.1) yields

$$f(\tilde{x}) + \frac{1}{2}\|\tilde{x} - \bar{x}\|^2 \leq f(\bar{x}).$$

As $\bar{x}$ is the point of minimizer of $(CP1)$ and $\tilde{x}$ is feasible to $(CP1)$, $f(\bar{x}) \leq f(\tilde{x})$. Thus the above inequality reduces to

$$\|\tilde{x} - \bar{x}\|^2 \leq 0,$$

which implies $\|\tilde{x} - \bar{x}\| = 0$, that is, $\tilde{x} = \bar{x}$. Hence, the sequence $x_k \to \bar{x}$ and thus there exists a $\bar{k} \in \mathbb{N}$ such that $x_k \in ri\ X \cap \mathbb{B}_\varepsilon(\bar{x})$ for every $k \geq \bar{k}$.

For $k \geq \bar{k}$, x_k is a point of minimizer of the penalized problem (P_k), which by Theorem 3.1 implies that

$$0 \in \partial F_k(x_k) + N_{X\cap\mathbb{B}_\varepsilon(\bar{x})}(x_k).$$

As $x_k \in ri\ X \cap \mathbb{B}_\varepsilon(\bar{x})$, by Proposition 2.39,

$$N_{X\cap\mathbb{B}_\varepsilon(\bar{x})}(x_k) = N_X(x_k) + N_{\mathbb{B}_\varepsilon(\bar{x})}(x_k).$$

Again, because $x_k \in \mathbb{B}_\varepsilon(\bar{x})$, by Example 2.38, $N_{\mathbb{B}_\varepsilon(\bar{x})}(x_k) = \{0\}$ and thus

$$0 \in \partial F_k(x_k) + N_X(x_k),\ \forall\ k \geq \bar{k}.$$

As $dom\ f = dom\ g_i = \mathbb{R}^n$, $i = 1, 2, \ldots, m$, by Theorem 2.69, f and g_i, $i = 1, 2, \ldots, m$ are continuous on $\mathbb{R}^n$. Applying the Sum Rule and the Chain Rule for the subdifferentials, Theorems 2.91 and 2.94, respectively, the above condition becomes

$$0 \in \partial f(x_k) + k \sum_{i=1}^{m} g_i^+(x_k) \partial g_i^+(x_k) + (x_k - \bar{x}) + N_X(x_k), \ \forall\ k \geq \bar{k},$$

which implies that for every $k \geq \bar{k}$, there exist $\xi_0^k \in \partial f(x_k)$ and $\xi_i^k \in \partial g_i(x_k)$, $i = 1, 2, \ldots, m$, such that

$$-\{\xi_0^k + \sum_{i=1}^{m} \alpha_i^k \xi_i^k + (x_k - \bar{x})\} \in N_X(x_k), \tag{5.3}$$

where $\alpha_i^k = k\beta_k g_i^+(x_k)$ and $\beta_k \in [0, 1]$ for $i = 1, 2, \ldots, m$. Denote

$$\gamma^k = \sqrt{1 + \sum_{i=1}^{m} (\alpha_i^k)^2}, \quad \lambda_0^k = \frac{1}{\gamma^k} \quad \text{and} \quad \lambda_i^k = \frac{\alpha_i^k}{\gamma^k},\ i = 1, 2, \ldots, m. \tag{5.4}$$

Observe that

$$(\lambda_0^k)^2 + \sum_{i=1}^{m} (\lambda_i^k)^2 = 1, \ \forall\ k \geq \bar{k}. \tag{5.5}$$

Therefore, the sequences $\{\lambda_0^k\}$ and $\{\lambda_i^k\}$, $i = 1, 2, \ldots, m$, are bounded sequences in $\mathbb{R}_+$ and thus by the Bolzano–Weierstrass Theorem, Proposition 1.3 have a convergent subsequence. Without loss of generality, let $\lambda_i^k \to \lambda_i$, $i = 0, 1, \ldots, m$. As $\alpha_i^k \geq 0$, $i = 1, 2, \ldots, m$ and $\gamma^k \geq 1$ for every $k \geq \bar{k}$, $\lambda_i^k \geq 0$ and thereby implying that $\lambda_i \geq 0$, $i = 0, 1, \ldots, m$. Also by condition (5.5), it is obvious that $\lambda_0, \lambda_1, \ldots, \lambda_m$ are not simultaneously zero. Now dividing (5.3) by γ^k leads to

$$-\{\lambda_0^k \xi_0^k + \sum_{i=1}^{m} \lambda_i^k \xi_i^k + \frac{1}{\gamma^k}(x_k - \bar{x})\} \in N_X(x_k). \tag{5.6}$$

As f and g_i, $i = 1, 2, \ldots, m$ are continuous at $x_k \in \mathbb{R}^n$, therefore by Proposition 2.82, $\partial f(x_k)$ and $\partial g_i(x_k)$, $i = 1, 2, \ldots, m$, are compact. Thus $\{\xi_i^k\}$, $i = 0, 1, \ldots, m$, are bounded sequences in $\mathbb{R}^n$ and hence by the Bolzano–Weierstrass Theorem have a convergent subsequence. Without loss of generality, let $\xi_i^k \to \xi_i$, $i = 0, 1, \ldots, m$. By the Closed Graph Theorem, Theorem 2.84, of the subdifferentials, $\xi_0 \in \partial f(\bar{x})$ and $\xi_i \in \partial g_i(\bar{x})$ for $i = 1, 2, \ldots, m$.

Taking the limit as $k \to +\infty$ in (5.6) along with the fact that the normal cone N_X has a closed graph yields

$$-\{\lambda_0 \xi_0 + \sum_{i=1}^{m} \lambda_i \xi_i\} \in N_X(\bar{x}),$$

which implies

$$0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}),$$

thereby establishing condition (i).

Now suppose that the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$ is nonempty. For $i \in \bar{I}$, corresponding to $\lambda_i > 0$, there exists a sequence $\lambda_i^k \to \lambda_i$. Therefore, for all k sufficiently large, $\lambda_i^k > 0$ and hence $\lambda_i \lambda_i^k > 0$ for $i \in \bar{I}$. By the condition (5.4), $\lambda_i g_i^+(x_k) > 0$ for sufficiently large k, which implies

$$\lambda_i g_i(x_k) > 0, \ \forall \ i \in \bar{I}.$$

Also, by condition (5.1), $f(x_k) < f(\bar{x})$ for sufficiently large k and hence condition (ii) is satisfied, thereby yielding the requisite result. □

Observe that the condition (ii) in the above theorem is a condition that replaces the complementary slackness condition in the Fritz John optimality condition. According to the condition (ii), if $\lambda_i > 0$, the corresponding constraint g_i is violated at the points arbitrarily close to $\bar{x}$. Thus the condition (ii) is called the *complementary violation condition* by Bertsekas and Ozdaglar [13].

Now let us consider, in particular, $X = \mathbb{R}^n$ and $g_i, \ i = 1, 2, \ldots, m$, to be affine in $(CP1)$. Then from the above theorem there exist $\lambda_i \geq 0$, $i = 0, 1, \ldots, m$, not all simultaneously zero, such that conditions (i) and (ii) hold. Due to affineness of $g_i, \ i = 1, 2, \ldots, m$, we have

$$g_i(x) = g_i(\bar{x}) + \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle, \ \forall \ x \in \mathbb{R}^n. \tag{5.7}$$

Suppose that $\lambda_0 = 0$. Then by condition (i) of Theorem 5.2,

$$0 = \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}),$$

which implies that

$$0 = \sum_{i=1}^{m} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle.$$

As all the scalars cannot be all simultaneously zero, the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$ is nonempty. By condition (ii), there exists a sequence $\{x_k\} \subset \mathbb{R}^n$ such that $g_i(x_k) > 0$ for $i \in \bar{I}$. Therefore, by (5.7), which along with the above condition for $x = x_k$, leads to

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = \sum_{i=1}^{m} \lambda_i g_i(x_k) = \sum_{i \in \bar{I}} \lambda_i g_i(x_k) > 0,$$

which implies that $g_i(\bar{x}) > 0$ for some $i \in \bar{I}$, thereby contradicting the feasibility of $\bar{x}$. Thus $\lambda_0 > 0$ and hence can be normalized to one, thereby leading to the KKT optimality condition.

Observe that in the case as discussed above, the KKT optimality condition holds without any assumption of constraint qualification. But if the convex programming problem is not of the above type, to ensure that $\lambda_0 \neq 0$, one has to impose some form of constraint qualification. In view of the enhanced Fritz John optimality conditions, Bertsekas [12] introduced the notion of pseudonormality, which is defined as follows.

Definition 5.3 A feasible point $\bar{x}$ of $(CP1)$ is said to be *pseudonormal* if there does not exist any $\lambda_i,\ i = 1, 2, \ldots, m$, and sequence $\{x_k\} \subset X$ such that

(i) $0 \in \displaystyle\sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x})$

(ii) $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$ and $\lambda_i = 0$ for $i \notin I(\bar{x})$. Recall that $I(\bar{x}) = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = 0\}$ denotes the active index set at $\bar{x}$.

(iii) $\{x_k\}$ converges to $\bar{x}$ and

$$\sum_{i=1}^{m} \lambda_i g_i(x_k) > 0,\ \forall\ k \in \mathbb{N}.$$

Below we present a result to show how the affineness of $g_i,\ i = 1, 2, \ldots, m$, or the Slater-type constraint qualification ensure the pseudonormality at a feasible point.

Theorem 5.4 *Consider the problem $(CP1)$ and let $\bar{x}$ be a feasible point of $(CP1)$. Then $\bar{x}$ is pseudonormal under either one of the following two criteria:*

(a) Linearity criterion: $X = \mathbb{R}^n$ and the functions $g_i,\ i = 1, 2, \ldots, m$, are affine.

(b) Slater-type constraint qualification: there exists a feasible point $\hat{x} \in X$ of $(CP1)$ such that $g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m$.

Proof. (a) Suppose on the contrary that $\bar{x}$ is not pseudonormal, which implies that there exist $\lambda_i,\ i = 1, 2, \ldots, m$, and $\{x_k\} \subset \mathbb{R}^n$ satisfying conditions (i), (ii), and (iii) in the Definition 5.3. By the affineness of $g_i,\ i = 1, 2, \ldots, m$, for every $x \in \mathbb{R}^n$,

$$g_i(x) = g_i(\bar{x}) + \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle,$$

which implies

$$\sum_{i=1}^{m} \lambda_i g_i(x) = \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) + \sum_{i=1}^{m} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle,\ \forall\ x \in \mathbb{R}^n. \tag{5.8}$$

By the conditions (i) and (ii) in the definition of pseudonormality,

$$0 = \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m,$$

thereby reducing the condition (5.8) to

$$\sum_{i=1}^{m} \lambda_i g_i(x) = 0,\ \forall\, x \in \mathbb{R}^n.$$

This is a contradiction of condition (iii) of Definition 5.3 at $\bar{x}$. Hence, $\bar{x}$ is pseudonormal.

(b) On the contrary, suppose that $\bar{x}$ is not pseudonormal. By the convexity of $g_i,\ i = 1, 2, \ldots, m$, for every $x \in \mathbb{R}^n$,

$$g_i(x) - g_i(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle,\ \forall\, \xi_i \in \partial g_i(\bar{x}),\ i = 1, 2, \ldots, m. \tag{5.9}$$

By condition (i) in the definition of pseudonormality, there exist $\bar{\xi}_i \in \partial g_i(\bar{x})$, $i = 1, 2, \ldots, m$, such that

$$\sum_{i=1}^{m} \lambda_i \langle \bar{\xi}_i, x - \bar{x} \rangle \geq 0,\ \forall\, x \in X.$$

The above inequality along with condition (ii) reduces the condition (5.9) to

$$\sum_{i=1}^{m} \lambda_i g_i(x) \geq 0,\ \forall\, x \in X. \tag{5.10}$$

As the Slater constraint qualification is satisfied at $\hat{x} \in X$,

$$\sum_{i=1}^{m} \lambda_i g_i(\hat{x}) < 0$$

if $\lambda_i > 0$ for some $i \in I(\bar{x})$. Thus, the condition (5.10) holds only if $\lambda_i = 0$ for $i = 1, 2, \ldots, m$. But then this contradicts condition (iii). Therefore, $\bar{x}$ is pseudonormal. $\square$

In Chapter 3 we derived the KKT optimality conditions under the Slater constraint qualification as well as the Abadie constraint qualification. For the convex programming problem (CP) considered in previous chapters, the feasible set C was given by (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}.$$

Recall that the Abadie constraint qualification is said to hold at $\bar{x} \in C$ if

$$T_C(\bar{x}) = \{d \in \mathbb{R}^n : g_i'(\bar{x}, d) \leq 0,\ \forall\, i \in I(\bar{x})\},$$

where $I(\bar{x})$ is the active index set at $\bar{x}$. But unlike the Slater constraint qualification, the Abadie constraint qualification need not imply pseudonormality. For better understanding, let us recall the example

$$C = \{x \in \mathbb{R} : |x| \leq 0, \ x \leq 0\}.$$

From the discussion in Chapter 3, we know that the Abadie constraint qualification is satisfied at $\bar{x} = 0$ but the Slater constraint qualification does not hold as the feasible set $C = \{0\}$. Observe that both constraints are active at $\bar{x}$. Taking the scalars $\lambda_1 = \lambda_2 = 1$ and the sequence $\{x_k\}$ as $\{1/k\}$, conditions (i), (ii), and (iii) in Definition 5.3 are satisfied. Thus, $\bar{x} = 0$ is not pseudonormal. The Abadie constraint qualification is also known as *quasiregularity* at $\bar{x}$. This condition was defined for $X = \mathbb{R}^n$. The notion of quasiregularity is implied by the concept of *quasinormality*. This concept was introduced by Hestenes [55] for the case $X = \mathbb{R}^n$. The notion of quasinormality is further implied by pseudonormality.

Now if $X \neq \mathbb{R}^n$, the quasiregularity at $\bar{x}$ is defined as

$$T_C(\bar{x}) = \{d \in \mathbb{R}^n : g_i'(\bar{x}, d) \leq 0, \ \forall\, i \in I(\bar{x})\} \cap T_X(\bar{x}).$$

The above condition was studied by Gould and Tolle [53] and Guignard [54]. It was shown by Ozdaglar [91] and Ozdaglar and Bertsekas [92] that under the regularity (Chapter 2 end notes) of the set X, pseudonormality implies quasiregularity. They also showed that unlike the case $X = \mathbb{R}^n$ where quasiregularity leads to KKT optimality conditions, the concept is not enough to derive KKT conditions when $X \neq \mathbb{R}^n$ unless some additional conditions are assumed. For more on quasiregularity and quasinormality, readers are advised to refer to the works of Bertsekas.

Next we establish the KKT optimality conditions under the pseudonormality assumptions at the point of minimizer.

Theorem 5.5 *Consider the convex programming problem* $(CP1)$*. Assume that* $\bar{x}$ *satisfies pseudonormality. Then* $\bar{x}$ *is a point of minimizer of* $(CP1)$ *if and only if there exist* $\lambda_i \geq 0, \ i = 1, \ldots, m$*, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}) \quad \text{ and } \quad \lambda_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m.$$

Proof. Observe that the complementary slackness condition is equivalent to condition (ii) in the definition of pseudonormality. Therefore, $\lambda_i = 0$ for every $i \notin I(\bar{x})$. Suppose that the multiplier λ_0 associated with the subdifferential of the objective function in the enhanced Fritz John optimality condition is zero. Therefore,

$$0 \in \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}),$$

that is, condition (i) of Definition 5.3 holds. As all $\lambda_i \geq 0$, $i = 0, 1, \ldots, m$, are not simultaneously zero, $\lambda_i > 0$ for some $i \in I(\bar{x})$ and thus $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$ is nonempty. Therefore, by condition (ii) of the enhanced Fritz John condition, there exists a sequence $\{x_k\} \subset X$ converging to $\bar{x}$ such that

$$\lambda_i g_i(x_k) > 0, \ \forall\, i \in \bar{I},$$

which implies

$$\sum_{i=1}^{m} \lambda_i g_i(x_k) > 0.$$

Thus condition (iii) in the definition of pseudonormality is satisfied, thereby implying that $\bar{x}$ is not pseudonormal. This contradicts the given hypothesis. Therefore, $\lambda_0 \neq 0$, thereby satisfying the KKT optimality conditions. The sufficiency can be worked out using the convexity of the objective function and the constraint functions along with the convexity of the set X as done in Chapter 3. $\square$

5.3 Enhanced Fritz John Conditions under Restrictions

Observe that in the problem $(CP1)$, the functions f and g_i, $i = 1, 2, \ldots, m$, are convex on $\mathbb{R}^n$. But if these functions are convex only over the closed convex set X, the line of proof of the above theorem breaks down. Bertsekas, Ozdaglar, and Tseng [14] gave an alternative version of the enhanced Fritz John optimality conditions, which is independent of the subdifferentials. The proof given by them, which we present below relies on the saddle point theory.

Theorem 5.6 *Consider the convex programming problem $(CP1)$ where the functions f and g_i, $i = 1, 2, \ldots, m$, are lsc and convex on the closed convex set $X \subset \mathbb{R}^n$ and let $\bar{x}$ be a point of minimizer of $(CP1)$. Then there exist $\lambda_i \geq 0$ for $i = 0, 1, \ldots, m$, not all simultaneously zero, such that*

(i) $\lambda_0 f(\bar{x}) = \min_{x \in X}\{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$

(ii) Consider the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$. If $\bar{I} \neq \emptyset$, then there exists a sequence $\{x_k\} \subset X$ that converges to $\bar{x}$ and is such that

$$\lim_{k \to \infty} f(x_k) = f(\bar{x}) \quad \textit{and} \quad \limsup_{k \to \infty} g_i(x_k) \leq 0, \ i = 1, 2, \ldots, m,$$

and for all k sufficiently large

$$f(x_k) < f(\bar{x}) \quad \textit{and} \quad g_i(x_k) > 0, \ \forall\, i \in \bar{I}.$$

Proof. For the positive integers k and r, consider the saddle point function $L_{k,r} : X \times \mathbb{R}^m_+ \to \mathbb{R}$ defined as

$$L_{k,r}(x, \alpha) = f(x) + \frac{1}{k^3}\|x - \bar{x}\|^2 + \sum_{i=1}^{m} \alpha_i g_i(x) - \frac{1}{2r}\|\alpha\|^2.$$

For fixed $\alpha_i \geq 0,\ i = 1, 2, \ldots, m$, by the lower semicontinuity and convexity of the functions f and $g_i,\ i = 1, 2, \ldots, m$, over X, $L_{k,r}(., \alpha)$ is an lsc convex function while for a fixed $x \in X$, $L_{k,r}(x, .)$ is strongly concave and quadratic in α. For every k, define the set

$$X_k = X \cap \bar{\mathbb{B}}_k(\bar{x}).$$

Observe that X_k is a compact set. As f and $g_i,\ i = 1, 2 \ldots, m$, are lsc convex on X, the functions are lsc, convex on X_k. Also, as $L_{k,r}(x, .)$ is strongly concave, it has a unique maximizer over $\mathbb{R}^m_+$ and thus for some $\beta \in \mathbb{R}$, the level set

$$\{\alpha \in \mathbb{R}^m_+ : L_{k,r}(\bar{x}, \alpha) \geq \beta\}$$

is nonempty and compact. Thus by condition (iii) of the Saddle Point Theorem, Proposition 4.1, $L_{k,r}$ has a saddle point over $X_k \times \mathbb{R}^m_+$, say $(x_{k,r}, \alpha_{k,r})$. By the saddle point definition,

$$L_{k,r}(x_{k,r}, \alpha) \leq L_{k,r}(x_{k,r}, \alpha_{k,r}) \leq L_{k,r}(x, \alpha_{k,r}),\ \forall\, x \in X_k,\ \forall\, \alpha \in \mathbb{R}^m_+. \quad (5.11)$$

As $L_{k,r}(., \alpha_{k,r})$ attains an infimum over X_k at $x_{k,r}$,

$$\begin{aligned}
L_{k,r}(x_{k,r}, \alpha_{k,r}) &= f(x_{k,r}) + \frac{1}{k^3}\|x_{k,r} - \bar{x}\|^2 + \sum_{i=1}^{m} \alpha_{k,r_i} g_i(x_{k,r}) - \frac{1}{2r}\|\alpha_{k,r}\|^2 \\
&\leq \inf_{x \in X_k} \{f(x) + \frac{1}{k^3}\|x - \bar{x}\|^2 + \sum_{i=1}^{m} \alpha_{k,r_i} g_i(x)\} \\
&\leq \inf_{x \in X_k, g_i(x) \leq 0, \forall i} \{f(x) + \frac{1}{k^3}\|x - \bar{x}\|^2 + \sum_{i=1}^{m} \alpha_{k,r_i} g_i(x)\} \\
&\leq \inf_{x \in X_k, g_i(x) \leq 0, \forall i} \{f(x) + \frac{1}{k^3}\|x - \bar{x}\|^2\}.
\end{aligned}$$

As $\bar{x} \in X_k$ and satisfies $g_i(\bar{x}) \leq 0,\ i = 1, 2, \ldots, m$, the above inequalities yield

$$L_{k,r}(x_{k,r}, \alpha_{k,r}) \leq f(\bar{x}). \quad (5.12)$$

Again from (5.11), $L_{k,r}(x_{k,r}, .)$ attains a supremum over $\alpha \in \mathbb{R}^m_+$ at $\alpha_{k,r}$. As a function of $\alpha \in \mathbb{R}^m_+$, $L_{k,r}(x_{k,r}, .)$ is strongly concave and quadratic, and thus, has a unique supremum at

$$\alpha_{k,r_i} = r g_i^+(x_{k,r}),\ i = 1, 2, \ldots, m. \quad (5.13)$$

We leave it to the readers to figure out how to compute α_{k,r_i}. Therefore,

$$L_{k,r}(x_{k,r}, \alpha_{k,r}) = f(x_{k,r}) + \frac{1}{k^3}\|x_{k,r} - \bar{x}\|^2 + \frac{r}{2}\|g^+(x_{k,r})\|^2, \tag{5.14}$$

which implies that

$$L_{k,r}(x_{k,r}, \alpha_{k,r}) \geq f(x_{k,r}). \tag{5.15}$$

From the conditions (5.12) and (5.15), we have

$$x_{k,r} \in \{x \in X_k : f(x) \leq f(\bar{x})\}.$$

As X_k is compact, the set $\{x \in X_k : f(x) \leq f(x_k)\}$ is bounded and thus $\{x_{k,r}\}$ forms a bounded sequence. In fact, we leave it to the readers to show that f is also coercive on X_k. Thus, by the Bolzano–Weierstrass Theorem, Proposition 1.3, for a fixed k the sequence $\{x_{k,r}\}$ has a convergent subsequence. Without loss of generality, let $\{x_{k,r}\}$ converge to $x_k \in \{x \in X_k : f(x) \leq f(\bar{x})\}$.

As f is convex and coercive on X_k, by the Weierstrass Theorem, Theorem 1.14, an infimum over X_k exists. Therefore for each k, the sequence $\{f(x_{k,r})\}$ is bounded below by $\inf_{x \in X_k} f(x)$. Also, by condition (5.12), $L_{k,r}(x_{k,r}, \alpha_{k,r})$ is bounded above by $f(\bar{x})$. Thus, from (5.14),

$$\limsup_{r \to \infty} g_i(x_{k,r}) \leq 0, \ \ i = 1, 2, \ldots, m,$$

which along with the lower semicontinuity of $g_i, \ i = 1, 2, \ldots, m$, implies that $g_i(x_k) \leq 0$ for $i = 1, 2, \ldots, m$, thereby yielding the feasibility of x_k for $(CP1)$. We urge the reader to work out the details. As $\bar{x}$ is the minimizer of $(CP1)$, $f(x_k) \geq f(\bar{x})$, which along with the conditions (5.12), (5.15), and the lower semicontinuity of f leads to

$$f(\bar{x}) \leq f(x_k) \leq \liminf_{r \to \infty} f(x_{k,r}) \leq \limsup_{r \to \infty} f(x_{k,r}) \leq f(\bar{x}),$$

which implies that for each k,

$$\lim_{r \to \infty} f(x_{k,r}) = f(\bar{x}).$$

By the conditions (5.12) and (5.14), we have for every $k \in \mathbb{N}$,

$$\lim_{r \to \infty} x_{k,r} = \bar{x}.$$

Further note that using the definition of $L_{k,r}(x_{k,r}, \alpha_{k,r})$ and using (5.12) and (5.15), for every k,

$$\lim_{r \to +\infty} \sum_{i=1}^{m} \alpha_{k,r_i} g_i(x_{k,r}) = 0.$$

Therefore, by the preceding conditions,

$$\lim_{r\to\infty}\{f(x_{k,r})-f(\bar{x})+\sum_{i=1}^{m}\alpha_{k,r_i}g_i(x_{k,r})\}=0. \tag{5.16}$$

Denote

$$\gamma^{k,r}=\sqrt{1+\sum_{i=1}^{m}(\alpha_{k,r_i})^2},\quad \lambda_0^{k,r}=\frac{1}{\gamma^{k,r}}\quad \text{and}\quad \lambda_i^{k,r}=\frac{\alpha_{k,r_i}}{\gamma^{k,r}},\ i=1,2,\ldots,m. \tag{5.17}$$

Dividing (5.16) by $\gamma^{k,r}>0$ leads to

$$\lim_{r\to\infty}\{\lambda_0^{k,r}f(x_{k,r})-\lambda_0^{k,r}f(\bar{x})+\sum_{i=1}^{m}\lambda_i^{k,r}g_i(x_{k,r})\}=0.$$

For each k, we fix an integer r_k such that

$$|\lambda_0^{k,r_k}f(x_{k,r_k})-\lambda_0^{k,r_k}f(\bar{x})+\sum_{i=1}^{m}\lambda_i^{k,r_k}g_i(x_{k,r_k})|\leq\frac{1}{k} \tag{5.18}$$

and

$$\|x_{k,r_k}-\bar{x}\|\leq\frac{1}{k},\quad |f(x_{k,r_k})-f(\bar{x})|\leq\frac{1}{k},\quad |g_i^+(x_{k,r_k})|\leq\frac{1}{k},\ i=1,2,\ldots,m. \tag{5.19}$$

Dividing the saddle point condition

$$L_{k,r_k}(x_{k,r_k},\alpha_{k,r_k})\leq L_{k,r_k}(x,\alpha_{k,r_k}),\ \forall\ x\in X_k$$

by γ^{k,r_k} yields

$$\begin{aligned}\lambda_0^{k,r_k}f(x_{k,r_k})+\frac{\lambda_0^{k,r_k}}{k^3}\|x_{k,r_k}-\bar{x}\|^2+\sum_{i=1}^{m}\lambda_i^{k,r_k}g_i(x_{k,r_k})&\\ \leq\lambda_0^{k,r_k}f(x)+\frac{1}{k^3\gamma^{k,r_k}}\|x-\bar{x}\|^2+\sum_{i=1}^{m}\lambda_i^{k,r_k}g_i(x),\ \forall\ x\in X_k.&\end{aligned}$$

As $\alpha_i^{k,r_k}\geq 0$, $i=1,2,\ldots,m$, from the condition (5.17), $\gamma^{k,r_k}\geq 1$ and $\lambda_i^{k,r_k}\geq 0$, $i=0,1,\ldots,m$, along with

$$(\lambda_0^{k,r_k})^2+\sum_{i=1}^{m}(\lambda_i^{k,r_k})^2=1.$$

Therefore, $\{\lambda_i^{k,r_k}\}$, $i=0,1,\ldots,m$, are bounded sequences in $\mathbb{R}_+$ and thus by the Bolzano–Weierstrass Theorem, Proposition 1.3, have a convergent subsequence. Without loss of generality, assume that $\lambda_i^{k,r_k}\to\lambda_i\geq 0$,

$i = 0, 1, \ldots, m$, not all simultaneously zero. Taking the limit as $k \to +\infty$ in the above inequality along with the condition (5.18) leads to

$$\lambda_0 f(\bar{x}) \leq \lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \ \forall \ x \in X,$$

which implies

$$\begin{aligned} \lambda_0 f(\bar{x}) &\leq \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\} \\ &\leq \inf_{x \in X, g_i(x) \leq 0, \forall i} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\} \\ &\leq \inf_{x \in X, g_i(x) \leq 0, \forall i} \lambda_0 f(x) \\ &= \lambda_0 f(\bar{x}). \end{aligned}$$

Therefore, $\lambda_i \geq 0, \ i = 0, 1, \ldots, m$, not all simultaneously zero, satisfy condition (i), that is,

$$\lambda_0 f(\bar{x}) = \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$$

Next suppose that the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$ is nonempty. Corresponding to $\lambda_i > 0$ for $i \in \bar{I}$, there is a sequence $\lambda_i^{k,r_k} \to \lambda_i$ such that $\lambda_i^{k,r_k} > 0, \ i = 1, 2, \ldots, m$, which along with the condition (5.13) implies

$$g_i(x_{k,r_k}) > 0, \ \forall \ i \in \bar{I}$$

for sufficiently large k. For each k, choosing r_k such that $x_{k,r_k} \neq \bar{x}$ and the condition (5.19) is satisfied, implies that

$$x_{k,r_k} \to \bar{x}, \quad f(x_{k,r_k}) \to f(\bar{x}), \quad g_i^+(x_{k,r_k}) \to 0, \ i = 1, 2, \ldots, m.$$

Also, by the condition (5.12),

$$f(x_{k,r_k}) \leq f(\bar{x}),$$

thereby proving (ii) and hence establishing the requisite result. □

Similar to the pseudonormality notion defined earlier, the notion is stated as below for the enhanced Fritz John conditions obtained in Theorem 5.6.

Definition 5.7 The constraint set of $(CP1)$ is said to be *pseudonormal* if there do not exist any scalars $\lambda_i \geq 0, \ i = 1, 2, \ldots, m$, and a vector $x' \in X$ such that

(i) $0 = \inf_{x \in X} \sum_{i=1}^{m} \lambda_i g_i(x)$,

(ii) $\sum_{i=1}^{m} \lambda_i g_i(x') > 0$.

For a better understanding of the above definition of pseudonormality, we recall the idea of proper separation from Definition 2.25. A hyperplane H is said to separate two convex sets F_1 and F_2 properly if

$$\sup_{x_1 \in F_1} \langle a, x_1 \rangle \leq \inf_{x_2 \in F_2} \langle a, x_2 \rangle \quad \text{and} \quad \inf_{x_1 \in F_1} \langle a, x_1 \rangle < \sup_{x_2 \in F_2} \langle a, x_2 \rangle.$$

Now consider a set $G = \{g(x) = (g_1(x), g_2(x), \ldots, g_m(x)) : x \in X\}$. Then from Definition 5.7 it is easy to observe that pseudonormality implies that there exists no hyperplane H that separates the set G and the origin $\{0\}$ properly.

Similar to Theorem 5.4, the pseudonormality of the constraint set can be derived under the Linearity criterion or the Slater constraint qualification.

Theorem 5.8 *Consider the problem $(CP1)$. Then the constraint set is pseudonormal under either one of the following two criteria:*

(a) Linearity criterion: $X = \mathbb{R}^n$ and the functions $g_i,\ i = 1, 2, \ldots, m$, are affine.

(b) Slater--type constraint qualification: there exists a feasible point $\hat{x} \in X$ of $(CP1)$ such that $g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m$.

Proof. (a) Suppose on the contrary that the constraint set is not pseudonormal, which implies that there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, and a vector $x' \in \mathbb{R}^n$ satisfying conditions (i) and (ii) in the Definition 5.7. Suppose that $\bar{x} \in \mathbb{R}^n$ is feasible to $(CP1)$, that is, $g_i(\bar{x}) \leq 0,\ i = 1, 2, \ldots, m$, which along with condition (i) yields

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = 0. \tag{5.20}$$

By the affineness of $g_i,\ i = 1, 2, \ldots, m$,

$$g_i(x) = g_i(\bar{x}) + \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle,\ \forall\ x \in \mathbb{R}^n,$$

which again by condition (i) and (5.20) implies

$$\sum_{i=1}^{m} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \geq 0,\ \forall\ x \in \mathbb{R}^n.$$

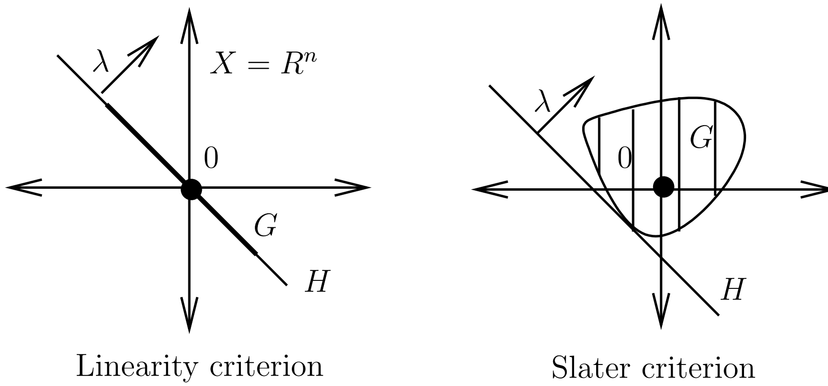

FIGURE 5.1: Pseudonormality.

By Definition 2.36 of the normal cone, $\sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) \in N_{\mathbb{R}^n}(\bar{x})$. As $\bar{x} \in \mathbb{R}^n$, by Example 2.38, the normal cone $N_{\mathbb{R}^n}(\bar{x}) = \{0\}$, which implies

$$\sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0.$$

This equality along with the condition (5.20) and the affineness of g_i, $i = 1, 2, \ldots, m$ implies that

$$\sum_{i=1}^{m} \lambda_i g_i(x) = 0, \ \forall \ x \in \mathbb{R}^n,$$

thereby contradicting condition (ii) in the definition of pseudonormality. Hence the constraint set is pseudonormal.

(b) Suppose on the contrary that the constraint set is not pseudonormal. As the Slater-type constraint qualification holds, there exists $\hat{x} \in X$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$, condition (i) is satisfied only if $\lambda_i = 0$, $i = 1, 2, \ldots, m$, which contradicts condition (ii) in Definition 5.7. Therefore, the constraint set is pseudonormal. $\square$

In case the Slater-type constraint qualification is satisfied, the set G intersects the orthant $\{x \in \mathbb{R}^m : x_i \leq 0, \ i = 1, 2, \ldots, m\}$ as shown in Figure 5.1. Then obviously condition (i) in the definition of pseudonormality does not hold; that is, there exists no hyperplane H passing through origin supporting G such that G lies in the positive orthant. Now when one has the linearity criterion, that is, $X = \mathbb{R}^n$ and $g_i, \ i = 1, 2, \ldots, m$, are affine, the set G is also affine (see Figure 5.1) and thus, condition (ii) is violated; that is, the hyperplane H does not contain the set G completely. In the linearity criterion, if X is a polyhedron instead of $X = \mathbb{R}^n$ along with $g_i, \ i = 1, 2, \ldots, m$, being affine,

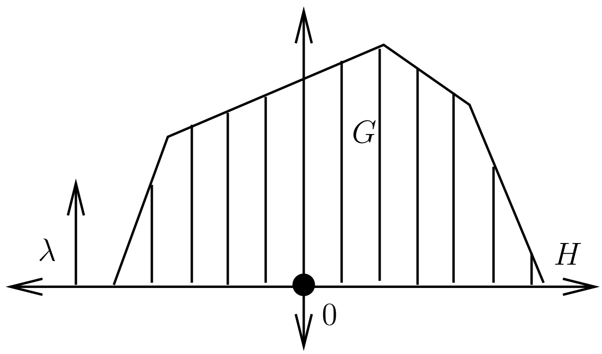

FIGURE 5.2: Not pseudonormal.

pseudonormality need not hold as shown in Figure 5.2. These observations were made by Bertsekas, Ozdaglar, and Tseng [14].

We end this section by establishing the KKT optimality conditions similar to Theorem 5.5, under the pseudonormality of the constraint set.

Theorem 5.9 *Consider the convex programming problem* $(CP1)$ *where the functions* f *and* $g_i,\ i = 1, 2, \ldots, m$ *are lsc and convex on the closed convex set* $X \subset \mathbb{R}^n$. *Assume that the constraint set is pseudonormal. Then* $\bar{x}$ *is a point of minimizer of* $(CP1)$ *if and only if there exist* $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, *such that*

$$f(\bar{x}) = \min_{x \in X}\{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\} \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

Proof. Suppose that in the enhanced Fritz John optimality condition, Theorem 5.6, $\lambda_0 = 0$. This implies

$$0 = \min_{x \in X} \sum_{i=1}^{m} \lambda_i g_i(x),$$

that is, $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, satisfies condition (i) in the definition of pseudonormality of the constraint set. As in the enhanced Fritz John condition $\lambda_i,\ i = 0, 1, \ldots, m$, are not all simultaneously zero, there exists at least one $i \in \{1, 2, \ldots, m\}$ such that $\lambda_i > 0$, that is, $\bar{I}$ is nonempty. Again by Theorem 5.6, there exist a sequence $\{x_k\} \subset X$ such that

$$g_i(x_k) > 0,\ \forall\ i \in \bar{I},$$

which implies

$$\sum_{i \in \bar{I}} \lambda_i g_i(x_k) = \sum_{i=1}^{m} \lambda_i g_i(x_k) > 0,$$

that is, satisfying condition (ii) in Definition 5.7, thereby contradicting the fact that the constraint sets are pseudonormal. Thus, $\lambda_0 \neq 0$ and hence can be taken in particular as one, thereby establishing the optimality condition. Using the optimality condition along with the feasibility of $\bar{x}$ leads to

$$0 \leq \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \leq 0,$$

that is,

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = 0.$$

As the sum of nonpositive terms is zero if every term is zero, the above equality leads to

$$\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m,$$

thereby establishing the complementary slackness condition.

Conversely, by the optimality condition,

$$f(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x),\ \forall\ x \in X.$$

In particular, for any x feasible to $(CP1)$, that is, $x \in X$ satisfying $g_i(x) \leq 0$, $i = 1, 2, \ldots, m$, the above inequality reduces to

$$f(\bar{x}) \leq f(x),$$

thus proving that $\bar{x}$ is a point of minimizer of $(CP1)$. $\square$

5.4 Enhanced Fritz John Conditions in the Absence of Optimal Solution

Up to now in this chapter, one observes two forms of enhanced Fritz John optimality conditions, one when the functions are convex over the whole space $\mathbb{R}^n$ while in the second scenario convexity of the functions is over the convex set $X \neq \mathbb{R}^n$. The results obtained in Section 5.3 are in a form similar to strong duality. In all the results of enhanced Fritz John and KKT optimality conditions, it is assumed that the point of minimizer exists. But what if the convex programming problem $(CP1)$ has an infimum that is not attained? In such a case is it possible to establish a Fritz John optimality condition that can then be extended to KKT optimality conditions under the pseudonormality condition? The answer is yes and we present a result from Bertsekas [12] and Bertsekas, Ozdaglar, and Tseng [14] to establish the enhanced Fritz John optimality conditions similar to those derived in Section 5.3. But in the absence of a point of minimizer of $(CP1)$, the multipliers are now dependent on the infimum, as one will observe in the theorem below.

Theorem 5.10 *Consider the convex programming problem* $(CP1)$ *where the functions* f *and* g_i, $i = 1, 2, \ldots, m$, *are convex on the convex set* $X \subset \mathbb{R}^n$ *and let* $f_{inf} < +\infty$ *be the infimum of* $(CP1)$*. Then there exist* $\lambda_i \geq 0$ *for* $i = 0, 1, \ldots, m$, *not all simultaneously zero, such that*

$$\lambda_0 f_{inf} = \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$$

Proof. If the infimum $f_{inf} = -\infty$, then by the condition

$$\inf_{x \in X} f(x) \leq \inf_{x \in X, g_i(x) \leq 0, \forall i} f(x) = f_{inf},$$

$\inf_{x \in X} f(x) = -\infty$. Thus for $\lambda_0 = 1$ and $\lambda_i = 0,\ i = 1, 2, \ldots, m$, the requisite condition is obtained.

Now suppose that f_{inf} is finite. To establish the Fritz John optimality condition we will invoke supporting hyperplane theorem. For that purpose, define a set in $\mathbb{R}^{m+1}$ as

$$\mathcal{M} = \{(d_0, d) \in \mathbb{R} \times \mathbb{R}^m : \text{there exists } x \in X \text{ such that} \\ f(x) \leq d_0,\ g_i(x) \leq d_i,\ i = 1, 2, \ldots, m\}.$$

We claim that $\mathcal{M}$ is a convex set. For $j = 1, 2$, consider $(d_0^j, d^j) \in \mathcal{M}$, which implies that there exist $x_j \in X$ such that

$$f(x_j) \leq d_0^j \quad \text{and} \quad g_i(x_j) \leq d_i^j,\ i = 1, 2, \ldots, m.$$

As X is a convex set, for every $\mu \in [0, 1]$, $y = \mu x_1 + (1 - \mu) x_2 \in X$. Also by the convexity of f and g_i, $i = 1, 2, \ldots, m$,

$$f(y) \leq \mu f(x_1) + (1 - \mu) f(x_2) \leq \mu d_0^1 + (1 - \mu) d_0^2,$$
$$g_i(y) \leq \mu g_i(x_1) + (1 - \mu) g_i(x_2) \leq \mu d_i^1 + (1 - \mu) d_i^2,\ i = 1, 2, \ldots, m,$$

which implies that $\mu(d_0^1, d^1) + (1 - \mu)(d_0^2, d^2) \in \mathcal{M}$ for every $\mu \in [0, 1]$. Hence $\mathcal{M}$ is a convex subset of $\mathbb{R} \times \mathbb{R}^m$.

Next we prove that $(f_{inf}, 0) \notin int\ \mathcal{M}$. On the contrary, suppose that $(f_{inf}, 0) \in int\ \mathcal{M}$, which by Definition 2.12 implies that there exists $\varepsilon > 0$ such that $(f_{inf} - \varepsilon, 0) \in \mathcal{M}$. Thus, there exists $x \in X$ such that

$$f(x) \leq f_{inf} - \varepsilon \quad \text{and} \quad g_i(x) \leq 0,\ i = 1, 2, \ldots, m.$$

From the above condition it is obvious that x is a feasible point of $(CP1)$, thereby contradicting the fact that f_{inf} is the infimum of the problem $(CP1)$.

Hence $(f_{inf}, 0) \notin int\ \mathcal{M}$. By the supporting hyperplane theorem, Theorem 2.26 (i), there exists $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$ with $(\lambda_0, \lambda) \neq (0, 0)$ such that

$$\lambda_0 f_{inf} \leq \lambda_0 d_0 + \sum_{i=1}^{m} \lambda_i d_i, \ \forall\ (d_0, d) \in \mathcal{M}. \tag{5.21}$$

Let $(d_0, d) = (d_0, d_1, \ldots, d_m) \in \mathcal{M}$. Then for $\alpha_i > 0$,

$$(d_0, \ldots, d_{i-1}, d_i + \alpha_i, d_{i+1}, \ldots, d_m) \in \mathcal{M}, \ i = 0, 1, \ldots, m.$$

If for some $i \in \{0, 1, \ldots, m\}$, $\lambda_i < 0$, then as the corresponding $\alpha_i \to +\infty$, it leads to a contradiction of (5.21). Therefore, $\lambda_i \geq 0$ for $i = 0, 1, \ldots, m$.

It is easy to observe that $(f(x), g(x)) = (f(x), g_1(x), g_2(x), \ldots, g_m(x)) \in \mathcal{M}$ for any $x \in X$. Therefore, the condition (5.21) becomes

$$\lambda_0 f_{inf} \leq \lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \ \forall\ x \in X,$$

which implies

$$\begin{aligned}
\lambda_0 f_{inf} &\leq \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\} \\
&\leq \inf_{x \in X, g_i(x) \leq 0, \forall i} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\} \\
&\leq \inf_{x \in X, g_i(x) \leq 0, \forall i} \lambda_0 f(x) \\
&= \lambda_0 f_{inf},
\end{aligned}$$

thereby leading to the Fritz John optimality condition, as desired. $\square$

Note that in Theorem 5.10, there is no complementary slackness condition.

Under the Slater-type constraint qualification, that is, there exists $\hat{x} \in X$ such that $g_i(\hat{x}) < 0, \ i = 1, 2, \ldots, m$, it can be ensured that $\lambda_0 \neq 0$. Otherwise if $\lambda_0 = 0$, from the Fritz John optimality condition, there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, not all simultaneously zero such that

$$\sum_{i=1}^{m} \lambda_i g_i(x) \geq 0, \ \forall\ x \in X,$$

which contradicts the Slater-type constraint qualification. This discussion can be stated as follows.

Theorem 5.11 *Consider the convex programming problem $(CP1)$ where the functions f and g_i, $i = 1, 2, \ldots, m$, are convex on the convex set $X \subset \mathbb{R}^n$ and let $f_{inf} < +\infty$ be the infimum of $(CP1)$. Assume that the Slater-type* *constraint qualification holds. Then there exist* $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, *such that*

$$f_{inf} = \inf_{x \in X} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}.$$

In Theorem 5.10, the Fritz John optimality condition is established in the duality form in the absence of any point of minimizer of $(CP1)$ but at the cost of the complementary slackness condition. Note that in Theorems 5.10 and 5.11, one requires the set X to be convex, but need not be closed. The enhanced Fritz John optimality condition similar to Theorem 5.6 has also been obtained in this scenario by Bertsekas, Ozdaglar, and Tseng [14] and Bertsekas [12]. The proof is similar to that of Theorem 5.6 but complicated as the point of minimizer does not exist.

Theorem 5.12 *Consider the convex programming problem* $(CP1)$ *where the functions* f *and* $g_i,\ i = 1, 2, \ldots, m$, *are lsc and convex on the closed convex set* $X \subset \mathbb{R}^n$ *and let* $f_{inf} < +\infty$ *be the infimum of* $(CP1)$. *Then there exist* $\lambda_i \geq 0$ *for* $i = 0, 1, \ldots, m$, *not all simultaneously zero, such that*

(i) $\lambda_0 f_{inf} = \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}$.

(ii) Consider the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$. *If* $\bar{I} \neq \emptyset$, *then there exists a sequence* $\{x_k\} \subset X$ *such that*

$$\lim_{k \to \infty} f(x_k) = f_{inf} \quad \textit{and} \quad \limsup_{k \to \infty} g_i(x_k) \leq 0,\ i = 1, 2, \ldots, m$$

and for all k *sufficiently large*

$$f(x_k) < f_{inf} \quad \textit{and} \quad g_i(x_k) > 0,\ \forall\, i \in \bar{I}.$$

Proof. If for every $x \in X$, $f(x) \geq f_{inf}$, then the result holds for $\lambda_0 = 1$ and $\lambda_i = 0,\ i = 1, 2, \ldots, m$.

Now suppose that there exists an $\bar{x} \in X$ such that $f(\bar{x}) < f_{inf}$, thereby implying that f_{inf} is finite. Consider the minimization problem

$$\min\ f(x) \quad \text{subject to} \quad g_i(x) \leq 0,\ i = 1, 2, \ldots, m, \quad x \in X_k. \qquad (CP1_k)$$

In $(CP1_k)$, X_k is a closed convex subset of $\mathbb{R}^n$ defined as

$$X_k = X \cap \bar{\mathbb{B}}_{\beta k}(0),\ \forall\, k \in \mathbb{N}$$

and $\beta > 0$ is chosen to be sufficiently large such that for every k, the constraint set

$$\{x \in X_k : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$$

is nonempty. As f and g_i, $i = 1, 2, \ldots, m$, are lsc convex on X, they are lsc convex and coercive on X_k. Thus by the Weierstrass Theorem, Theorem 1.14, the problem $(CP1_k)$ has a point of minimizer, say $\bar{x}_k$. As $k \to \infty$, $X_k \to X$ and thus $f(\bar{x}_k) \to f_{inf}$. Because $X_k \subset X$, $f_{inf} \leq f(\bar{x}_k)$. Define $\delta_k = f(\bar{x}_k) - f_{inf}$. Observe that $\delta_k \geq 0$ for every k. If $\delta_k = 0$ for some k, then $\bar{x}_k \in X_k \subset X$ is a point of minimizer of $(CP1)$ and the result holds by Theorem 5.6 with $f_{inf} = f(\bar{x}_k)$.

Now suppose that $\delta_k > 0$ for every k. For positive integers k and positive scalars r, consider the function $L_{k,r} : X_k \times \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$L_{k,r}(x, \alpha) = f(x) + \frac{\delta_k^2}{4k^2}\|x - \bar{x}_k\|^2 + \sum_{i=1}^{m} \alpha_i g_i(x) - \frac{\|\alpha\|^2}{2r}.$$

Observe that the above function is similar to the saddle point function considered in Theorem 5.6 except that the term $\dfrac{1}{k^3}\|x - \bar{x}\|^2$ is now replaced by $\dfrac{\delta_k^2}{4k^2}\|x - \bar{x}_k\|^2$. In Theorem 5.6, $\bar{x}$ is a point of minimizer of $(CP1)$ whereas here the infimum is not attained and thus the term involves $\bar{x}_k$, the point of minimizer of the problem $(CP1_k)$ and δ_k.

Now working along the lines of proof of Theorem 5.6, $L_{k,r}$ has a saddle point over $X_k \times \mathbb{R}^m_+$, say $(x_{k,r}, \alpha_{k,r})$, which by the saddle point definition implies

$$L_{k,r}(x_{k,r}, \alpha) \leq L_{k,r}(x_{k,r}, \alpha_{k,r}) \leq L_{k,r}(x, \alpha_{k,r}), \ \forall\, x \in X_k, \ \forall\, \alpha \in \mathbb{R}^m_+. \quad (5.22)$$

As $L_{k,r}(., \alpha_{k,r})$ attains an infinimum over X_k at $x_{k,r}$,

$$L_{k,r}(x_{k,r}, \alpha_{k,r}) \leq f(\bar{x}_k). \quad (5.23)$$

Also, from (5.22), $L(x_{k,r}, \alpha)$ attains a supremum over $\alpha \in \mathbb{R}^m_+$ at

$$\alpha_{k,r_i} = r g_i^+(x_{k,r}), \ i = 1, 2, \ldots, m. \quad (5.24)$$

Therefore,

$$L_{k,r}(x_{k,r}, \alpha_{k,r}) \geq f(x_{k,r}). \quad (5.25)$$

Further, as in the proof of Theorem 5.6,

$$\lim_{r \to \infty} f(x_{k,r}) = f(\bar{x}_k).$$

Note that in the proof, the problem $(CP1_k)$ is considered instead of $(CP1)$ as in Theorem 5.6 and hence the condition obtained involves the point of minimizer of $(CP1_k)$, that is, $\bar{x}_k$. Now as $\delta_k = f(\bar{x}_k) - f_{inf}$, the above equality leads to

$$\lim_{r \to \infty} f(x_{k,r}) = f_{inf} + \delta_k. \quad (5.26)$$

Now before continuing with the proof to obtain the multipliers for the Fritz John optimality condition, we present a lemma from Bertsekas, Ozdaglar, and Tseng [14].

Lemma 5.13 *For sufficiently large k and every $r \leq 1/\sqrt{\delta_k}$,*

$$f(x_{k,r}) \leq f_{inf} - \frac{\delta_k}{2}. \tag{5.27}$$

Furthermore, there exists a scalar $r_k \geq 1/\sqrt{\delta_k}$ such that

$$f(x_{k,r_k}) = f_{inf} - \frac{\delta_k}{2}. \tag{5.28}$$

Proof. Define $\delta = f_{inf} - f(\bar{x})$, where $\bar{x} \in X$ is such that $f(\bar{x}) < f_{inf}$. For sufficiently large k, $\bar{x} \in X_k$. As $\bar{x}_k$ is the point of minimizer of the problem $(CP1_k)$, $f(\bar{x}_k) \geq f_{inf}$ such that $f(\bar{x}_k) \to f_{inf}$, thus for sufficiently large k,

$$f(\bar{x}_k) - f_{inf} < f_{inf} - f(\bar{x}),$$

which implies $\delta_k < \delta$. By the convexity of f over X and that of $X_k \subset X$, for $\lambda_k \in [0,1]$,

$$\begin{aligned} f(y_k) &\leq \lambda_k f(\bar{x}) + (1-\lambda_k) f(\bar{x}_k) \\ &= \lambda_k (f_{inf} - \delta) + (1-\lambda_k)(f_{inf} + \delta_k) \\ &= f_{inf} - \lambda_k(\delta_k + \delta) + \delta_k, \end{aligned}$$

where $y_k = \lambda_k \bar{x} + (1-\lambda_k)\bar{x}_k$. Because $0 \leq \delta_k < \delta$, $0 \leq \dfrac{2\delta_k}{\delta_k + \delta} < 1$. Substituting $\lambda_k = \dfrac{2\delta_k}{\delta_k + \delta}$ in the above condition yields

$$f(y_k) \leq f_{inf} - \delta_k. \tag{5.29}$$

Again by the convexity assumptions of g_i, $i = 1, 2, \ldots, m$, along with the feasibility of $\bar{x}_k$ for $(CP1_k)$, for $\lambda_k = \dfrac{2\delta_k}{\delta_k + \delta}$,

$$\begin{aligned} g_i(y_k) &\leq \lambda_k g_i(\bar{x}) + (1-\lambda_k) g_i(\bar{x}_k) \\ &\leq \frac{2\delta_k}{\delta_k + \delta} g_i(\bar{x}), \quad i = 1, 2, \ldots, m. \end{aligned} \tag{5.30}$$

From the saddle point condition (5.22) along with (5.24) and (5.25),

$$\begin{aligned} f(x_{k,r}) &\leq L_{k,r}(x_{k,r}, \alpha_{k,r}) \\ &= \inf_{x \in X_k} \{ f(x) + \frac{\delta_k^2}{4k^2} \|x - \bar{x}_k\|^2 + \frac{r}{2} \|g^+(x)\|^2 \}. \end{aligned}$$

As $x, \bar{x}_k \in X_k \subset \bar{\mathbb{B}}_{\beta k}(0)$,

$$\|x - \bar{x}_k\| \leq \|x\| + \|\bar{x}\| \leq 2\beta k,$$

thereby reducing the preceding inequality to

$$f(x_{k,r}) \leq f(x) + (\beta\delta_k)^2 + \frac{r}{2}\|g^+(x)\|^2, \ \forall\, x \in X_k.$$

In particular, taking $x = y_k \in X_k$ in the above condition, which along with (5.29) and (5.30) implies that for sufficiently large k,

$$f(x_{k,r}) \leq f_{inf} - \delta_k + (\beta\delta_k)^2 + \frac{2r\delta_k^2}{(\delta_k+\delta)^2}\|g^+(\bar{x})\|^2.$$

For sufficiently large k, $\delta_k \to 0$ and for every $r \leq 1/\sqrt{\delta_k}$, the above inequality reduces to (5.27).

Now by the saddle point condition (5.22), which along with (5.24) implies that

$$\begin{aligned} L_{k,r}(x_{k,r},\alpha_{k,r}) &= f(x_{k,r}) + \frac{\delta_k^2}{4k^2}\|x_{k,r}-\bar{x}_k\|^2 + \frac{r}{2}\|g^+(x_{k,r})\|^2 \\ &= \inf_{x\in X_k}\{f(x) + \frac{\delta_k^2}{4k^2}\|x-\bar{x}_k\|^2 + \frac{r}{2}\|g^+(x)\|^2\}. \end{aligned}$$

Consider $\bar{r} > 0$. Then for every $r \geq \bar{r}$,

$$\begin{aligned} L_{k,\bar{r}}(x_{k,\bar{r}},\alpha_{k,\bar{r}}) &= \inf_{x\in X_k}\{f(x) + \frac{\delta_k^2}{4k^2}\|x-\bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x)\|^2\} \\ &\leq f(x_{k,r}) + \frac{\delta_k^2}{4k^2}\|x_{k,r}-\bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x_{k,r})\|^2 \\ &\leq f(x_{k,r}) + \frac{\delta_k^2}{4k^2}\|x_{k,r}-\bar{x}_k\|^2 + \frac{r}{2}\|g^+(x_{k,r})\|^2 \\ &= L_{k,r}(x_{k,r},\alpha_{k,r}) \\ &\leq f(x_{k,\bar{r}}) + \frac{\delta_k^2}{4k^2}\|x_{k,\bar{r}}-\bar{x}_k\|^2 + \frac{r}{2}\|g^+(x_{k,\bar{r}})\|^2. \end{aligned}$$

Thus as $r \downarrow \bar{r}$, $L_{k,r}(x_{k,r},\alpha_{k,r}) \to L_{k,\bar{r}}(x_{k,\bar{r}},\alpha_{k,\bar{r}})$.

Now for $r \leq \bar{r}$,

$$\begin{aligned} & f(x_{k,\bar{r}}) + \frac{\delta_k^2}{4k^2}\|x_{k,\bar{r}}-\bar{x}_k\|^2 + \frac{r}{2}\|g^+(x_{k,\bar{r}})\|^2 \\ &\quad \leq f(x_{k,\bar{r}}) + \frac{\delta_k^2}{4k^2}\|x_{k,\bar{r}}-\bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x_{k,\bar{r}})\|^2 \\ &\quad = L_{k,\bar{r}}(x_{k,\bar{r}},\alpha_{k,\bar{r}}) \\ &\quad \leq f(x_{k,r}) + \frac{\delta_k^2}{4k^2}\|x_{k,r}-\bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x_{k,r})\|^2 \\ &\quad = f(x_{k,r}) + \frac{\delta_k^2}{4k^2}\|x_{k,r}-\bar{x}_k\|^2 + \frac{r}{2}\|g^+(x_{k,r})\|^2 + \frac{\bar{r}-r}{2}\|g^+(x_{k,r})\|^2 \\ &\quad = L_{k,r}(x_{k,r},\alpha_{k,r}) + \frac{\bar{r}-r}{2}\|g^+(x_{k,r})\|^2 \\ &\quad \leq f(x_{k,\bar{r}}) + \frac{\delta_k^2}{4k^2}\|x_{k,\bar{r}}-\bar{x}_k\|^2 + \frac{r}{2}\|g^+(x_{k,\bar{r}})\|^2 + \frac{\bar{r}-r}{2}\|g^+(x_{k,r})\|^2. \end{aligned}$$

For every k, as $g_i,\ i = 1, 2, \ldots, m$, is lsc and coercive on X_k, $\{g_i(x_{k,r})\}$ is bounded below by $\inf\limits_{x \in X_k} g_i(x)$, which exists by the Weierstrass Theorem, Theorem 1.14. Therefore as $r \uparrow \bar{r}$, $L_{k,r}(x_{k,r}, \alpha_{k,r}) \to L_{k,\bar{r}}(x_{k,\bar{r}}, \alpha_{k,\bar{r}})$, which along with the previous case of $r \downarrow \bar{r}$ leads to the continuity of $L_{k,r}(x_{k,r}, \alpha_{k,r})$ in r, that is, as $r \to \bar{r}$, $L_{k,r}(x_{k,r}, \alpha_{k,r}) \to L_{k,\bar{r}}(x_{k,\bar{r}}, \alpha_{k,\bar{r}})$.

By the conditions (5.23) and (5.25) $x_{k,r}$ belongs to the compact set $\{x \in X_k : f(x) \leq f(\bar{x}_k)\}$ for every k and therefore $\{x_{k,r}\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, as $r \to \bar{r}$, it has convergent subsequence. Without loss of generality, let $x_{k,r} \to \hat{x}_k$, where $\hat{x}_k \in \{x \in X_k : f(x) \leq f(\bar{x}_k)\}$. The continuity of $L_{k,r}(x_{k,r}, \alpha_{k,r})$ in r along with the lower semicontinuity of f and $g_i,\ i = 1, 2, \ldots, m$, leads to

$$
\begin{aligned}
L_{k,\bar{r}}(x_{k,\bar{r}}, \alpha_{k,\bar{r}}) &= \lim_{r \to \bar{r}} L_{k,r}(x_{k,r}, \alpha_{k,r}) \\
&= \lim_{r \to \bar{r}} \{f(x_{k,r}) + \frac{\delta_k^2}{4k^2}\|x_{k,r} - \bar{x}_k\|^2 + \frac{r}{2}\|g^+(x_{k,r})\|^2\} \\
&\geq f(\hat{x}_k) + \frac{\delta_k^2}{4k^2}\|\hat{x}_k - \bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(\hat{x}_k)\|^2 \\
&\geq \inf_{x \in X_k} \{f(x) + \frac{\delta_k^2}{4k^2}\|x - \bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x)\|^2\} \\
&= L_{k,\bar{r}}(x_{k,\bar{r}}, \alpha_{k,\bar{r}}),
\end{aligned}
$$

which implies $\hat{x}_k$ is the point of minimizer of

$$
f(x) + \frac{\delta_k^2}{4k^2}\|x - \bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x)\|^2
$$

over X_k. As a strict convex function has unique point of minimizer and $f(x) + \frac{\delta_k^2}{4k^2}\|x - \bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x)\|^2$ is strictly convex, $\hat{x}_k = x_{k,\bar{r}}$.

We claim that $f(x_{k,r}) \to f(x_{k,\bar{r}})$ as $r \to \bar{r}$. As f is lsc, we will prove the upper semicontinuity of f in r. On the contrary, suppose that $f(x_{k,\bar{r}}) < \limsup_{r \to \bar{r}} f(x_{k,r})$. As $r \to \bar{r}$, $L_{k,r}(x_{k,r}, \alpha_{k,r}) \to L_{k,\bar{r}}(x_{k,\bar{r}}, \alpha_{k,\bar{r}})$ and $x_{k,r} \to \hat{x}_k = x_{k,\bar{r}}$, which implies that

$$
\begin{aligned}
f(x_{k,\bar{r}}) &+ \frac{\delta_k^2}{4k^2}\|x_{k,\bar{r}} - \bar{x}_k\|^2 + \liminf_{r \to \bar{r}} \frac{\bar{r}}{2}\|g^+(x_{k,r})\|^2 \\
&< \limsup_{r \to \bar{r}} L_{k,r}(x_{k,r}, \alpha_{k,r}) \\
&= L_{k,\bar{r}}(x_{k,\bar{r}}, \alpha_{k,\bar{r}}) \\
&= f(x_{k,\bar{r}}) + \frac{\delta_k^2}{4k^2}\|x_{k,\bar{r}} - \bar{x}_k\|^2 + \frac{\bar{r}}{2}\|g^+(x_{k,\bar{r}})\|^2.
\end{aligned}
$$

But the above inequality is a contradiction of the lower semicontinuity of $g_i,\ i = 1, 2, \ldots, m$. Therefore, $f(x_{k,r})$ is continuous in r.

Now by (5.26), for sufficiently large k,

$$
\lim_{r \to +\infty} f(x_{k,r}) = f_{inf} + \delta_k.
$$

Therefore, taking $\varepsilon = \dfrac{3\delta_k}{2}$, for r sufficiently large,

$$|f(x_{k,r}) - (f_{inf} + \delta_k)| < \frac{3\delta_k}{2},$$

which implies that

$$f_{inf} - \frac{\delta_k}{2} < f(x_{k,r}) < f_{inf} + \frac{5\delta_k}{2}. \tag{5.31}$$

For $r \leq \dfrac{1}{\sqrt{\delta_k}}$, by (5.27),

$$f(x_{k,r}) \leq f_{inf} - \frac{\delta_k}{2}.$$

Now for $r = \dfrac{1}{\sqrt{\delta_k}}$, we have two possibilities:

(i) $f(x_{k,r}) = f_{inf} - \dfrac{\delta_k}{2}$,

(ii) $f(x_{k,r}) < f_{inf} - \dfrac{\delta_k}{2}$.

If (i) holds, then we are done with $r = r_k$. If (ii) holds, then it contradicts

$$f_{inf} - \frac{\delta_k}{2} < f(x_{k,r})$$

and thus, r must satisfy $r > \dfrac{1}{\sqrt{\delta_k}}$. As $f(x_{k,r})$ is continuous in r, by the Intermediate Value Theorem, there exists $r_k \geq 1/\sqrt{\delta_k}$ such that

$$f(x_{k,r}) = f_{inf} - \frac{\delta_k}{2},$$

that is, (5.28) holds. $\square$

Now we continue proving the theorem. From the conditions (5.23) and (5.25),

$$\begin{aligned} f(x_{k,r}) \leq L_{k,r}(x_{k,r}, \alpha_{k,r}) &\leq \inf_{x \in X_k} \{f(x) + \frac{\delta_k^2}{4k^2}\|x - \bar{x}_k\|^2 + \sum_{i=1}^m \alpha_{k,r_i} g_i(x)\} \\ &= f(x_{k,r}) + \frac{\delta_k^2}{4k^2}\|x_{k,r} - \bar{x}_k\|^2 + \sum_{i=1}^m \alpha_{k,r_i} g_i(x_{k,r}) \\ &\leq f(\bar{x}_k). \end{aligned}$$

For $r = r_k \geq \frac{1}{\sqrt{\delta_k}}$, the above condition along with (5.28) and the fact that as $k \to +\infty$, $f(\bar{x}_k) \to f_{inf}$ and $\delta_k \to 0$ imply that

$$\lim_{k\to\infty} f(x_{k,r_k}) - f_{inf} + \frac{\delta_k^2}{4k^2}\|x_{k,r_k} - \bar{x}_k\|^2 + \sum_{i=1}^m \alpha_{k,r_k i} g_i(x_{k,r_k}) = 0. \quad (5.32)$$

Define

$$\gamma_k = \sqrt{1 + \sum_{i=1}^m \alpha_{k,r_k i}^2}, \quad \lambda_0^k = \frac{1}{\gamma_k}, \quad \lambda_i^k = \frac{\alpha_{k,r_k i}}{\gamma_k}, \ i = 1, 2, \ldots, m. \quad (5.33)$$

As $\alpha_{k,r_k} \in \mathbb{R}^m_+$, $\delta_k \geq 1$ for every k. Therefore, dividing (5.32) by γ_k along with the relation (5.33) leads to

$$\lim_{k\to\infty} \lambda_0^k f(x_{k,r_k}) - \lambda_0^k f_{inf} + \frac{\delta_k^2 \lambda_0^k}{4k^2}\|x_{k,r_k} - \bar{x}_k\|^2 + \sum_{i=1}^m \lambda_i^k g_i(x_{k,r_k}) = 0. \quad (5.34)$$

By the saddle point condition (5.22),

$$L_{k,r_k}(x_{k,r_k}, \alpha_{k,r_k}) \leq L_{k,r_k}(x, \alpha_{k,r_k}), \ \forall\, x \in X_k.$$

Dividing the above inequality throughout by γ_k along with the fact that $\|x - \bar{x}_k\| \leq 2\beta k$ for every $x \in X_k$ implies that

$$\begin{aligned}\lambda_0^k f(x_{k,r_k}) + \frac{\delta_k^2 \lambda_0^k}{4k^2}\|x_{k,r_k} - \bar{x}_k\|^2 + \sum_{i=1}^m \lambda_i^k g_i(x_{k,r_k}) \\ \leq \lambda_0^k f(x) + \frac{\delta_k^2 \lambda_0^k}{4k^2}\|x - \bar{x}_k\|^2 + \sum_{i=1}^m \lambda_i^k g_i(x).\end{aligned}$$

From (5.33), for every k, $\lambda_i^k \geq 0$, $i = 0, 1, \ldots, m$, such that

$$(\lambda_0^k)^2 + \sum_{i=1}^m (\lambda_i^k)^2 = 1.$$

Therefore $\{\lambda_i^k\}$, $i = 1, 2, \ldots, m$ are bounded sequences and hence, by the Bolzano–Weierstrass Theorem, Proposition 1.3, have convergent subsequences. Without loss of generality, assume that $\lambda_i^k \to \lambda_i$, $i = 1, 2, \ldots, m$, with $\lambda_i \geq 0$, $i = 0, 1, \ldots, m$, not all simultaneously zero. Taking the limit as $k \to +\infty$ in the above inequality along with (5.34) leads to

$$\lambda_0 f_{inf} \leq \lambda_0 f(x) + \sum_{i=1}^m \lambda_i g_i(x), \ \forall\, x \in X.$$

Therefore,

$$\begin{aligned}\lambda_0 f_{inf} &\leq \inf_{x\in X}\{\lambda_0 f(x)+\sum_{i=1}^m \lambda_i g_i(x)\}\\ &\leq \inf_{x\in X, g_i(x)\leq 0, \forall\ i}\{\lambda_0 f(x)+\sum_{i=1}^m \lambda_i g_i(x)\}\\ &\leq \inf_{x\in X, g_i(x)\leq 0, \forall i}\lambda_0 f(x)\\ &= \lambda_0 f_{inf},\end{aligned}$$

which leads to condition (i).

Now dividing the condition (5.24) by γ_k, which along with (5.33) leads to

$$\lambda_i^k = \frac{r_k g_i^+(x_{k,r_k})}{\gamma_k},\ i=1,2,\dots,m.$$

As $k\to+\infty$,

$$\lambda_i = \lim_{k\to\infty}\frac{r_k g_i^+(x_{k,r_k})}{\gamma_k},\ i=1,2,\dots,m.$$

In the beginning of the proof, we assumed that there exists $\bar{x}\in X$ satisfying $f(\bar{x}) < f_{inf}$, which along with the condition (i) implies that the index set $\bar{I}=\{i\in\{1,2,\dots,m\}:\lambda_i>0\}$ is nonempty. Otherwise, if $\bar{I}$ is empty, then

$$\inf_{x\in X}\lambda_0 f(x)\leq \lambda_0 f(\bar{x}) < \lambda_0 f_{inf},$$

which is a contradiction to condition (i). For $i\in\bar{I}$, $\lambda_i>0$, which implies there exists a sequence $\lambda_i^k\to\lambda_i$ such that $\lambda_i^k>0$. Therefore, $g_i^+(x_{k,r_k})>0$, that is,

$$g_i(x_{k,r_k})>0,\ \forall\ i\in\bar{I}.$$

In particular, for $r=r_k\geq \dfrac{1}{\sqrt{\delta_k}}$, conditions (5.23) and (5.24) yield

$$\begin{aligned} f(x_{k,r_k})+\frac{r_k}{2}\|g^+(x_{k,r_k})\|^2 &\leq f(x_{k,r_k})+\frac{\delta_k^2}{4k^2}\|x_{k,r_k}-\bar{x}_k\|^2+\frac{r_k}{2}\|g^+(x_{k,r_k})\|^2\\ &\leq f(\bar{x}_k),\end{aligned}$$

which along with (5.28) and the relation $\delta_k = f(\bar{x}_k)-f_{inf}\geq 0$ implies that

$$r_k\|g^+(x_{k,r_k})\|^2\leq 3\delta_k.$$

As $k\to+\infty$, $\delta_k\to 0$ and $r_k\geq 1/\sqrt{\delta_k}\to\infty$, the above inequality leads to $g^+(x_{k,r_k})\to 0$, that is,

$$\limsup_{k\to\infty} g_i(x_{k,r_k})\leq 0,\ i=1,2,\dots,m.$$

Also, from the condition (5.28),

$$f(x_{k,r_k}) < f_{inf} \quad \text{and} \quad \lim_{k\to\infty} f(x_{k,r_k}) = f_{inf}.$$

Thus, the condition (ii) is satisfied by the sequence $\{x_{k,r_k}\} \subset X$, thereby yielding the desired result. $\square$

Under the Slater-type constraint qualification, the multiplier λ_0 can be ensured to be nonzero and hence can be normalized to one.

5.5 Enhanced Dual Fritz John Optimality Conditions

In this chapter we emphasize the enhanced Fritz John conditions. As observed in Section 5.4, we dealt with the situation where the infimum of the original problem $(CP1)$ exists but is not attained. Those results were extended to the dual scenario where the dual problem has a supremum but not attained by Bertsekas, Ozdaglar and Tseng [14]. Now corresponding to the problem $(CP1)$, the associated dual problem is

$$\sup\ w(\lambda) \quad \text{subject to} \quad \lambda \in \mathbb{R}^m_+ \tag{DP1}$$

where $w(\lambda) = \inf\limits_{x\in X} L(x,\lambda)$ with

$$L(x,\lambda) = \begin{cases} f(x) + \displaystyle\sum_{i=1}^{m} \lambda_i g_i(x), & \lambda \in \mathbb{R}^m_+, \\ -\infty, & \text{otherwise}. \end{cases}$$

Before presenting the enhanced dual Fritz John optimality condition, we first prove a lemma that will be required in establishing the theorem.

Lemma 5.14 *Consider the convex programming problem $(CP1)$ where the functions f and g_i, $i = 1, 2, \ldots, m$, are lsc and convex on the convex set $X \subset \mathbb{R}^n$ and $(DP1)$ is the associated dual problem. Let $f_{inf} < +\infty$ be the infimum of $(CP1)$ and for every $\delta > 0$, assume that*

$$f_\delta = \inf_{x\in X, g_i(x)\le\delta, \forall i} f(x).$$

Then the supremum of $(DP1)$, w_{sup}, satisfies $f_\delta \le w_{sup}$ for every $\delta > 0$ and

$$w_{sup} = \lim_{\delta\downarrow 0} f_\delta.$$

Proof. For the problem $(CP1)$, as the infimum f_{inf} exists and $f_{inf} < +\infty$, the feasible set of $(CP1)$ is nonempty, that is, there exists $\bar{x} \in X$ satisfying $g_i(\bar{x}) \le 0$, $i = 1, 2, \ldots, m$. Thus for $\delta > 0$, the problem

$$\inf \ f(x) \quad \text{subject to} \quad g_i(x) \leq \delta, \ i = 1, 2, \dots, m, \quad x \in X, \qquad (CP1_\delta)$$

satisfies the Slater-type constraint qualification as $\bar{x} \in X$ with $g_i(\bar{x}) < \delta$, $i = 1, 2, \dots, m$. Therefore, by Theorem 5.11, there exist $\lambda_i^\delta \geq 0$, $i = 1, 2, \dots, m$, such that

$$\begin{aligned} f_\delta &= \inf_{x \in X} \{f(x) + \sum_{i=1}^m \lambda_i^\delta g_i(x) - \delta \sum_{i=1}^m \lambda_i^\delta\} \\ &\leq \inf_{x \in X} \{f(x) + \sum_{i=1}^m \lambda_i^\delta g_i(x)\} \\ &= w(\lambda^\delta) \\ &\leq \sup_{\lambda \in \mathbb{R}^m_+} w(\lambda) \ = \ w_{sup}. \end{aligned}$$

Therefore, for every $\delta > 0$, $f_\delta \leq w_{sup}$ and hence

$$\lim_{\delta \downarrow 0} f_\delta \leq w_{sup}. \qquad (5.35)$$

Now as $\delta \to 0$, the feasible region of $(CP1_\delta)$ decreases and thus f_δ is nondecreasing as $\delta \downarrow 0$ and for every $\delta > 0$, $f_\delta \leq f_{inf}$. This leads to two cases, either $\lim_{\delta \to 0} f_\delta > -\infty$ or $\lim_{\delta \to 0} f_\delta = -\infty$.

If $\lim_{\delta \to 0} f_\delta > -\infty$, then $f_\delta > -\infty$ for every $\delta > 0$ sufficiently small. For those $\delta > 0$, choosing $x_\delta \in X$ such that $g_i(x_\delta) \leq \delta$, $i = 1, 2, \dots, m$, and $f(x_\delta) \leq f_\delta + \delta$. Such x_δ are called *almost δ-solution* of $(CP1)$, the concept that will be dealt with in Chapter 10. Therefore for $\lambda \in \mathbb{R}^m_+$,

$$\begin{aligned} w(\lambda) &= \inf_{x \in X} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x)\} \\ &\leq f(x_\delta) + \sum_{i=1}^m \lambda_i g_i(x_\delta) \\ &\leq f_\delta + \delta + \delta \sum_{i=1}^m \lambda_i. \end{aligned}$$

Taking the limit as $\delta \to 0$ in the above inequality leads to

$$w(\lambda) \leq \lim_{\delta \to 0} f_\delta, \ \forall \ \lambda \in \mathbb{R}^m_+,$$

which implies $w_{sup} \leq \lim_{\delta \to 0} f_\delta$.

If $\lim_{\delta \to 0} f_\delta = -\infty$, then for $\delta > 0$, choose $x_\delta \in X$ such that $g_i(x_\delta) \leq \delta$, $i = 1, 2, \dots, m$, and $f(x_\delta) \leq -1/\delta$. As in the previous case, for $\lambda \in \mathbb{R}^m_+$,

$$w(\lambda) \leq \frac{-1}{\delta} + \delta \sum_{i=1}^m \lambda_i,$$

which leads to $w(\lambda) = -\infty$ for every $\lambda \in \mathbb{R}^m_+$ as $\delta \downarrow 0$ and hence, $w_{sup} = -\infty = \lim_{\delta \to 0} f_\delta$. From both these cases along with the condition (5.35), the requisite result is established. $\square$

Finally, we present the enhanced dual Fritz John optimality conditions obtained by Bertsekas, Ozdaglar, and Tseng [14], which are expressed with respect to the supremum of the dual problem $(DP1)$.

Theorem 5.15 *Consider the convex programming problem $(CP1)$ where the functions f and g_i, $i = 1, 2, \ldots, m$, are lsc and convex on the closed convex set $X \subset \mathbb{R}^n$ and $(DP1)$ is the associated dual problem. Let $f_{inf} < +\infty$ be the infimum of $(CP1)$ and $w_{sup} > -\infty$ be the supremum of $(DP1)$. Then there exist $\lambda_i \geq 0$ for $i = 0, 1, \ldots, m$, not all simultaneously zero, such that*

(i) $\lambda_0 w_{sup} = \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}$.

(ii) Consider the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$. If $\bar{I} \neq \emptyset$, then there exists a sequence $\{x_k\} \subset X$ such that

$$\lim_{k \to \infty} f(x_k) = w_{sup} \quad \text{and} \quad \limsup_{k \to \infty} g_i(x_k) \leq 0,\ i = 1, 2, \ldots, m,$$

and for all k sufficiently large

$$f(x_k) < w_{sup} \quad \text{and} \quad g_i(x_k) > 0,\ \forall\, i \in \bar{I}.$$

Proof. By the weak duality, $w_{sup} \leq f_{inf}$, which along with the hypothesis implies that f_{inf} and w_{sup} are finite. For $k = 1, 2, \ldots$, consider the problem:

$$\min\ f(x) \quad \text{subject to} \quad g_i(x) \leq \frac{1}{k^4},\ i = 1, 2, \ldots, m, \quad x \in X. \qquad (CP1_k)$$

By Lemma 5.14, the infimum f^k_{inf} of $(CP1_k)$ satisfies the condition $f^k_{inf} \leq w_{sup}$ for every k. For each k, consider $\hat{x}_k \in X$ such that

$$f(\hat{x}_k) \leq w_{sup} + \frac{1}{k^2} \quad \text{and} \quad g_i(\hat{x}_k) \leq \frac{1}{k^4},\ i = 1, 2, \ldots, m. \tag{5.36}$$

Now consider another problem:

$$\min\ f(x) \quad \text{subject to} \quad g_i(x) \leq \frac{1}{k^4},\ i = 1, 2, \ldots, m, \quad x \in \hat{X}_k, \quad (C\hat{P}1_k)$$

where $\hat{X}_k = X \cap \{x \in \mathbb{R}^n : \|x\| \leq k(\max_{j=1,\ldots,k} \|\hat{x}_j\| + 1)\}$ is a compact set. By the lower semicontinuity and convexity of f and g_i, $i = 1, 2, \ldots, m$, over X, the functions are lsc convex and coercive on $\hat{X}_k$. Therefore, by the Weierstrass Theorem, Theorem 1.14, $(C\hat{P}1_k)$ has a point of minimizer, say $\bar{x}_k$. From (5.36), $\hat{x}_k$ is feasible for $(C\hat{P}1_k)$ which leads to

$$f(\bar{x}_k) \leq f(\hat{x}_k) \leq w_{sup} + \frac{1}{k^2}. \tag{5.37}$$

For every k, define the Lagrangian function as

$$L_k(x,\alpha) = f(x) + \sum_{i=1}^{m} \alpha_i g_i(x) - \frac{\|\alpha\|^2}{2k}$$

and the set

$$X_k = \hat{X}_k \cap \{x \in \mathbb{R}^n : g_i(x) \leq k,\ i = 1,2,\ldots,m\}. \tag{5.38}$$

For a fixed $\alpha \in \mathbb{R}^m_+$, $L_k(.,\alpha)$ is lsc convex and coercive on X_k by the lower semicontinuity convexity and coercivity of f and g_i, $i = 1,2,\ldots,m$, on $\hat{X}_k$ whereas for a given $x \in X_k$, $L_k(x,.)$ is quadratic negative definite in α. Then by the Saddle Point Theorem, Proposition 4.1, L_k has a saddle point over $X_k \times \mathbb{R}^m_+$, say (x_k,α_k), that is,

$$L_k(x_k,\alpha) \leq L_k(x_k,\alpha_k) \leq L(x,\alpha_k),\ \forall\ x \in X_k,\ \forall\ \alpha \in \mathbb{R}^m_+.$$

Because $L_k(x_k,.)$ is quadratic negative definite, it attains a supremum over $\mathbb{R}^m_+$ at

$$\alpha_i^k = k g_i^+(x_k),\ i = 1,2,\ldots,m. \tag{5.39}$$

Also, as $L_k(.,\alpha_k)$ attains the infimum over X_k at x_k, which along with (5.37), (5.38) and (5.39) implies

$$\begin{aligned} L_k(x_k,\alpha_k) &= f(x_k) + \sum_{i=1}^{m} \alpha_i^k g_i(x_k) - \frac{\|\alpha\|^2}{2k} \\ &\leq f(x_k) + \sum_{i=1}^{m} \alpha_i^k g_i(x_k) \\ &= \inf_{x \in X_k} \{f(x) + k\sum_{i=1}^{m} g_i^+(x_k) g_i(x)\} \\ &\leq \inf_{x \in X_k, g_i(x) \leq \frac{1}{k^4}, \forall i} \{f(x) + k\sum_{i=1}^{m} g_i^+(x_k) g_i(x)\}. \end{aligned}$$

As $x_k \in X_k$, $g_i(x_k) \leq k$ for $i = 1,2,\ldots,m$. Therefore, the above inequality leads to

$$\begin{aligned} L_k(x_k,\alpha_k) &\leq \inf_{x \in X_k, g_i(x) \leq \frac{1}{k^4}, \forall i} \{f(x) + \frac{m}{k^2}\} \\ &= f(\bar{x}_k) + \frac{m}{k^2} \\ &\leq w_{sup} + \frac{m+1}{k^2}. \end{aligned} \tag{5.40}$$

Due to the finiteness of w_{sup}, there exists a sequence $\{\mu_k\} \subset \mathbb{R}^m_+$ satisfying

$$w(\mu_k) \to w_{sup} \quad \text{and} \quad \frac{\|\mu_k\|^2}{2k} \to 0, \tag{5.41}$$

which is ensured by choosing μ_k as the point of maximizer of the problem

$$\max\ w(\alpha) \quad \text{subject to} \quad \|\alpha\| \leq k^{1/3},\ \alpha \in \mathbb{R}^m_+.$$

Thus for every k,

$$\begin{aligned}
L_k(x_k,\alpha_k) &= \sup_{\alpha\in\mathbb{R}^m_+}\inf_{x\in X_k} L_k(x,\alpha) \\
&\geq \sup_{\alpha\in\mathbb{R}^m_+}\inf_{x\in X} L_k(x,\alpha) \\
&= \sup_{\alpha\in\mathbb{R}^m_+}\{\inf_{x\in X}\{f(x)+\sum_{i=1}^m \alpha_i g_i(x)\} - \frac{\|\alpha\|^2}{2k}\} \\
&= \sup_{\alpha\in\mathbb{R}^m_+}\{w(\alpha) - \frac{\|\alpha\|^2}{2k}\} \\
&\geq w(\mu_k) - \frac{\|\mu_k\|^2}{2k}.
\end{aligned} \tag{5.42}$$

From the conditions (5.40) and (5.42),

$$\begin{aligned}
w(\mu_k) - \frac{\|\mu_k\|^2}{2k} &\leq f(x_k) + \sum_{i=1}^m \alpha_i^k g_i(x_k) - \frac{\|\alpha_k\|^2}{2k} \\
&\leq f(x_k) + \sum_{i=1}^m \alpha_i^k g_i(x_k) \\
&\leq w_{sup} + \frac{m+1}{k^2}.
\end{aligned} \tag{5.43}$$

Taking the limit as $k \to +\infty$ in the above inequality, which along with (5.41) implies that

$$\lim_{k\to\infty}\{f(x_k) - w_{sup} + \sum_{i=1}^m \alpha_i^k g_i(x_k)\} = 0. \tag{5.44}$$

Define

$$\gamma_k = \sqrt{1+\sum_{i=1}^m (\alpha_i^k)^2}, \quad \lambda_0^k = \frac{1}{\gamma_k} \quad \text{and} \quad \lambda_i^k = \frac{\alpha_i^k}{\gamma_k},\ i = 1,2,\ldots,m. \tag{5.45}$$

As $\alpha_k \in \mathbb{R}^m_+$, from the above condition it is obvious that $\gamma_k \geq 1$ for every k and thus dividing (5.44) by γ_k yields

$$\lim_{k\to\infty}\{\lambda_0^k(f(x_k) - w_{sup}) + \sum_{i=1}^m \lambda_i^k g_i(x_k)\} = 0. \tag{5.46}$$

As x_k minimizes $L_k(.,\alpha_k)$ over X_k,

$$f(x_k) + \sum_{i=1}^m \alpha_i^k g_i(x_k) \leq f(x) + \sum_{i=1}^m \alpha_i^k g_i(x),\ \forall\ x \in X_k,$$

which on dividing throughout by γ_k leads to

$$\lambda_0^k f(x_k) + \sum_{i=1}^m \lambda_i^k g_i(x_k) \leq \lambda_0^k f(x) + \sum_{i=1}^m \lambda_i^k g_i(x), \ \forall\, x \in X_k.$$

From the condition (5.45),

$$(\lambda_0^k)^2 + \sum_{i=1}^m (\lambda_i^k)^2 = 1,$$

which implies that the sequences $\{\lambda_i^k\} \subset \mathbb{R}_+,\ i = 0, 1, \ldots, m$, are bounded and thus by Bolzano–Weierstrass Theorem, Proposition 1.3, have a convergent subsequence. Without loss of generality, let $\lambda_i^k \to \lambda_i$ with $\lambda_i \geq 0,\ i = 0, 1, \ldots, m$, not all simultaneously zero. Therefore, as $k \to +\infty$ in the preceding inequality, which along with (5.46) yields

$$\lambda_0 w_{sup} \leq \lambda_0 f(x) + \sum_{i=1}^m \lambda_i g_i(x), \ \forall\, x \in X,$$

which leads to

$$\lambda_0 w_{sup} \leq \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^m \lambda_i g_i(x)\}. \tag{5.47}$$

If $\lambda_0 > 0$, then from the above inequality (5.47),

$$w_{sup} \leq \inf_{x \in X} \{f(x) + \sum_{i=1}^m \frac{\lambda_i}{\lambda_0} g_i(x)\} = w(\lambda/\lambda_0) \leq w_{sup},$$

thereby satisfying condition (i).

If $\lambda_0 = 0$, then the relation (5.45) reduces to

$$0 \leq \inf_{x \in X} \sum_{i=1}^m \lambda_i g_i(x).$$

As f_{inf} exists and is finite, the feasible set of $(CP1)$ is nonempty, which implies that there exists $x \in X$ satisfying $g_i(x) \leq 0,\ i = 1, 2, \ldots, m$. Therefore, the above condition becomes

$$0 = \inf_{x \in X} \sum_{i=1}^m \lambda_i g_i(x).$$

Therefore for both cases, condition (i) holds, that is,

$$\lambda_0 w_{sup} = \inf_{x \in X} \{\lambda_0 f(x) + \sum_{i=1}^m \lambda_i g_i(x)\}.$$

Now suppose that the index set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$ is nonempty. Dividing the condition (5.39) throughout by γ_k and using (5.45),

$$\lambda_i^k = \frac{kg_i^+(x_k)}{\delta_k}, \ i = 1, 2, \ldots, m.$$

As $k \to +\infty$, $\lambda_i^k \to \lambda_i, \ i = 1, 2, \ldots, m$, thereby reducing the above equality to

$$\lambda_i = \lim_{k\to\infty} \frac{kg_i^+(x_k)}{\delta_k}, \ i = 1, 2, \ldots, m.$$

For any $i \in \bar{I}$, $\lambda_i > 0$, which implies for sufficiently large k, $g_i^+(x_k) > 0$, that is,

$$g_i(x_k) > 0, \ \forall\, i \in \bar{I}.$$

From the inequalities (5.43), for every k,

$$k(f(x_k) - w_{sup}) + k\sum_{i=1}^{m} \alpha_i^k g_i(x_k) \leq \frac{m+1}{k}.$$

By the condition (5.39), $\sum_{i=1}^{m} \alpha_i^k g_i(x_k) = \dfrac{1}{k}\|\alpha_k\|^2$. Therefore, the above inequality becomes

$$k(f(x_k) - w_{sup}) + \sum_{i=1}^{m} (\alpha_i^k)^2 \leq \frac{m+1}{k}.$$

Dividing the above inequality throughout by γ_k^2, which along with (5.45) implies that

$$\frac{k(f(x_k) - w_{sup})}{\gamma_k^2} + \sum_{i=1}^{m} (\lambda_i^k)^2 \leq \frac{m+1}{k\gamma_k^2},$$

which as $k \to +\infty$ yields

$$\limsup_{k\to\infty} \frac{k(f(x_k) - w_{sup})}{\gamma_k^2} \leq -\sum_{i=1}^{m} \lambda_i^2. \tag{5.48}$$

As $\bar{I}$ is nonempty, the above inequality leads to

$$\limsup_{k\to\infty} \frac{k(f(x_k) - w_{sup})}{\gamma_k^2} < 0,$$

which for sufficiently large k implies that $f(x_k) < w_{sup}$.

Now from (5.41) and (5.43),

$$\lim_{k\to\infty}\{f(x_k) - w_{sup} + \sum_{i=1}^{m} \alpha_i^k g_i(x_k)\} - \lim_{k\to\infty} \frac{\|\alpha_k\|^2}{2k} = 0,$$

which by the condition (5.44) implies that

$$\lim_{k\to\infty} \frac{\|\alpha_k\|^2}{2k} = 0. \tag{5.49}$$

The condition (5.39) along with (5.41) and (5.43) leads to

$$\lim_{k\to\infty}(f(x_k) - w_{sup}) + \frac{\|\alpha_k\|^2}{2k} = 0,$$

which together with (5.48) implies that $f(x_k) \to w_{sup}$. Also, (5.49) along with (5.39) and (5.48) yields

$$\lim_{k\to\infty} k \sum_{i=1}^{m} (g_i^+(x_k))^2 = 0,$$

which shows that

$$\limsup_{k\to\infty} g_i(x_k) \leq 0.$$

Thus for nonempty $\bar{I}$, the sequence $\{x_k\} \subset X$ satisfies condition (ii), thereby establishing the desired result. $\square$

Chapter 6

Optimality without Constraint Qualification

6.1 Introduction

In the last few chapters we saw how fundamental the role of constraint qualification is like the Slater constraint qualification in convex optimization. In Chapter 3 we saw that a relaxation of the Slater constraint qualification to the Abadie constraint qualification leads to an asymptotic version of the KKT conditions for the nonsmooth convex programming problems. Thus it is interesting to ask whether it is possible to develop necessary and sufficient optimality conditions for (CP) without any constraint qualifications. Recently a lot of work has been done in this respect in the form of *sequential optimality conditions*. But to the best of our knowledge the first step in this direction was taken by Ben-Tal, Ben-Israel, and Zlobec [7]. They obtained the necessary and sufficient optimality conditions in the smooth scenario in the absence of constraint qualifications. This work was extended to the nonsmooth scenario by Wolkowicz [112]. All these studies involved direction sets, which we will discuss below. So before moving on with the discussion of the results derived by Ben-Tal, Ben-Israel, and Zlobec [7], and Wolkowicz [112], we present the notion of direction sets. Before that we introduce the definition of a blunt cone.

A set $K \subset \mathbb{R}^n$ is said to be a *cone* (Definition 2.18) if

$$\lambda x \in K \quad \text{whenever} \quad \lambda \geq 0 \text{ and } x \in K,$$

whereas K is a *blunt cone* if K is a cone without origin, that is,

$$0 \notin K \quad \text{and} \quad \lambda x \in K \quad \text{if} \quad x \in K \text{ and } \lambda > 0.$$

For example, $\mathbb{R}^2_+ \setminus \{(0,0)\}$ is a blunt cone while the set $K \subset \mathbb{R}^2$ given as $K = \{(x,y) \in \mathbb{R}^2 : x = y\}$ is not a blunt cone.

Definition 6.1 Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a given function and let $\bar{x} \in \mathbb{R}^n$ be any given point. Then the set

$$D_\phi^{relation}(\bar{x}) = \{d \in \mathbb{R}^n : \text{there exists } \bar{\alpha} > 0 \text{ such that}$$
$$\phi(\bar{x} + \alpha d) \ relation\ \phi(\bar{x}),\ \forall\ \alpha \in (0, \bar{\alpha}]\},$$

where the *relation* can be $=$, $\leq$, $<$, $\geq$, or $>$.

In particular, the set $D^{=}_{\phi}$ is called the *cone of directions of constancy* that was considered by Ben-Tal, Ben-Israel, and Zlobec [7]. The other direction sets were introduced in the work of Wolkowicz [112]. We present certain examples of computing explicitly the set $D^{=}_{\phi}(\bar{x})$ from Ben-Tal, Ben-Israel, and Zlobec [7]. For a strictly convex function $\phi : \mathbb{R}^n \to \mathbb{R}$, $D^{=}_{\phi}(\bar{x}) = \{0\}$ for any $\bar{x} \in \mathbb{R}^n$. Another interesting example from Ben-Tal, Ben-Israel, and Zlobec [7] is the cone of the directions of constancy for the so-called *faithfully convex function* given as

$$\phi(x) = h(Ax + b) + \langle a, x \rangle + \beta,$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is strictly convex, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, $a \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$. The class of faithfully convex functions is quite broad, comprising all the strictly convex functions and quadratic convex functions. See Rockafellar [98] for more details. In the case of faithfully convex functions,

$$D^{=}_{\phi}(\bar{x}) = Null\left(\begin{bmatrix} A \\ a^T \end{bmatrix}\right) = \{d \in \mathbb{R}^n : Ad = 0,\ \langle a, d \rangle = 0\},$$

where $Null(S)$ is the null space of the matrix S. It is obvious that the null space is contained in $D^{=}_{\phi}$. For the sake of completeness, we provide an explanation for the reverse containment. We consider the following cases.

1. $Ad = 0$: Then by the definition of direction of constancy, $\langle a, d \rangle = 0$.

2. $\langle a, d \rangle = 0$: Suppose that $d \in D^{=}_{\phi}(\bar{x})$, which implies there exists $\bar{\alpha} > 0$ such that

   $$h(A\bar{x} + \alpha Ad + b) = h(A\bar{x} + b),\ \forall\ \alpha \in (0, \bar{\alpha}].$$

   Suppose $Ad \neq 0$, then $A\bar{x} + \alpha Ad + b \neq A\bar{x} + b$ for every $\alpha \in (0, \bar{\alpha}]$. Now two cases arise. If $h(A\bar{x} + \hat{\alpha}Ad + b) = h(A\bar{x} + b)$ for some $\hat{\alpha} \in (0, \bar{\alpha}]$, then by the strict convexity of h, for every $\lambda \in (0, 1)$,

   $$h(A\bar{x} + \lambda\hat{\alpha}Ad + b) < (1 - \lambda)h(A\bar{x} + b) + \lambda h(A\bar{x} + \hat{\alpha}Ad + b),$$

   which implies

   $$h(A\bar{x} + \alpha Ad + b) < h(A\bar{x} + b),\ \forall\ \alpha \in (0, \hat{\alpha})$$

   and hence, $d \notin D^{=}_{\phi}(\bar{x})$.

   The second case is that $h(A\bar{x}+\alpha Ad+b) \neq h(A\bar{x}+b)$ for every $\alpha \in (0, \bar{\alpha}]$. Then again it implies that $d \notin D^{=}_{\phi}(\bar{x})$, which violates our assumption. Therefore, for d to be a direction of constancy, $Ad = 0$.

3. $Ad \neq 0,\ \langle a, d\rangle \neq 0$: This implies $d \neq 0$. We will show that ϕ is strictly convex on the line segment $[\bar{x}, \bar{x} + \bar{\alpha}d]$. Consider $x_i = \bar{x} + \alpha_i d$, $i = 1, 2$, where $\alpha_i \in [0, \bar{\alpha}]$ and $\alpha_1 \neq \alpha_2$. Therefore $x_1 \neq x_2$. By the strict convexity of h, for every $\lambda \in (0, 1)$,

$$h(A(\lambda x_1 + (1-\lambda)x_2) + b) < \lambda h(Ax_1 + b) + (1-\lambda)h(Ax_2 + b),$$

   and by linearity of $\langle a, .\rangle$,

$$\langle a, \lambda x_1 + (1-\lambda)x_2\rangle = \lambda\langle a, x_1\rangle + (1-\lambda)\langle a, x_2\rangle.$$

   Combining the above two conditions,

$$\phi(\lambda x_1 + (1-\lambda)x_2) < \lambda\phi(x_1) + (1-\lambda)\phi(x_2),\ \forall\ \lambda \in (0, 1).$$

   This condition holds for every $x_1, x_2 \in [\bar{x}, \bar{x} + \alpha d]$ and thus ϕ is strictly convex on $[\bar{x}, \bar{x} + \alpha d]$. Hence as mentioned earlier, $D^{=}_{\phi}(\bar{x}) = \{0\}$ for the strictly convex function ϕ. But this contradicts the fact that $d \neq 0$.

Combining the above cases, we have

$$D^{=}_{\phi}(\bar{x}) = \{d \in \mathbb{R}^n : Ad = 0,\ \langle a, d\rangle = 0\}.$$

Below we present some results on the direction sets that will be required in deriving the optimality conditions from Ben-Tal, Ben-Israel, and Zlobec [7], Ben-Tal and Ben-Israel [6], and Wolkowicz [112].

Proposition 6.2 *(i) Consider a function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Then*

$$D^{=}_{\phi}(\bar{x}) \subset \{d \in \mathbb{R}^n : \phi'(\bar{x}, d) = 0\}.$$

(ii) Consider a differentiable convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Then $D^{=}_{\phi}(\bar{x})$ is a convex cone.

(iii) Consider a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Then $D^{\leq}_{\phi}(\bar{x})$ is a convex cone while $D^{<}_{\phi}(\bar{x})$ is a convex blunt open cone. Also

$$D^{\leq}_{\phi}(\bar{x}) = \{d \in \mathbb{R}^n : \phi'(\bar{x}, d) \leq 0\} \quad \textit{and} \quad D^{<}_{\phi}(\bar{x}) = \{d \in \mathbb{R}^n : \phi'(\bar{x}, d) < 0\}.$$

(iv) Consider a convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Assume that $D^{<}_{\phi}(\bar{x}) \neq \emptyset$ (equivalently $0 \notin \partial\phi(\bar{x})$). Then

$$(D^{\leq}_{\phi}(\bar{x}))^{\circ} = \textit{cone}\ \partial\phi(\bar{x}).$$

Proof. (i) Consider $d \in D^{=}_{\phi}(\bar{x})$, which implies there exists $\bar{\alpha} > 0$ such that

$$\phi(\bar{x} + \alpha d) = \phi(\bar{x}),\ \forall\ \alpha \in (0, \bar{\alpha}].$$

Therefore, from the above condition,

$$\lim_{\alpha \downarrow 0} \frac{\phi(\bar{x} + \alpha d) - \phi(\bar{x})}{\alpha} = 0,$$

which implies $\phi'(\bar{x}, d) = 0$, thereby yielding the desired result.

(ii) Consider $d \in D^{=}_{\phi}(\bar{x})$, which implies there exists $\bar{\alpha} > 0$ such that

$$\phi(\bar{x} + \alpha d) = \phi(\bar{x}), \ \forall \ \alpha \in (0, \bar{\alpha}].$$

The above condition can be rewritten as

$$\phi(\bar{x} + \alpha' d') = \phi(\bar{x}), \ \forall \ \alpha' \in (0, \bar{\alpha}'],$$

where $d' = \lambda d$, $\alpha' = \dfrac{\alpha}{\lambda}$ and $\bar{\alpha}' = \dfrac{\bar{\alpha}}{\lambda}$ for any $\lambda > 0$. Also $0 \in D^{=}_{\phi}(\bar{x})$. Therefore, $\lambda d \in D^{=}_{\phi}(\bar{x})$ for every $\lambda \geq 0$ and hence $D^{=}_{\phi}(\bar{x})$ is a cone.

Now consider $d_1, d_2 \in D^{=}_{\phi}(\bar{x})$. Then for $i = 1, 2$, there exists $\bar{\alpha}_i > 0$ such that

$$\phi(\bar{x} + \alpha_i d_i) = \phi(\bar{x}), \ \forall \ \alpha_i \in (0, \bar{\alpha}_i].$$

Taking $\bar{\alpha} = \min\{\bar{\alpha}_1, \bar{\alpha}_2\} > 0$, for $i = 1, 2$ the above condition becomes

$$\phi(\bar{x} + \alpha d_i) = \phi(\bar{x}), \ \forall \ \alpha \in (0, \bar{\alpha}]. \tag{6.1}$$

For any $\lambda \in [0, 1]$, consider $d = \lambda d_1 + (1 - \lambda) d_2$. The convexity of ϕ along with (6.1) on d_1 and d_2 yields

$$\begin{aligned} \phi(\bar{x} + \alpha d) &= \phi(\lambda(\bar{x} + \alpha d_1) + (1 - \lambda)(\bar{x} + \alpha d_2)) \\ &\leq \lambda \phi(\bar{x} + \alpha d_1) + (1 - \lambda)\phi(\bar{x} + \alpha d_2), \ \forall \ \alpha \in (0, \bar{\alpha}], \end{aligned}$$

that is,

$$\phi(\bar{x} + \alpha d) \leq \phi(\bar{x}), \ \forall \ \alpha \in (0, \bar{\alpha}]. \tag{6.2}$$

Again, by the convexity of ϕ for the differentiable case, for every $\alpha \in (0, \bar{\alpha}]$,

$$\begin{aligned} \phi(\bar{x} + \alpha d) &\geq \phi(\bar{x}) + \alpha \langle \nabla \phi(\bar{x}), d \rangle \\ &= \phi(\bar{x}) + \alpha \lambda \langle \nabla \phi(\bar{x}), d_1 \rangle + \alpha(1 - \lambda)\langle \nabla \phi(\bar{x}), d_2 \rangle. \end{aligned} \tag{6.3}$$

For a differentiable convex function, $\phi'(\bar{x}, d) = \langle \nabla \phi(\bar{x}), d \rangle$ for any $d \in \mathbb{R}^n$. Thus the relation in (i) becomes

$$D^{=}_{\phi}(\bar{x}) \subset \{d \in \mathbb{R}^n : \langle \nabla \phi(\bar{x}), d \rangle = 0\},$$

which reduces the inequality (6.3) to

$$\phi(\bar{x} + \alpha d) \geq \phi(\bar{x}), \ \forall \ \alpha \in (0, \bar{\alpha}]$$

as $d_1, d_2 \in D^=_\phi(\bar{x})$. This inequality along with the condition (6.2) implies that $d \in D^=_\phi(\bar{x})$, thereby leading to the convexity of $D^=_\phi$.

(iii) We will prove the result for $D^<_\phi(\bar{x})$. Consider $d \in D^<_\phi(\bar{x})$, which implies there exists $\bar{\alpha} > 0$ such that

$$\phi(\bar{x} + \alpha d) < \phi(\bar{x}), \ \forall\ \alpha \in (0, \bar{\alpha}].$$

As done in (ii), the above inequality can be rewritten as

$$\phi(\bar{x} + \alpha' d') < \phi(\bar{x}), \ \forall\ \alpha' \in (0, \bar{\alpha}'],$$

where $d' = \lambda d$, $\alpha' = \dfrac{\alpha}{\lambda}$ and $\bar{\alpha}' = \dfrac{\bar{\alpha}}{\lambda}$ for any $\lambda > 0$. Note that $0 \notin D^<_\phi(\bar{x})$. Therefore, $\lambda d \in D^<_\phi(\bar{x})$ for every $\lambda > 0$ and hence $D^<_\phi(\bar{x})$ is a blunt cone.

Now consider $d_1, d_2 \in D^<_\phi(\bar{x})$. Working along the lines of the proof in (ii), for $i = 1, 2$,

$$\phi(\bar{x} + \alpha d_i) < \phi(\bar{x}), \ \forall\ \alpha \in (0, \bar{\alpha}]. \tag{6.4}$$

For any $\lambda \in [0, 1]$, let $d = \lambda d_1 + (1 - \lambda) d_2$. The convexity of ϕ along with the condition (6.4) on d_1 and d_2 yields

$$\begin{aligned} \phi(\bar{x} + \alpha d) &\leq \lambda \phi(\bar{x} + \alpha d_1) + (1 - \lambda) \phi(\bar{x} + \alpha d_2) \\ &< \phi(\bar{x}), \ \forall\ \alpha \in (0, \bar{\alpha}], \end{aligned}$$

thereby implying the convexity of $D^<_\phi$. From Definition 6.1, it is obvious that $D^<_\phi$ is open by using the continuity of ϕ.

Consider $d \in D^<_\phi(\bar{x})$, which implies that there exists $\bar{\alpha} > 0$ such that

$$\phi(\bar{x} + \alpha d) < \phi(\bar{x}), \ \forall\ \alpha \in (0, \bar{\alpha}].$$

By the convexity of ϕ, for every $\alpha \in (0, \bar{\alpha}]$,

$$\alpha \langle \xi, d \rangle \leq \phi(\bar{x} + \alpha d) - \phi(\bar{x}) < 0, \ \forall\ \xi \in \partial \phi(\bar{x}).$$

As *dom* $\phi = \mathbb{R}^n$, by Theorem 2.79 and Proposition 2.83, the directional derivative is the support function of the subdifferential, which along with the compactness of $\partial \phi$ is attained at some $\xi \in \partial \phi(\bar{x})$. Thus $\phi'(\bar{x}, d) < 0$, which leads to

$$D^<_\phi(\bar{x}) \subset \{d \in \mathbb{R}^n : \phi'(\bar{x}, d) < 0\}. \tag{6.5}$$

Now consider $d \in \mathbb{R}^n$ such that $\phi'(\bar{x}, d) < 0$, that is,

$$\lim_{\alpha \downarrow 0} \frac{\phi(\bar{x} + \alpha d) - \phi(\bar{x})}{\alpha} < 0.$$

Therefore, there exists $\bar{\alpha} > 0$ such that

$$\phi(\bar{x} + \alpha d) < \phi(\bar{x}), \ \forall\ \alpha \in (0, \bar{\alpha}],$$

which implies $d \in D_\phi^<(\bar{x})$, thereby establishing the equality in the relation (6.5). Working along the above lines of proof, readers are advised to prove the result for $D_\phi^{\leq}(\bar{x})$.

(iv) As $D_\phi^<(\bar{x})$ is nonempty, (iii) implies that

$$\phi'(\bar{x}, d) < 0, \ \forall \ d \in D_\phi^<(\bar{x}).$$

Because *dom* $\phi = \mathbb{R}^n$, by Theorem 2.79, the directional derivative acts as a support function of the subdifferential, which along with the above relation is equivalent to $0 \notin \partial\phi(\bar{x})$. Therefore, by Proposition 3.4, *cone* $\partial\phi(\bar{x})$ is closed. The proof can be worked along the lines of Proposition 3.9 by replacing $S(\bar{x})$ by $D_\phi^{\leq}(\bar{x})$ and *cl* $\widehat{S}(\bar{x})$ by the closed set *cone* $\partial\phi(\bar{x})$. $\square$

Note that unlike (iii) where the relation holds as equality, one is able to prove only inclusion in (i) and not equality. For example, consider the strict convex function $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(x) = x^2$. For $\bar{x} = 0$, $D_\phi^=(\bar{x}) = \{0\}$ and $\nabla\phi(\bar{x}) = 0$. Observe that

$$\{d \in \mathbb{R} : \langle \nabla\phi(\bar{x}), d\rangle = 0\} = \mathbb{R} \neq \{0\} = D_\phi^=(\bar{x}).$$

Hence, the equality need not hold in (i) even for a differentiable function. Also, for a differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$, if there are n linearly independent vectors $d_i \in D_\phi^=(\bar{x}), \ i = 1, 2, \ldots, n$, then $\nabla\phi(\bar{x}) = 0$. Observe that one needs the differentiability assumption only in (ii). A careful look at the proof of (ii) shows that to prove the reverse inequality in (6.2), we make use of (i) under differentiability. So if ϕ is nondifferentiable, to prove the result one needs to assume that for some $\xi \in \partial\phi(\bar{x})$, $\phi'(\bar{x}, d) = \langle \xi, d\rangle = 0$ for every $d \in D_\phi^=(\bar{x})$. For a better understanding, we illustrate with an example from Ben-Tal, Ben-Israel, and Zlobec [7]. Consider a convex nondifferentiable function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\phi(x_1, x_2) = \max\{x_1, x_2\}.$$

For $\bar{x} = (0, 0)$, $\partial\phi(\bar{x}) = co \ \{(1, 0), (0, 1)\}$ and

$$D_\phi^=(\bar{x}) = \{(d, 0) \in \mathbb{R}^2 : d \leq 0\} \cup \{(0, d) \in \mathbb{R}^2 : d \leq 0\},$$

which is not convex. Note that $\langle (\bar{\xi}_1, \bar{\xi}_2), (d, 0)\rangle = 0$ for $\bar{\xi} = (0, 1)$ whereas $\langle (\tilde{\xi}_1, \tilde{\xi}_2), (0, d)\rangle = 0$ for $\tilde{\xi} = (1, 0)$, that is, $\phi'(\bar{x}, d) = 0$ for $\bar{\xi}, \ \tilde{\xi} \in \partial\phi(\bar{x})$ with $\bar{\xi} \neq \tilde{\xi}$.

With all these discussions on the direction sets, we move on to study the work done by Ben-Tal, Ben-Israel, and Zlobec [7].

6.2 Geometric Optimality Condition: Smooth Case

Ben-Tal, Ben-Israel, and Zlobec [7] established the necessary and sufficient optimality conditions for (CP) with the feasible set C given by (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\},$$

in the absence of any constraint qualifications in the smooth scenario. The result relates the point of minimizer of (CP) with the inconsistency of a system. We present the result below. Throughout we will assume that the *active index set* $I(\bar{x}) = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = 0\}$ is nonempty.

Theorem 6.3 *Consider the convex programming problem (CP) with C given by (3.1). Let f and $g_i,\ i = 1, 2, \ldots, m$, be differentiable convex functions. Then $\bar{x}$ is a point of minimizer of (CP) if and only if for every subset $\Omega \subset I(\bar{x})$, the system*

$$\left.\begin{array}{ll} \langle \nabla f(\bar{x}, d\rangle < 0, & \\ \langle \nabla g_i(\bar{x}), d\rangle < 0, & i \in \Omega, \\ d \in D_i^{=}(\bar{x}), & i \in \Omega^* = I(\bar{x}) \backslash \Omega \end{array}\right\} \qquad (CP_\Omega)$$

is inconsistent where $D_i^{=}(\bar{x}) = D_{g_i}^{=}(\bar{x})$ for $i \in \Omega^*$. It is important to note that for $\Omega = I(\bar{x})$, $\Omega^* = \emptyset$ and then by convention we will consider $d \in \mathbb{R}^n$.

Proof. We will prove that the negation of the result, that is, $\bar{x}$ is not a point of minimizer of (CP) if and only if there exists some subset $\Omega \subset I(\bar{x})$ such that the system (CP_Ω) is consistent. Suppose that $\bar{x}$ is not a point of minimizer of (CP), which implies that there exists a feasible point $\tilde{x} \in C$ of (CP) such that $f(\tilde{x}) < f(\bar{x})$. Therefore, by the convexity of the differentiable functions f and $g_i,\ i \in I(\bar{x})$, Theorem 2.81,

$$\begin{array}{rcl} \langle \nabla f(\bar{x}), \tilde{x} - \bar{x}\rangle & \leq & f(\tilde{x}) - f(\bar{x}) < 0, \\ \langle \nabla g_i(\bar{x}), \tilde{x} - \bar{x}\rangle & \leq & g_i(\tilde{x}) - g_i(\bar{x}) \leq 0,\ i \in I(\bar{x}), \end{array}$$

which implies for $d = \tilde{x} - \bar{x}$, the system

$$\begin{array}{rcl} \langle \nabla f(\bar{x}), d\rangle & < & 0, \\ \langle \nabla g_i(\bar{x}), d\rangle & \leq & 0,\ i \in I(\bar{x}). \end{array}$$

Define the subset Ω of $I(\bar{x})$ as

$$\Omega = \{i \in I(\bar{x}) : \langle \nabla g_i(\bar{x}), d\rangle < 0\}.$$

Therefore, d satisfies the system

$$\begin{array}{rcl} \langle \nabla f(\bar{x}), d\rangle & < & 0, \\ \langle \nabla g_i(\bar{x}), d\rangle & < & 0,\ i \in \Omega, \\ \langle \nabla g_i(\bar{x}), d\rangle & = & 0,\ i \in \Omega^*. \end{array}$$

We claim that for every $i \in \Omega^*$,

$$D_i^{=}(\bar{x}) = \{d \in \mathbb{R}^n : \langle \nabla g_i(\bar{x}), d \rangle = 0\}.$$

By Proposition 6.2 (i),

$$D_i^{=}(\bar{x}) \subset \{d \in \mathbb{R}^n : g_i'(\bar{x}, d) = 0\} = \{d \in \mathbb{R}^n : \langle \nabla g_i(\bar{x}), d \rangle = 0\}. \tag{6.6}$$

Thus, to establish our claim, we will prove the reverse inclusion in the condition (6.6). Consider any $i \in \Omega^*$. Define a differentiable convex function $G_i : \mathbb{R} \to \mathbb{R}$ as $G_i(\lambda) = g_i(\bar{x} + \lambda d)$. Therefore,

$$\begin{aligned} \nabla G_i(\lambda) &= \lim_{\delta \downarrow 0} \frac{G_i(\lambda + \delta) - G_i(\lambda)}{\delta} \\ &= \lim_{\delta \downarrow 0} \frac{g_i(\bar{x} + (\lambda + \delta)d) - g_i(\bar{x} + \lambda d)}{\delta}, \end{aligned}$$

which for $\lambda = 0$ along with the fact that $i \in \Omega^*$ implies that

$$\nabla G_i(\lambda) = \lim_{\delta \downarrow 0} \frac{g_i(\bar{x} + (\lambda + \delta)d)}{\delta} = \langle \nabla g_i(\bar{x}), d \rangle = 0.$$

By Proposition 2.75, ∇G_i is a nondecreasing over $\lambda > 0$, that is,

$$\nabla G_i(\lambda) \geq \nabla G_i(0) = 0, \ \forall \ \lambda > 0.$$

Therefore, G_i is a nondecreasing function over $\lambda > 0$, which implies that $\lambda = 0$ is a point of minimizer of G_i. Hence,

$$g_i(\bar{x} + \lambda d) = G_i(\lambda) \geq G_i(0) = 0, \ \forall \ \lambda > 0 \tag{6.7}$$

as $i \in \Omega^* \subset I(\bar{x})$. As $\tilde{x} = \bar{x} + d$ is feasible to (CP), for $i \in \Omega^*$, $g_i(\bar{x} + d) \leq 0$. Thus, for $\lambda = 1$, the condition (6.7) reduces to $g_i(\bar{x} + d) = 0$. By the convexity of g_i,

$$\begin{aligned} g_i(\bar{x} + \lambda d) &= g_i((1 - \lambda)\bar{x} + \lambda(\bar{x} + d)) \\ &\leq (1 - \lambda)g_i(\bar{x}) + \lambda g_i(\bar{x} + d) = 0, \ \forall \ \lambda \in (0, 1), \end{aligned}$$

which by (6.7) yields

$$g_i(\bar{x} + \lambda d) = 0, \ \forall \ \lambda \in (0, 1].$$

Thus, $d \in D_i^{=}(\bar{x})$. Because $d \in \{d \in \mathbb{R}^n : \langle \nabla g_i(\bar{x}), d \rangle = 0\}$ was arbitrary,

$$D_i^{=}(\bar{x}) \supset \{d \in \mathbb{R}^n : \langle \nabla g_i(\bar{x}), d \rangle = 0\},$$

thereby proving the claim. As the claim holds for every $i \in \Omega^*$, $\bar{x}$ is not a point of minimizer of (CP) implies that the system (CP_Ω) is consistent.

Conversely, suppose that the system (CP_Ω) is consistent for some subset $\Omega \subset I(\bar{x})$, that is,

$$\langle \nabla f(\bar{x}, d\rangle < 0, \tag{6.8}$$

$$\langle \nabla g_i(\bar{x}), d\rangle < 0, \quad i \in \Omega, \tag{6.9}$$

$$d \in D_i^{=}(\bar{x}), \quad i \in \Omega^* = I(\bar{x})\backslash\Omega. \tag{6.10}$$

From the inequality (6.8),

$$\lim_{\alpha \downarrow 0} \frac{f(\bar{x} + \alpha d) - f(\bar{x})}{\alpha} < 0,$$

which implies there exists $\bar{\alpha}_f > 0$ such that

$$f(\bar{x} + \alpha d) < f(\bar{x}), \ \forall\ \alpha \in (0, \bar{\alpha}_f]. \tag{6.11}$$

Similarly, from the condition (6.9), there exist $\bar{\alpha}_i > 0,\ i \in \Omega$ such that

$$g_i(\bar{x} + \alpha d) < g_i(\bar{x}) = 0, \ \forall\ \alpha \in (0, \bar{\alpha}_i],\ i \in \Omega. \tag{6.12}$$

From (6.10), $d \in D_i^{=}(\bar{x}),\ i \in \Omega^*$, which by Definition 6.1 implies that there exist $\bar{\alpha}_i > 0,\ i \in \Omega^*$ such that

$$g_i(\bar{x} + \alpha d) = g_i(\bar{x}) = 0, \ \forall\ \alpha \in (0, \bar{\alpha}_i],\ i \in \Omega^*. \tag{6.13}$$

For $i \notin I(\bar{x})$, $g_i(\bar{x}) < 0$. As $g_i,\ i \notin I(\bar{x})$ is continuous on $\mathbb{R}^n$, there exist $\bar{\alpha}_i > 0,\ i \in I(\bar{x})$ such that

$$g_i(\bar{x} + \alpha d) < 0, \ \forall\ \alpha \in (0, \bar{\alpha}_i],\ i \in I(\bar{x}). \tag{6.14}$$

Define $\bar{\alpha} = \min\{\bar{\alpha}_f, \bar{\alpha}_1, \ldots, \bar{\alpha}_m\}$. Therefore, the conditions (6.12), (6.13), and (6.14) hold for $\bar{\alpha}$ as well, which implies $\bar{x} + \bar{\alpha}d \in C$, that is, feasible for (CP). By the strict inequality (6.11),

$$f(\bar{x} + \bar{\alpha}d) < f(\bar{x}),$$

thereby leading to the fact that $\bar{x}$ is not a point of minimizer of (CP), as desired. $\square$

We illustrate the above result by the following example. Consider the convex programming problem

$$\begin{array}{rl} \min & -x_1 + x_2 \\ \text{subject to} & x_1 + x_2 + 1 \leq 0, \\ & x_2^2 \leq 0. \end{array}$$

Observe that $\bar{x} = (-1, 0)$ is the point of minimizer of the above problem. The KKT optimality condition at $\bar{x}$ is given by

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is not satisfied by any $\lambda_i \geq 0,\ i = 1, 2$. For $\bar{x}$, $I(\bar{x}) = \{1, 2\}$ with

$$\begin{array}{rcl} D_1^{=}(\bar{x}) & = & \{(d_1, d_2) \in \mathbb{R}^2 : d_1 + d_2 = 0\}, \\ D_2^{=}(\bar{x}) & = & \{(d_1, d_2) \in \mathbb{R}^2 : d_2 = 0\}. \end{array}$$

Now consider the following systems as in Theorem 6.3:

$$\left.\begin{array}{l} -d_1 + d_2 < 0, \\ d_1 + d_2 = 0, \\ d_2 = 0. \end{array}\right\} \qquad (CP_{\emptyset})$$

$$\left.\begin{array}{l} -d_1 + d_2 < 0, \\ d_1 + d_2 < 0, \\ d_2 = 0. \end{array}\right\} \qquad (CP_1)$$

$$\left.\begin{array}{l} -d_1 + d_2 < 0, \\ 0 < 0, \\ d_1 + d_2 = 0. \end{array}\right\} \qquad (CP_2)$$

$$\left.\begin{array}{l} -d_1 + d_2 < 0, \\ d_1 + d_2 < 0, \\ 0 < 0. \end{array}\right\} \qquad (CP_{I(\bar{x})})$$

Observe that all four systems are inconsistent. Therefore, by the above theorem, $\bar{x}$ is the point of minimizer of the problem.

Now if we consider $\tilde{x} = (-2, 0)$, which is feasible for the problem, $I(\tilde{x}) = \{2\}$ with $D_2^{=}(\tilde{x}) = D_2^{=}(\bar{x})$. For $\tilde{x}$, the system

$$\left.\begin{array}{l} -d_1 + d_2 < 0, \\ 0 < 0. \end{array}\right\} \qquad (CP_{I(\tilde{x})})$$

is inconsistent whereas

$$\left.\begin{array}{l} -d_1 + d_2 < 0, \\ d_2 = 0. \end{array}\right\} \qquad (CP_{\emptyset})$$

is consistent. Thus, by Theorem 6.3, $\tilde{x}$ is not the point of minimizer.

Theorem 6.3 was expressed in terms of the inconsistency of a system for every subset $\Omega \subset I(\bar{x})$. Next we present the result of Ben-Tal, Ben-Israel, and Zlobec [7] in terms of the Fritz John type optimality conditions. But before establishing that result, we state the *Dubovitskii–Milyutin Theorem*, which acts as a tool in the proof.

Proposition 6.4 *Consider open blunt convex cones $C_1, C_2, \ldots, C_m$ and convex cone C_{m+1}. Then*

$$\bigcap_{i=1}^{m+1} C_i = \emptyset$$

if and only if there exists $y_i \in C_i^\circ,\ i = 1, 2, \ldots, m$, *not all simultaneously zero such that*

$$y_1 + y_2 + \ldots + y_m + y_{m+1} = 0.$$

Theorem 6.5 *Consider the convex programming problem* (CP) *with* C *given by (3.1). Let* f *and* $g_i,\ i = 1, 2, \ldots, m$, *be differentiable convex functions. Then* $\bar{x}$ *is a point of minimizer of* (CP) *if and only if for every subset* $\Omega \subset I(\bar{x})$ *the system*

$$\left.\begin{array}{l} 0 \in \lambda_0 \nabla f(\bar{x}) + \sum_{i\in\Omega} \lambda_i \nabla g_i(\bar{x}) + (D^{=}_{\Omega^*}(\bar{x}))^\circ, \\ \lambda_0 \geq 0,\ \lambda_i \geq 0,\ i \in \Omega,\ \textit{not all simultaneously zeros} \end{array}\right\} \qquad (CP'_\Omega)$$

is consistent, where

$$D^{=}_{\Omega^*}(\bar{x}) = \begin{cases} \bigcap_{i\in\Omega^*} D^{=}_i(\bar{x}), & \textit{if}\ \ \Omega^* \neq \emptyset, \\ \mathbb{R}^n, & \textit{if}\ \ \Omega^* = \emptyset. \end{cases}$$

Proof. From Theorem 6.3, $\bar{x}$ is a point of minimum of (CP) if and only if for every subset $\Omega \subset I(\bar{x})$, the system (CP_Ω) is inconsistent, which by the differentiability of f and $g_i,\ i \in \Omega$, along with Proposition 6.2 (iii) is equivalent to

$$D^{<}_f(\bar{x}) \cap \left(\bigcap_{i\in\Omega} D^{<}_i(\bar{x}) \right) \cap D^{=}_{\Omega^*}(\bar{x}) = \emptyset,$$

where $D^{<}_i(\bar{x}) = D^{<}_{g_i}(\bar{x}),\ i \in \Omega$. By Proposition 6.2 (c), $D^{<}_f(\bar{x})$ and $D^{<}_i(\bar{x})$, $i \in \Omega$ are open blunt convex cones while $D^{=}_{\Omega^*}(\bar{x})$, being the intersection of convex cones, is itself a convex cone. Applying Propositions 6.2 (iv) and 2.80,

$$(D^{<}_\phi(\bar{x}))^\circ = \{y \in \mathbb{R}^n : y = \mu \nabla \phi(\bar{x}),\ \mu \geq 0\}.$$

Now applying the Dubovitskii–Milyutin Theorem, Proposition 6.4, is equivalent to the existence of multipliers $\lambda_0 \geq 0,\ \lambda_i \geq 0,\ i \in \Omega$, not all simultaneously zero such that

$$0 \in \lambda_0 \nabla f(\bar{x}) + \sum_{i\in\Omega} \lambda_i \nabla g_i(\bar{x}) + (D^{=}_{\Omega^*}(\bar{x}))^\circ,$$

thereby leading to the requisite result. $\square$

Ben-Tal, Ben-Israel, and Zlobec [7] also dealt with the strictly convex case. For more details, one can go through [7]. Observe that taking $\Omega = I(\bar{x})$ in Theorem 6.5, the system (CP'_Ω) reduces to the standard Fritz John optimality condition. Similarly in Theorem 6.3, the system (CP_Ω) becomes

$$\begin{aligned} &\langle \nabla f(\bar{x}), d \rangle < 0, \\ &\langle \nabla g_i(\bar{x}), d \rangle < 0, \quad i \in I(\bar{x}). \end{aligned}$$

Similar to the notion of constraint qualification, they define the following concept of regularization condition under which the result holds for $\Omega = I(\bar{x})$, and the other subsets of $I(\bar{x})$ need not be considered.

Definition 6.6 A condition is called a *regularization condition* at a point $\bar{x}$ if, when assumed along with the convexity and differentiability conditions of f and $g_i,\ i = 1, 2, \ldots, m$, the family $\{(CP_\Omega) : \Omega \subset I(\bar{x})\}$ can be replaced by a single system $(CP_{I(\bar{x})})$. Thus, $\bar{x}$ is a point of minimum of (CP) if and only if $(CP_{I(\bar{x})})$ is inconsistent or $(CP'_{I(\bar{x})})$ is consistent. In this case, the Fritz John optimality condition is necessary as well as sufficient to characterize the point of minimum of (CP).

In the example considered in this section, there is no regularization condition because in the case of $\tilde{x}$ we need to verify the inconsistency, all the possible systems other than $(CP_{I(\tilde{x})})$ only to check that $\tilde{x}$ is not the point of minimizer.

As observed in Chapter 3, under the Slater constraint qualification, the KKT optimality conditions is necessary as well as sufficient to check whether a point is optimal or not. It has been shown in Ben-Tal, Ben-Israel, and Zlobec [7] that the Slater constraint qualification acts as a regularization condition for (CP). We present the result below.

Proposition 6.7 *Consider the convex programming problem (CP) with C given by (3.1). Let f and $g_i,\ i = 1, 2, \ldots, m$, be differentiable convex functions. Then the Slater constraint qualification, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m$, is a regularization condition for (CP).*

Proof. We prove the result by establishing the negation. From the definition of regularization condition, it is equivalent to verifying that $\bar{x}$ is not a point of minimizer of (CP) if and only if the system $(CP_{I(\bar{x})})$ is consistent. If $(CP_{I(\bar{x})})$ is consistent, then by Theorem 6.3, $\bar{x}$ is not a point of minimizer of (CP).

Conversely, suppose that $\bar{x}$ is not a point of minimizer for (CP). Again, by Theorem 6.3, there exists a subset $\bar{\Omega} \in I(\bar{x})$ and $\bar{d} \in \mathbb{R}^n$ such that the system

$$\left.\begin{aligned} \langle \nabla f(\bar{x}), \bar{d}\rangle &< 0, & & \\ \langle \nabla g_i(\bar{x}), \bar{d}\rangle &< 0, & i &\in \bar{\Omega}, \\ \bar{d} &\in D_i^{=}(\bar{x}), & i &\in \bar{\Omega}^* = I(\bar{x})\backslash\bar{\Omega}. \end{aligned}\right\} \qquad (CP_{\bar{\Omega}})$$

By Proposition 6.2 (i),

$$\langle \nabla g_i(\bar{x}), \bar{d}\rangle = 0,\ i \in \bar{\Omega}^*. \tag{6.15}$$

As $\hat{x}$ satisfies the Slater constraint qualification, applying Theorem 2.81 to $g_i,\ i \in I(\bar{x})$,

$$\langle \nabla g_i(\bar{x}), \hat{x} - \bar{x}\rangle \leq g_i(\hat{x}) - g_i(\bar{x}) < 0,\ i \in I(\bar{x}).$$

Define $\tilde{d} = \bar{d} + \alpha(\hat{x} - \bar{x})$ for $\alpha > 0$ sufficiently small. Then using the condition (6.15), the system

$$\left.\begin{array}{l}\langle \nabla f(\bar{x}, \tilde{d}\rangle < 0, \\ \langle \nabla g_i(\bar{x}), \tilde{d}\rangle < 0, \quad i \in I(\bar{x}),\end{array}\right\} \qquad (CP_{I(\bar{x})})$$

is consistent for $\tilde{d}$, thereby leading to the desired result. □

Note that in the example considered in this section, the regularization condition did not hold. As a matter of fact, the Slater constraint qualification was not satisfied.

6.3 Geometric Optimality Condition: Nonsmooth Case

The work of Ben-Tal, Ben-Israel, and Zlobec [7] was extended by Wolkowicz [112] to nonsmooth convex scenario. The latter not only studied the optimality conditions by avoiding constraint qualifications, but also gave a geometrical interpretation to what he termed as *badly behaved constraints.* Before discussing the contributions of Wolkowicz [112] toward the convex programming problem (CP) with the feasible set C given by (3.1), we will define some notations. The *equality set* is given by

$$I^{=} = \{i \in \{1, 2, \ldots, m\} : g_i(x) = 0, \ \forall\, x \in C\}.$$

For $\bar{x} \in C$, define

$$I^{<}(\bar{x}) = I(\bar{x}) \backslash I^{=},$$

where $I(\bar{x})$ is the active index set at $\bar{x}$. Observe that while $I^{<}(\bar{x})$ depends on $\bar{x}$, $I^{=}$ is independent of any $x \in C$. Using the direction notations presented in the beginning of this chapter, Wolkowicz [112] defined the set of badly behaved constraints.

Definition 6.8 For $\bar{x} \in C$, the set of *badly behaved constraints* is given by

$$I^{b}(\bar{x}) = \{i \in I^{=} : (D_i^{>}(\bar{x}) \cap S(\bar{x})) \backslash cl \bigcap_{i \in I^{=}} D_i^{=}(\bar{x}) \neq \emptyset\},$$

where

$$S(\bar{x}) = \{d \in \mathbb{R}^n : g_i'(\bar{x}, d) \leq 0, \ \forall\, i \in I(\bar{x})\}.$$

Recall that we introduced the set $S(\bar{x})$ in Section 3.3 and proved in Proposition 3.9 that

$$(S(\bar{x}))^{\circ} = cl\ \widehat{S}(\bar{x}),$$

where

$$\widehat{S}(\bar{x}) = \{\sum_{i \in I(\bar{x})} \lambda_i \xi_i : \lambda_i \geq 0, \ \xi_i \in \partial g_i(\bar{x}), \ i \in I(\bar{x})\}.$$

The set $I^b(\bar{x})$ is the set of constraints that create problems in KKT conditions. A characterization of the above set in terms of the directional derivative was stated by Wolkowicz [112] without proof. We present the result with proof for a better understanding.

Theorem 6.9 *Consider the convex programming problem (CP) with C given by (3.1). Let $i^* \in I^=$. Then $i^* \in I^b(\bar{x})$ if and only if the system*

$$\left.\begin{array}{l} g'_{i^*}(\bar{x}, d) = 0, \\ g'_i(\bar{x}, d) \leq 0, \ \forall\, i \in I(\bar{x})\backslash i^*, \\ d \notin D^=_{i^*}(\bar{x}) \cup cl \bigcap\limits_{i \in I^=} D^=_i(\bar{x}). \end{array}\right\} \qquad (CP_b)$$

is consistent.

Proof. Suppose that $i^* \in I^b(\bar{x})$, which implies there exists $d^* \in \mathbb{R}^n$ such that

$$\left\{\begin{array}{l} d^* \in D^>_{i^*}(\bar{x}), \\ d^* \in S(\bar{x}), \\ d^* \notin cl \bigcap\limits_{i \in I^=} D^=_i(\bar{x}). \end{array}\right.$$

As $d^* \in D^>_{i^*}(\bar{x})$, $d^* \notin D^=_{i^*}(\bar{x})$, which along with the last condition implies

$$d^* \notin D^=_{i^*}(\bar{x}) \cup cl \bigcap_{i \in I^=} D^=_i(\bar{x}). \tag{6.16}$$

Also, as $d^* \in D^>_{i^*}(\bar{x})$, by Definition 6.1 there exists $\alpha^* > 0$ such that

$$g_{i^*}(\bar{x} + \alpha d^*) > g_{i^*}(\bar{x}), \ \forall\, \alpha \in (0, \alpha^*].$$

Therefore,

$$\lim_{\alpha \downarrow 0} \frac{g_{i^*}(\bar{x} + \alpha d^*) - g_{i^*}(\bar{x})}{\alpha} \geq 0,$$

which implies

$$g'_{i^*}(\bar{x}, d^*) \geq 0. \tag{6.17}$$

Because $d^* \in S(\bar{x})$,

$$g'_i(\bar{x}, d^*) \leq 0, \ \forall\, i \in I(\bar{x}). \tag{6.18}$$

In particular, taking $i^* \in I^= \subseteq I(\bar{x})$ in the above inequality along with (6.17) yields

$$g'_{i^*}(\bar{x}, d^*) = 0. \tag{6.19}$$

Combining the conditions (6.16), (6.18), and (6.19) together imply that d^* solves the system (CP_b), thereby leading to its consistency.

Conversely, suppose that (CP_b) is consistent, which implies there exists $d^* \in \mathbb{R}^n$ such that

$$\left.\begin{array}{l} g'_{i^*}(\bar{x}, d^*) = 0, \\ g'_i(\bar{x}, d^*) \leq 0, \ \forall\, i \in I(\bar{x}) \backslash i^*, \\ d^* \notin D^{=}_{i^*}(\bar{x}) \cup cl \displaystyle\bigcap_{i \in I^{=}} D^{=}_i(\bar{x}). \end{array}\right\}$$

The first equality condition can be expressed as two inequalities given by

$$g'_{i^*}(\bar{x}, d^*) \leq 0 \quad \text{and} \quad g'_{i^*}(\bar{x}, d^*) \geq 0. \tag{6.20}$$

As $i^* \in I^{=} \subseteq I(\bar{x})$ along with the above condition yields

$$d^* \in S(\bar{x}). \tag{6.21}$$

Also, from the inequality (6.20), there exists $\alpha^* > 0$ such that

$$g_{i^*}(\bar{x} + \alpha d^*) \geq g_{i^*}(\bar{x}), \ \forall\, \alpha \in (0, \alpha^*].$$

As $d^* \notin D^{=}_{i^*}(\bar{x})$, the above inequality holds as a strict inequality and hence

$$d^* \in D^{>}_{i^*}(\bar{x}). \tag{6.22}$$

The conditions (6.21) and (6.22) along with the fact that $d^* \notin cl \bigcap_{i \in I^{=}} D^{=}_i(\bar{x})$ implies that $d^* \in I^b(\bar{x})$, thereby establishing the desired result. □

Observe that if $D^{=}_{i^*}(\bar{x}) = \{d \in \mathbb{R}^n : g'_{i^*}(\bar{x}, d) = 0\}$, then by the above characterization of the badly behaved constraints, $i^* \notin I^b(\bar{x})$. The class of functions that are *never badly behaved* includes the class of all continuous linear functionals and the classical *distance function*. For more on badly behaved constraints, one can refer to Wolkowicz [112].

Before moving any further, we present a few results from Wolkowicz [112] that act as a tool in the derivation of the characterization for the point of minimum.

Proposition 6.10 *Consider the convex programming problem (CP) with C given by (3.1). Suppose that $\bar{x} \in C$. Then*

(i) $\displaystyle\bigcap_{i \in I(\bar{x})} D^{\leq}_i(\bar{x}) = \bigcap_{i \in I^{=}} D^{=}_i(\bar{x}) \cap \bigcap_{i \in I^{<}(\bar{x})} D^{\leq}_i(\bar{x}).$

(ii) $\displaystyle\bigcap_{i \in I^{=}} D^{=}_i(\bar{x}) \cap \bigcap_{i \in I^{<}(\bar{x})} D^{<}_i(\bar{x}) \neq \emptyset.$

Furthermore, suppose that the set Ω satisfies $I^b(\bar{x}) \subset \Omega \subset I^{=}$. If

$$\textit{either} \quad co \bigcap_{i \in \Omega} D^{=}_i(\bar{x}) \ \textit{is closed} \quad \textit{or} \quad \Omega = I^{=},$$

then

(iii) $T_C(\bar{x}) = cl \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x})$.

(iv) $cl\ co \bigcap_{i \in \Omega} D_i^{=}(\bar{x}) \cap S(\bar{x}) = cl \bigcap_{i \in \Omega} D_i^{=}(\bar{x}) \cap S(\bar{x}) = cl \bigcap_{i \in I^=} D_i^{=}(\bar{x}) \cap S(\bar{x})$.

(v) $T_C(\bar{x}) = cl\ co \bigcap_{i \in \Omega} D_i^{=}(\bar{x}) \cap S(\bar{x})$.

(vi) $-co \bigcup_{i \in I^<(\bar{x})} \partial g_i(\bar{x}) \cap (\bigcap_{i \in \Omega} D_i^{=}(\bar{x}))^\circ = \emptyset$.

Proof. (i) Observe that $I(\bar{x}) = I^= \cup I^<(\bar{x})$, which implies

$$\begin{aligned}\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}) &= \{d \in \mathbb{R}^n : \text{there exists } \bar{\alpha} > 0 \text{ such that} \\ &\qquad\qquad g_i(\bar{x} + \alpha d) \leq g_i(\bar{x}),\ \forall\ i \in I(\bar{x})\} \\ &= \bigcap_{i \in I^=} D_i^{\leq}(\bar{x}) \cap \bigcap_{i \in I^<(\bar{x})} D_i^{\leq}(\bar{x}).\end{aligned}$$

For any $d \in D_i^{\leq}(\bar{x})$, there exists $\bar{\alpha} > 0$ such that

$$g_i(\bar{x} + \alpha d) \leq g_i(\bar{x}) = 0,\ \ \alpha \in (0, \bar{\alpha}],$$

which implies $\bar{x} + \alpha d \in C$ for every $\alpha \in (0, \bar{\alpha}]$. As for every $i \in I^=$, $g_i(x) = 0$ for every feasible point $x \in C$ of (CP), thereby implying that for every $i \in I^=$,

$$g_i(\bar{x} + \alpha d) = g_i(\bar{x}) = 0,\ \ \alpha \in (0, \bar{\alpha}],$$

which implies $D_i^{\leq}(\bar{x}) = D_i^{=}(\bar{x})$ for every $i \in I^=$. Therefore, by this condition,

$$\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}) = \bigcap_{i \in I^=} D_i^{=}(\bar{x}) \cap \bigcap_{i \in I^<(\bar{x})} D_i^{\leq}(\bar{x}),$$

as desired.

(ii) If $I(\bar{x}) = \emptyset$, the result holds trivially by (i). Suppose that $I^=$ and $I^<(\bar{x})$ are nonempty. Then corresponding to any $i \in I^<$, there exists some $\hat{x} \in C$ such that $g_i(\hat{x}) < 0$. By the convexity of g_i, for every $\lambda \in (0, 1]$,

$$g_i(\bar{x} + \lambda(\hat{x} - \bar{x})) \leq \lambda g_i(\hat{x}) + (1 - \lambda) g_i(\bar{x}) < 0 = g_i(\bar{x}),$$

which implies that $\hat{d} = \hat{x} - \bar{x} \in D_i^{<}(\bar{x})$. Also, suppose that there is some $j \in I^<(\bar{x})$, $j \neq i$, then corresponding to j there exists some $\tilde{x} \in C$ such that $g_j(\tilde{x}) < 0$. Then as before, $\tilde{d} = \tilde{x} - \bar{x} \in D_j^{<}(\bar{x})$. Now if i and j are such that

$$g_i(\tilde{x}) = 0 \quad \text{and} \quad g_j(\hat{x}) = 0,$$

then by the convexity of g_i and g_j, for every $\lambda \in (0,1)$,

$$g_i(\lambda\hat{x} + (1-\lambda)\tilde{x}) \leq \lambda g_i(\hat{x}) + (1-\lambda)g_i(\tilde{x}) < 0,$$
$$g_j(\lambda\hat{x} + (1-\lambda)\tilde{x}) \leq \lambda g_j(\hat{x}) + (1-\lambda)g_j(\tilde{x}) < 0,$$

which implies for $\lambda \in (0,1)$, $(\lambda\hat{x} + (1-\lambda)\tilde{x}) - \bar{x} = \lambda\hat{d} + (1-\lambda)\tilde{d}$ such that

$$\lambda\hat{d} + (1-\lambda)\tilde{d} \in D_i^{<}(\bar{x}) \quad \text{and} \quad \lambda\hat{d} + (1-\lambda)\tilde{d} \in D_j^{<}(\bar{x}).$$

Proceeding as above, there exists $\bar{d} \in \mathbb{R}^n$ such that

$$d \in D_i^{<}(\bar{x}), \ \forall\, i \in I^{<}(\bar{x}) \tag{6.23}$$

with corresponding $\bar{\alpha} > 0$ such that $\bar{x}+\alpha d \in C$ for every $\alpha \in (0,\bar{\alpha}]$. Therefore, for every $i \in I^{=}$,

$$g_i(\bar{x} + \alpha d) = 0 = g_i(\bar{x}), \ \forall\, \alpha \in (0,\bar{\alpha}],$$

that is,

$$d \in D_i^{=}(\bar{x}), \ \forall\, i \in I^{=},$$

which along with the condition (6.23) proves the desired result.

(iii) Consider a feasible point $x \in C$ of (CP) that implies

$$g_i(x) \leq 0, \ \forall\, i \in I(\bar{x}).$$

By the convexity of $g_i, \ i \in I(\bar{x})$, for every $\lambda \in (0,1]$,

$$g_i(\bar{x} + \lambda(x-\bar{x})) \leq \lambda g_i(x) + (1-\lambda)g_i(\bar{x}) \leq 0 = g_i(\bar{x}), \ \forall\, i \in I(\bar{x}).$$

Therefore, $x - \bar{x} \in \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x})$ for every $x \in C$, which implies

$$(C - \bar{x}) \subset \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}).$$

As $\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x})$ is a cone,

$$\mathit{cone}\ (C - \bar{x}) \subset \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}). \tag{6.24}$$

Suppose that $d \in \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x})$, which implies there exists $\bar{\alpha} > 0$ such that

$$g_i(\bar{x} + \alpha d) \leq g_i(\bar{x}) = 0, \ \forall\, \alpha \in (0,\bar{\alpha}], \ \forall\, i \in I(\bar{x}).$$

For $i \notin I(\bar{x})$, $g_i(\bar{x}) < 0$ and thus, there exists some $\alpha' > 0$ such that for any $d \in \mathbb{R}^n$,

$$g_i(\bar{x} + \alpha d) < 0, \ \forall\, \alpha \in (0,\alpha'), \ \forall\, i \notin I(\bar{x}).$$

Therefore, by the preceding inequalities, $x' = \bar{x} + \alpha d \in C$ for $\alpha \in (0, \alpha^*]$, where $\alpha^* = \min\{\bar{\alpha}, \alpha'\}$, which implies $\alpha d \in C - \bar{x}$, thereby leading to $d \in cone\ (C - \bar{x})$, which along with the condition (6.24) yields

$$\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}) = cone\ (C - \bar{x}).$$

By Theorem 2.35,

$$T_C(\bar{x}) = cl\ cone\ (C - \bar{x}) = cl \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}),$$

hence establishing the result.

(iv) By the given hypothesis $\Omega \subset I^=$, which implies that the containment relation

$$cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl\ co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \quad (6.25)$$

holds. To establish the result, we will prove the following:

(1) $cl\ co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl\ (co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}))$.

If $co\ \bigcap_{i \in \Omega} D_i^=(\bar{x})$ is closed, then

$$cl\ co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) = co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl\ (co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x})),$$

thereby establishing the above condition.

If $\Omega = I^=$, we prove

$$cl\ co \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl\ (co \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x})). \quad (6.26)$$

As $S(\bar{x})$ is a closed convex set and $\bigcap_{i \in I^=} D_i^=(\bar{x}) \subset S(\bar{x})$,

$$cl\ co \bigcap_{i \in I^=} D_i^=(\bar{x}) \subset S(\bar{x}).$$

Also $S(x) = \bigcap_{i \in I^=} S_i(\bar{x}) \cap \bigcap_{i \in I^<(\bar{x})} S_i(\bar{x})$, where

$$S_i(\bar{x}) = \{d \in \mathbb{R}^n : g_i'(\bar{x}, d) \leq 0\}.$$

Therefore, establishing (6.26) is equivalent to proving

$$cl\ co \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap \bigcap_{i \in I^<(\bar{x})} S_i(\bar{x}) \subset cl\ (co \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap \bigcap_{i \in I^<(\bar{x})} S_i(\bar{x})).$$

By condition (ii), there exists $d \in \mathbb{R}^n$ such that

$$d \in \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap \bigcap_{i \in I^<(\bar{x})} D_i^<(\bar{x}) \subset co \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap int \bigcap_{i \in I^<(\bar{x})} S_i(\bar{x}),$$

which yields the above condition.

(2) $co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) = \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x})$.

By Proposition 6.2 (i) and (iii),

$$\bigcap_{i \in \Omega} D_i^=(\bar{x}) \subset \bigcap_{i \in \Omega} D_i^{\leq}(\bar{x}).$$

Because $D_i^{\leq}(\bar{x})$ is convex,

$$co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \subset \bigcap_{i \in \Omega} D_i^{\leq}(\bar{x}). \tag{6.27}$$

As $\Omega \subset I^=$, for every feasible point $x \in C$, $g_i(x) = 0, \ i \in \Omega$. For any $d \in D_i^{\leq}(\bar{x})$, $i \in \Omega$, there exists $\bar{\alpha}_i > 0$ such that

$$g_i(\bar{x} + \alpha d) \leq g_i(\bar{x}) = 0, \ \forall \ \alpha \in (0, \bar{\alpha}_i],$$

which implies $\bar{x} + \alpha d \in C$. Therefore, for any $i \in \Omega$,

$$g_i(\bar{x} + \alpha d) = 0, \ \forall \ \alpha \in (0, \bar{\alpha}_i],$$

thereby implying that $d \in D_i^{\leq}(\bar{x}), \ i \in \Omega$. Thus, the condition (6.27) becomes

$$co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \subset \bigcap_{i \in \Omega} D_i^=(\bar{x}) \subset co \bigcap_{i \in \Omega} D_i^=(\bar{x}).$$

The above relation implies that $\bigcap_{i \in \Omega} D_i^=(\bar{x})$ is convex, thereby leading to

$$co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) = \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}),$$

as desired. Note that, in particular, for $\Omega = I^=$, $\bigcap_{i \in I^=} D_i^=(\bar{x})$ is convex.

(3) $cl \ (\bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x})) \subset cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x})$.

Suppose that $\Omega \subsetneq I^=$. We claim that

$$\bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}).$$

Assume on the contrary that there exists $d \in \mathbb{R}^n$ such that

$$d \in \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \setminus (cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x})).$$

By the given hypothesis, there exists $\tilde{\Omega} \subset I^{=} \setminus \Omega$ such that

$$d \in S(\bar{x}), \quad d \in \bigcap_{i \in I^{=} \setminus \tilde{\Omega}} D_i^{=}(\bar{x})$$

but

$$d \notin D_i^{=}(\bar{x}) \cup cl \bigcap_{i \in I^{=}} D_i^{=}(\bar{x}), \ \forall\, i \in \tilde{\Omega}.$$

By the hypothesis $I^b(\bar{x}) \subset \Omega \subset I^{=}$, $\tilde{\Omega} \subset I^{=} \setminus \Omega \subset I^{=} \setminus I^b(\bar{x})$, which implies $\tilde{\Omega} \not\subset I^b(\bar{x})$. By invoking Theorem 6.9, the system (CP_b) is inconsistent and thus

$$g_i'(\bar{x}, d) < 0, \ \forall\, i \in \tilde{\Omega}.$$

Therefore,

$$d \in \bigcap_{i \in \tilde{\Omega}} D_i^{<}(\bar{x}) \cap \bigcap_{i \in I^{=} \setminus \tilde{\Omega}} D_i^{=}(\bar{x}). \tag{6.28}$$

By (ii), as

$$\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}) \cap \bigcap_{i \in I^{<}(\bar{x})} D_i^{<}(\bar{x}) \neq \emptyset,$$

there exists $\bar{d} \in \mathbb{R}^n$ such that

$$\bar{d} \in \bigcap_{i \in I^{=}} D_i^{=}(\bar{x}) \cap \bigcap_{i \in I^{<}(\bar{x})} D_i^{<}(\bar{x}). \tag{6.29}$$

Define $d_\lambda = \lambda d + (1-\lambda)\bar{d}$. By condition (6.28), for $i \in \tilde{\Omega}$ there exists $\bar{\alpha}_i > 0$ such that

$$g_i(\bar{x} + \alpha d) < g_i(\bar{x}), \ \forall\, \alpha \in (0, \bar{\alpha}_i]. \tag{6.30}$$

As $\tilde{\Omega} \subset I^{=}$, by condition (6.29), for $i \in \tilde{\Omega}$ there exists $\hat{\alpha}_i > 0$ such that

$$g_i(\bar{x} + \alpha d) = g_i(\bar{x}), \ \forall\, \alpha \in (0, \hat{\alpha}_i]. \tag{6.31}$$

Denote $\alpha_i = \min\{\bar{\alpha}_i, \hat{\alpha}_i\}$. By the convexity of $g_i,\ i \in \tilde{\Omega}$ along with conditions (6.30) and (6.31), for $\lambda \in (0,1]$,

$$g_i(\bar{x} + \alpha d_\lambda) \leq \lambda g_i(\bar{x} + \alpha d) + (1-\lambda) g_i(\bar{x} + \alpha \bar{d}) < g_i(\bar{x}), \ \forall\, \alpha \in (0, \alpha_i],$$

which implies

$$d_\lambda \in \bigcap_{i \in \tilde{\Omega}} D_i^{<}(\bar{x}), \ \forall\, \lambda \in (0,1]. \tag{6.32}$$

Again from (6.28),

$$d \in \bigcap_{i \in I^{=} \setminus \tilde{\Omega}} D_i^{=}(\bar{x}) \subset \bigcap_{i \in I^{=} \setminus \tilde{\Omega}} D_i^{\leq}(\bar{x}),$$

and from (6.29),

$$\bar{d} \in \bigcap_{i \in I^=} D_i^=(\bar{x}) \subset \bigcap_{i \in I^= \setminus \tilde{\Omega}} D_i^=(\bar{x}) \subset \bigcap_{i \in I^= \setminus \tilde{\Omega}} D_i^{\leq}(\bar{x}).$$

Because $D_i^{\leq}(\bar{x}),\ i \in I^= \setminus \tilde{\Omega}$ are convex sets,

$$d_\lambda \in \bigcap_{i \in I^= \setminus \tilde{\Omega}} D_i^{\leq}(\bar{x}),\ \forall\, \lambda \in (0,1). \tag{6.33}$$

By Theorem 2.69, $g_i,\ i \in I^<(\bar{x})$, is continuous on $\mathbb{R}^n$, which along with condition (6.29) implies that there exists $\beta \in (0,1)$ such that

$$d_\lambda \in \bigcap_{i \in I^<(\bar{x})} D_i^<(\bar{x}),\ \lambda \in (0,\beta]. \tag{6.34}$$

Observe that

$$I(\bar{x}) = I^<(\bar{x}) \cup I^= \setminus \tilde{\Omega} \cup \tilde{\Omega},$$

which along with (i) leads to

$$\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}) \cap \bigcap_{i \in \tilde{\Omega}} D_i^<(\bar{x}) = \bigcap_{i \in I^<(\bar{x})} D_i^{\leq}(\bar{x}) \cap \bigcap_{i \in I^= \setminus \tilde{\Omega}} D_i^{\leq}(\bar{x}) \cap \bigcap_{i \in \tilde{\Omega}} D_i^<(\bar{x}).$$

Therefore, combining (6.32), (6.33), and (6.34) along with the above relation yields

$$d_\lambda \in \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}) \cap \bigcap_{i \in \tilde{\Omega}} D_i^<(\bar{x}).$$

As $\tilde{\Omega} \subset I^=$, which along with (i) implies

$$\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}) = \bigcap_{i \in I^<(\bar{x})} D_i^{\leq}(\bar{x}) \cap \bigcap_{i \in I^=} D_i^=(\bar{x}) \subset \bigcap_{i \in I^<(\bar{x})} D_i^{\leq}(\bar{x}) \cap \bigcap_{i \in \tilde{\Omega}} D_i^=(\bar{x}).$$

Thus,

$$d_\lambda \in \bigcap_{i \in \tilde{\Omega}} D_i^=(\bar{x}),$$

which is a contradiction to

$$d_\lambda \in \bigcap_{i \in \tilde{\Omega}} D_i^<(\bar{x}).$$

Therefore,

$$\bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}).$$

Because $cl \bigcap_{i \in I^=} D_i^=(\bar{x})$ and $S(\bar{x})$ are closed sets,

$$cl\ (\bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x})) \subset cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}),$$

thereby establishing the desired result when $\Omega \subsetneq I^=$.

If $\Omega = I^=$,

$$\bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}),$$

thus yielding the desired condition as before.

From the conditions (1) through (3), it is easy to observe that

$$\begin{aligned} cl\ co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x}) \quad &\subset \quad cl\ (co \bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x})) \\ &= \quad cl\ (\bigcap_{i \in \Omega} D_i^=(\bar{x}) \cap S(\bar{x})) \ \subset \ cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}), \end{aligned}$$

which along with (6.25) yields the requisite result.

(v) Using (iii) and (iv), it is enough to show that

$$cl \bigcap_{i \in I(\bar{x})} D_i^{\leqq}(\bar{x}) = cl\ (\bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x})).$$

From (ii) and Proposition 6.2 (iii), it is obvious that

$$cl \bigcap_{i \in I(\bar{x})} D_i^{\leqq}(\bar{x}) \subset cl\ (\bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x})). \tag{6.35}$$

To prove the result, we claim that

$$\bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}) \subset cl \bigcap_{i \in I(\bar{x})} D_i^{\leqq}(\bar{x}). \tag{6.36}$$

Suppose that $d \in \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x})$. By (ii), there exists $\bar{d} \in \mathbb{R}^n$ such that

$$\bar{d} \in \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap \bigcap_{i \in I^<(\bar{x})} D_i^<(\bar{x}).$$

Denote $d_\lambda = \lambda d + (1 - \lambda)\bar{d}$. Therefore, by Theorem 2.79 and Proposition 6.2, for every $\xi \in \bigcup_{i \in I^<(\bar{x})} \partial g_i(\bar{x})$,

$$\langle \xi, d_\lambda \rangle = \lambda \langle \xi, d \rangle + (1 - \lambda) \langle \xi, \bar{d} \rangle < 0, \ \forall \ \lambda \in [0, 1),$$

which again by Theorem 2.79 implies that for every $i \in I^{<}(\bar{x})$,

$$g_i'(\bar{x}, d_\lambda) < 0, \ \forall \ \lambda \in [0, 1).$$

Therefore, by Proposition 6.2 (iii),

$$d_\lambda \in \bigcap_{i \in I^{<}(\bar{x})} D_i^{<}(\bar{x}), \ \forall \ \lambda \in [0, 1). \tag{6.37}$$

Also, by the convexity of $\bigcap_{i \in I^{=}} D_i^{=}(\bar{x})$,

$$d_\lambda \in \bigcap_{i \in I^{=}} D_i^{=}(\bar{x}), \ \forall \ \lambda \in [0, 1). \tag{6.38}$$

Thus, by the relations (6.37) and (6.38), along with (i), we obtain

$$d_\lambda \in \bigcap_{i \in I^{<}(\bar{x})} D_i^{<}(\bar{x}) \cap \bigcap_{i \in I^{=}} D_i^{=}(\bar{x}) \subset \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}), \ \forall \ \lambda \in [0, 1).$$

As the limit $\lambda \to 1$, $d_\lambda \to d$, which implies $d \in cl \ \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x})$, thus proving (6.36), which yields that

$$cl \ (\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}) \cap S(\bar{x})) \subset cl \bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}).$$

The above condition along with (6.35) establishes the desired result.

(vi) Define

$$F = -co \bigcup_{i \in I^{<}(\bar{x})} \partial g_i(\bar{x}).$$

We will prove the result by contradiction. Assume that

$$F \cap (\bigcap_{i \in \Omega} D_i^{=}(\bar{x}))^\circ \neq \emptyset,$$

which implies there exists

$$\xi \in F \cap (\bigcap_{i \in \Omega} D_i^{=}(\bar{x}))^\circ.$$

As $\xi \in F$, there exists $\xi_i \in \partial g_i(\bar{x})$ and $\lambda_i \geq 0, \ i \in I^{<}(\bar{x})$ with $\sum_{i \in I^{<}(\bar{x})} \lambda_i = 1$ such that

$$\xi = -\sum_{i \in I^{<}(\bar{x})} \lambda_i \xi_i.$$

By Proposition 2.31,

$$\begin{aligned}\bigcap_{i\in\Omega} D_i^{=}(\bar{x}) &\subset \{\xi\}^\circ = \{d \in \mathbb{R}^n : \langle \xi, d\rangle \leq 0\},\\ -F^\circ &\subset -\{\xi\}^\circ = \{d \in \mathbb{R}^n : \langle \xi, d\rangle \geq 0\}.\end{aligned}$$

Therefore,

$$-F^\circ \cap \bigcap_{i\in\Omega} D_i^{=}(\bar{x}) \subset \{d \in \mathbb{R}^n : \langle \xi, d\rangle = 0\}.$$

By (ii), there exists

$$\begin{aligned}\hat{d} &\in \bigcap_{i\in I^{=}} D_i^{=}(\bar{x}) \cap \bigcap_{i\in I^{<}(\bar{x})} D_i^{<}(\bar{x})\\ &\subset \bigcap_{i\in\Omega} D_i^{=}(\bar{x}) \cap -F^\circ \subset \{d \in \mathbb{R}^n : \langle \xi, d\rangle = 0\},\end{aligned}$$

that is, $\langle \xi, \hat{d}\rangle = 0$. As $\hat{d} \in \bigcap_{i\in I^{<}(\bar{x})} D_i^{<}(\bar{x})$, there exists $\bar{\alpha}_i > 0,\ i \in I^{<}(\bar{x})$, such that

$$g_i(\bar{x} + \alpha_i \hat{d}) < 0,\ \forall\ \alpha_i \in (0, \bar{\alpha}_i],\ \forall\ i \in I^{<}(\bar{x}).$$

By the convexity of $g_i,\ i \in I^{<}(\bar{x})$, for every $\alpha_i \in (0, \bar{\alpha}_i]$,

$$\alpha_i \langle \xi_i, \hat{d}\rangle \leq g_i(\bar{x} + \alpha_i \hat{d}) - g_i(\bar{x}) < 0,\ \forall\ i \in I^{<}(\bar{x}),$$

which implies

$$\langle \xi, d\rangle = \sum_{i\in I^{<}(\bar{x})} \lambda_i \langle \xi_i, d\rangle < 0,$$

which is a contradiction, thereby leading to the requisite result. □

Wolkowicz [112] derived a certain characterization in form of the KKT type optimality conditions. But before presenting that result, we present a lemma that will be required to prove the result.

Lemma 6.11 *Consider the convex programming problem (CP) with C given by (3.1). Suppose that $\bar{x} \in C$ and $F \subset \mathbb{R}^n$ any nonempty set. Then the statement*

$\bar{x}$ is a point of minimizer of (CP) if and only if the system

$$\left.\begin{aligned}&0 \in \partial f(\bar{x}) + \sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) + F\\ &\lambda_i \geq 0,\ i \in I(\bar{x})\end{aligned}\right\} \tag{6.39}$$

is consistent

holds for any objective function f if and only if F satisfies

$$N_C(\bar{x}) = \widehat{S}(\bar{x}) + F. \tag{6.40}$$

Proof. Suppose that the statement is satisfied for any fixed objective function. We will prove the condition (6.40). Consider $\xi \in N_C(\bar{x})$ and define the objective function as $f(x) = -\langle \xi, x \rangle$. Then $\xi \in -\partial f(\bar{x}) \cap N_C(\bar{x})$, which implies

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}).$$

By the optimality conditions for (CP), Theorem 3.1 (ii), $\bar{x}$ is a point of minimizer of (CP). Therefore, by (6.39) along with $\partial f(\bar{x}) = \{-\xi\}$ leads to

$$\xi \in \widehat{S}(\bar{x}) + F,$$

that is,

$$N_C(\bar{x}) \subset \widehat{S}(\bar{x}) + F. \tag{6.41}$$

Now suppose that $\xi \in \widehat{S}(\bar{x}) + F$, which implies there exist $\xi_i \in \partial g_i(\bar{x})$ and $\lambda_i \geq 0$ for $i \in I(\bar{x})$ such that

$$\xi - \sum_{i \in I(\bar{x})} \lambda_i \xi_i \in F.$$

Again define the objective function as $f(x) = -\langle \xi, x \rangle$, which implies $\partial f(\bar{x}) = \{-\xi\}$. By the above condition it is obvious that the condition (6.39) is satisfied and thus by the statement, $\bar{x}$ is a point of minimizer of (CP). Applying Theorem 3.1, $-\xi \in N_C(\bar{x})$, which implies

$$\widehat{S}(\bar{x}) + F \subset N_C(\bar{x}).$$

The above containment along with the relation (6.41) yields the desired condition (6.40).

Conversely, suppose that (6.40) holds. By Theorem 3.1 (ii), $\bar{x}$ is a point of minimizer of (CP) if and only if

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}),$$

which by (6.40) is equivalent to

$$0 \in \partial f(\bar{x}) + \widehat{S}(\bar{x}) + F,$$

that is, the system (6.39) is consistent, thereby completing the proof. $\square$

As mentioned in the beginning of this section, $(S(\bar{x}))^\circ = cl\, \widehat{S}(\bar{x})$ by Proposition 3.9. Therefore, if $\widehat{S}(\bar{x})$ is closed, condition (6.40) becomes

$$N_C(\bar{x}) = (S(\bar{x}))^\circ + F.$$

A similar result as the above theorem was studied by Gould and Tolle [53] under the assumption of differentiability of the functions but not necessarily convex.

Applying the above lemma along with some additional conditions, Wolkowicz [112] established KKT type optimality conditions. We present the result below.

Theorem 6.12 *Consider the convex programming problem* (CP) *with* C *given by (3.1) and* $\bar{x} \in C$. *Suppose that the set* Ω *satisfies*

$$I^b(\bar{x}) \subset \Omega \subset I^=$$

and both the sets

$$co \bigcap_{i\in\Omega} D_i^=(\bar{x}) \quad \text{and} \quad \widehat{S}(\bar{x}) + (\bigcap_{i\in\Omega} D_i^=(\bar{x}))^\circ$$

are closed. Then $\bar{x}$ *is a point of minimizer of* (CP) *if and only if the system*

$$\left.\begin{array}{l} 0 \in \partial f(\bar{x}) + \displaystyle\sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) + (\bigcap_{i\in\Omega} D_i^=(\bar{x}))^\circ, \\ \lambda_i \geq 0, \ i \in I(\bar{x}), \end{array}\right\} \tag{6.42}$$

is consistent.

Proof. Observe that the system (6.42) is obtained, in particular, by taking $F = (\bigcap_{i\in\Omega} D_i^=(\bar{x}))^\circ$ in Lemma 6.11. Thus, to establish the result, it is sufficient to prove that

$$N_C(\bar{x}) = \widehat{S}(\bar{x}) + (\bigcap_{i\in\Omega} D_i^=(\bar{x}))^\circ. \tag{6.43}$$

By Proposition 6.10 (v),

$$T_C(\bar{x}) = S(\bar{x}) \cap cl\ co \bigcap_{i\in\Omega} D_i^=(\bar{x}),$$

which by Propositions 2.31 and 3.9 imply that

$$\begin{aligned} N_C(\bar{x}) &= cl\ ((S(\bar{x})^\circ + (cl\ co \bigcap_{i\in\Omega} D_i^=(\bar{x}))^\circ) \\ &= cl\ (\widehat{S}(\bar{x}) + (\bigcap_{i\in\Omega} D_i^=(\bar{x}))^\circ). \end{aligned}$$

The closedness assumption leads to the condition (6.43), thereby yielding the requisite result. $\square$

In the above theorem, the closedness conditions on the sets

$$co \bigcap_{i\in\Omega} D_i^=(\bar{x}) \quad \text{and} \quad \widehat{S}(\bar{x}) + (\bigcap_{i\in\Omega} D_i^=(\bar{x}))^\circ$$

act as a constraint qualification. If, in particular, we choose $\Omega = I^{=}$, then the closedness conditions are no longer needed. In fact,

$$N_C(\bar{x}) = \widehat{S}(\bar{x}) + (\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}))^{\circ}$$

is always satisfied. Below we present the result for this particular case.

Theorem 6.13 *$\bar{x}$ is a minimum of (CP) if and only if the system*

$$\left.\begin{array}{l} 0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}) + (\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}))^{\circ}, \\ \lambda_i \geq 0, \ i \in I(\bar{x}), \end{array}\right\}$$

is consistent.

Proof. By Theorem 3.1 (ii), $\bar{x} \in C$ is a point of minimizer if and only if

$$0 \in \partial f(\bar{x}) + N_C(\bar{x}).$$

In order to establish the result, it is enough to show that

$$N_C(\bar{x}) = \widehat{S}(\bar{x}) + (\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}))^{\circ}. \tag{6.44}$$

Observe that *int* $D_i^{\leq}(\bar{x}) = D_i^{<}(\bar{x})$ for every $i \in I^{<}(\bar{x})$. Thus, invoking Propositions 2.31 and 6.10 implies

$$N_C(\bar{x}) = T_C(\bar{x})^{\circ} = \sum_{i \in I^{<}(\bar{x})} (D_i^{\leq}(\bar{x}))^{\circ} + (\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}))^{\circ}.$$

Again by Proposition 6.10 (ii), $D_i^{<}(\bar{x}) \neq \emptyset$, which along with Proposition 6.2 (iv) yields

$$N_C(\bar{x}) = \{\sum_{i \in I^{<}(\bar{x})} \lambda_i \partial g_i(\bar{x}) : \lambda_i \geq 0, \ i \in I^{<}(\bar{x})\} + (\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}))^{\circ}.$$

Choosing $\lambda_i = 0, \ i \in I^{=}$, the above condition leads to

$$N_C(\bar{x}) \subset \widehat{S}(\bar{x}) + (\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}))^{\circ}. \tag{6.45}$$

By Propositions 3.9, 2.31, and 6.10 imply that

$$\widehat{S}(\bar{x}) \subset (S(\bar{x}))^{\circ} = (\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}))^{\circ} = N_C(\bar{x}). \tag{6.46}$$

Again, by Proposition 6.10,

$$(\bigcap_{i \in I^{=}} D_i^{=}(\bar{x}))^{\circ} \subset (\bigcap_{i \in I(\bar{x})} D_i^{\leq}(\bar{x}))^{\circ} = N_C(\bar{x}).$$

As $N_C(\bar{x})$ is a closed convex cone, the above relation along with (6.46) leads to

$$\widehat{S}(\bar{x}) + (\bigcap_{i \in I^=} D_i^=(\bar{x}))^\circ \subset N_C(\bar{x}),$$

which together with (6.45) yields the desired condition (6.44). $\square$

In all these discussions, the notion of constraint qualification was not considered. Observe that in Theorem 6.13, instead of the standard KKT optimality conditions, Wolkowicz [112] derived KKT type optimality conditions involving the set $\bigcap_{i \in I^=} D_i^=(\bar{x})$. The system reduces to the standard KKT optimality conditions if $(\bigcap_{i \in I^=} D_i^=(\bar{x}))^\circ = \{0\}$, that is, $F = \{0\}$ in system (6.39) of Lemma 6.11. Similar to the regularization condition of Ben-Tal, Ben-Israel, and Zlobec [7], Wolkowicz [112] introduced the notion of regular point and weakest constraint qualification.

Definition 6.14 A feasible point $\bar{x} \in C$ of (CP) is a *regular point* if for any objective function f, the system (6.39) holds for $F = \{0\}$. A constraint qualification that is satisfied if and only if $\bar{x}$ is a regular point is known as the *weakest constraint qualification.*

For the differentiable case, Gould and Tolle [52, 53] showed that the Abadie constraint qualification, that is,

$$T_C(\bar{x}) = S(\bar{x})$$

is a weakest constraint qualification. Under the differentiability of the functions $g_i,\ i \in I(\bar{x})$, the set $\widehat{S}(\bar{x})$ is closed, which along with the Abadie constraint qualification is equivalent to

$$N_C(\bar{x}) = \widehat{S}(\bar{x}),$$

which is a weakest constraint qualification. For the nonsmooth case, as discussed in Theorem 3.10, the Abadie constraint qualification along with the assumption that $\widehat{S}(\bar{x})$ is closed leads to the standard KKT conditions. In fact, the Abadie constraint qualification is equivalent to the emptiness of the class of badly behaved constraints $I^b(\bar{x})$. We present the result below.

Proposition 6.15 *Let $\bar{x} \in C$. Then $T_C(\bar{x}) = S(\bar{x})$ if and only if $I^b(\bar{x}) = \emptyset$.*

Proof. Suppose that $I^b(\bar{x}) = \emptyset$. Therefore by Proposition 6.10 (iii) and (v), it is obvious that $T_C(\bar{x}) = S(\bar{x})$.

Conversely, let $I^b(\bar{x}) \neq \emptyset$, which implies there exists $i^* \in I^b(\bar{x})$ such that $i^* \in I^=$ and there exists

$$v^* \in (D_{i^*}^{>}(\bar{x}) \cap S(\bar{x})) \backslash cl \bigcap_{i \in I^=} D_i^=(\bar{x}).$$

Again by Proposition 6.10,

$$v^* \notin cl \bigcap_{i \in I^=} D_i^=(\bar{x}) \cap S(\bar{x}) = T_C(\bar{x}),$$

which implies $T_C(\bar{x}) \neq S(\bar{x})$, thereby proving the result. $\square$

Now we illustrate by examples the above result. Consider

$$C = \{x \in \mathbb{R}^n : x^2 \leq 0,\ x \leq 0\}$$

with $g_1(x) = x^2$ and $g_2(x) = x$. Observe that $C = \{0\}$. For $\bar{x} = 0$, $T_C(\bar{x}) = \{0\}$, and $I(\bar{x}) = I^= = \{1, 2\}$. Here,

$$\begin{array}{rcl} S(\bar{x}) & = & \{v \in \mathbb{R} : g_1'(\bar{x}, v) \leq 0,\ g_2'(\bar{x}, v) \leq 0\} \\ & = & \{v \in \mathbb{R} : \langle \nabla g_1(\bar{x}), v\rangle \leq 0,\ \langle \nabla g_2(\bar{x}), v\rangle \leq 0\} \\ & = & \{v \in \mathbb{R} : v \leq 0\}. \end{array}$$

Thus, $T_C(\bar{x}) \neq S(\bar{x})$, thereby showing that the Abadie constraint qualification is not satisfied. Also by the definitions of the cones of directions, we have

$$\begin{array}{l} D_1^>(\bar{x}) = \{v \in \mathbb{R} : v \neq 0\}, \\ D_2^>(\bar{x}) = \{v \in \mathbb{R} : v > 0\}, \\ D_1^=(\bar{x}) = \{0\} = D_2^=(\bar{x}). \end{array}$$

Observe that $I^b(\bar{x}) = \{1\}$, that is, the set of badly behaved constraints is nonempty.

Next let us consider the set

$$C = \{x \in \mathbb{R} : |x| \leq 0,\ x \leq 0\}.$$

Recall from Chapter 3 that the Abadie constraint qualification is satisfied at $\bar{x} = 0$ with $S(\bar{x}) = \{0\}$. Here also, the cones of directions are the same as that of the previous example but now $D_i^>(\bar{x}) \cap S(\bar{x}) = \emptyset$, thereby showing that the set of badly behaved constraints $I^b(\bar{x})$ is empty.

Wolkowicz [112] gave an equivalent characterization of the regular point with Abadie constraint qualification and the set of badly behaved constraints $I^b(\bar{x})$. We state the result below. The proof can be worked out using Theorem 3.10 and Proposition 6.15.

Theorem 6.16 *Consider the convex programming problem (CP) with C given by (3.1) and let $\bar{x} \in C$. Then the following are equivalent:*

(i) $\bar{x}$ is a regular point,

(ii) Abadie constraint qualification holds at $\bar{x}$ and $\widehat{S}(\bar{x})$ is closed,

(iii) $I^b(\bar{x})$ is empty and $\widehat{S}(\bar{x})$ is closed.

In Chapter 3 we derived the optimality conditions not only under the Abadie constraint qualification, but also the Slater constraint qualification. It was observed by Wolkowicz [112] that the Slater constraint qualification is a weakest constraint qualification with respect to the Fritz John optimality condition, which we present below.

Theorem 6.17 *Consider the convex programming problem* (CP) *with* C *given by (3.1). Then the Slater constraint qualification is a weakest constraint qualification.*

Proof. By Definition 6.14, the Slater constraint qualification is a weakest constraint qualification if and only if $\bar{x}$ is a regular point. Consider the Fritz John optimality condition for (CP); that is, if $\bar{x} \in C$ is a point of minimizer of (CP), then there exist $\lambda_i \geq 0,\ i \in \{0\} \cup I(\bar{x})$, not all simultaneously zero such that

$$0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}).$$

Suppose that the Slater constraint qualification is satisfied, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0,\ i = 1, 2, \ldots, m$. We claim that $\lambda_0 \neq 0$. On the contrary, assume that $\lambda_0 = 0$. Then the above condition implies that there exist $\lambda_i \geq 0,\ i \in I(\bar{x})$, not all simultaneously zero, such that

$$0 \in \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}),$$

which implies that there exist $\xi_i \in \partial g_i(\bar{x}),\ i \in I(\bar{x})$, such that

$$0 = \sum_{i \in I(\bar{x})} \lambda_i \xi_i. \tag{6.47}$$

By the convexity of $g_i,\ i \in I(\bar{x})$,

$$\langle \xi_i, \hat{x} - \bar{x} \rangle \leq g_i(\hat{x}) - g_i(\bar{x}) < 0,\ \forall\ \xi_i \in \partial g_i(\bar{x}),$$

which along with the condition (6.47) leads to a contradiction. Thus, $\lambda_0 \neq 0$ and hence can be normalized to one, thereby leading to the KKT optimality conditions.

Observe that the KKT optimality condition holds at $\bar{x}$ if the system

$$\left.\begin{array}{l} 0 \in \partial f(\bar{x}) + \displaystyle\sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}), \\ \lambda_i \geq 0,\ i \in I(\bar{x}), \end{array}\right\}$$

is consistent for any f, which is equivalent to the inconsistency of the system

$$\left.\begin{array}{l} 0 \in \displaystyle\sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}), \\ \lambda_i \geq 0,\ i \in I(\bar{x}),\ \text{not all simultaneously zero.} \end{array}\right\}$$

Thus, the inconsistency of the above system is equivalent to

$$0 \notin cone\ co \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}). \tag{6.48}$$

We claim that the above condition is equivalent to the Slater constraint qualification. Suppose that the condition (6.48) holds. Because $dom\ g_i = \mathbb{R}^n$, $i \in I(\bar{x})$, by Proposition 2.83, $\partial g_i(\bar{x})$ is a nonempty compact set. As $I(\bar{x}) \subset \{1, 2, \ldots, m\}$ is finite, $\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})$ is also nonempty compact. Also, as

$$\bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}) \subset cone\ co \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

the condition (6.48) implies that

$$0 \notin \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}).$$

Invoking Proposition 3.4,

$$cone\ co \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})$$

is a closed set. Invoking the Strict Separation Theorem, Theorem 2.26 (iii), there exists $\bar{d} \in \mathbb{R}^n$ and $\bar{d} \neq 0$ such that

$$\langle z, \bar{d} \rangle < 0, \ \forall\ z \in cone\ co \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}).$$

In particular, for $\xi_i \in \partial g_i(\bar{x}),\ i \in I(\bar{x})$, the above inequality leads to

$$\langle \xi, \bar{d} \rangle < 0.$$

As $dom\ g_i = \mathbb{R}^n,\ i \in I(\bar{x})$, by Theorem 2.79, for $i \in I(\bar{x})$,

$$\max_{\xi_i \in \partial g_i(\bar{x})} \langle \xi_i, \bar{d} \rangle = g_i'(\bar{x}, \bar{d}) < 0,$$

which implies

$$\lim_{\lambda \downarrow 0} \frac{g_i(\bar{x} + \lambda \bar{d}) - g_i(\bar{x})}{\lambda} = \lim_{\lambda \downarrow 0} \frac{g_i(\bar{x} + \lambda \bar{d})}{\lambda} < 0.$$

Therefore, for every $\lambda > 0$,

$$g_i(\bar{x} + \lambda \bar{d}) < 0, \ \forall\ i \in I(\bar{x}). \tag{6.49}$$

For $i \notin I(\bar{x})$, $g_i(\bar{x}) < 0$. Because $dom\ g_i = \mathbb{R}^n,\ i \notin I(\bar{x})$, by Theorem 2.69, $g_i,\ i \notin I(\bar{x})$ is continuous over $\mathbb{R}^n$. Thus, there exists $\bar{\lambda} > 0$ such that

$$g_i(\bar{x} + \bar{\lambda} d) < 0, \ \forall\ d \in \mathbb{R}^n.$$

In particular, for $d = \bar{d}$, the above inequality becomes

$$g_i(\bar{x} + \bar{\lambda}\bar{d}) < 0, \ \forall \ i \notin I(\bar{x}). \tag{6.50}$$

Combining (6.49) and (6.50), for $\bar{x} + \bar{\lambda}\bar{d} \in \mathbb{R}^n$,

$$g_i(\bar{x} + \bar{\lambda}\bar{d}) < 0, \ \forall \ i = 1, 2, \ldots, m,$$

which implies that the Slater constraint qualification holds.

Conversely, suppose that the Slater constraint qualification holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0, \ i \in I$. By Definition 2.77 of subdifferentiability, for any $\xi_i \in \partial g_i(\bar{x}), \ i \in I(\bar{x})$,

$$\langle \xi, \hat{x} - \bar{x} \rangle \leq g_i(\hat{x}) - g_i(\bar{x}) = g_i(\hat{x}) < 0,$$

which implies that

$$\langle z, \hat{x} - \bar{x} \rangle < 0, \ \forall \ z \in \mathit{cone}\ \mathit{co} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}).$$

Therefore, $z \neq 0$ for any $z \in \mathit{cone}\ \mathit{co} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})$, thereby establishing (6.48). Hence, the Slater constraint qualification is a weakest constraint qualification. $\square$

In both these approaches, one makes use of the direction sets to establish optimality conditions in the absence of any constraint qualification for the convex programming problem (CP). More recently, Jeyakumar and Li [69] studied a class of sublinear programming problems involving separable sublinear constraints in the absence of any constraint qualification, which we discuss in the next section.

6.4 Separable Sublinear Case

As already mentioned, the sublinear programming problem considered by Jeyakumar and Li [69] involved separable sublinear constraints. So before moving ahead with the problem, we state the concept of separable sublinear function.

Definition 6.18 A sublinear function $p : \mathbb{R}^n \to \mathbb{R}$ is called a *separable sublinear function* if

$$p(x) = \sum_{j=1}^{n} p_j(x_j)$$

with each $p_j : \mathbb{R} \to \mathbb{R}, \ j = 1, 2, \ldots, n$ being a sublinear function.

The sublinear programming problem studied by Jeyakumar and Li [69] is

$$\min\ p_0(x) \quad \text{subject to} \quad p_i(x) \leq b_i,\ i = 1, 2, \ldots, m \tag{SP}$$

where $p_0 : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, $p_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$, is a separable sublinear function and $b_i \in \mathbb{R},\ i = 1, 2, \ldots, m$. Before establishing the optimality conditions for (SP), we first present the *Farkas' Lemma* derived by Jeyakumar and Li [69]. Farkas' Lemma acts as a tool in the study of optimality conditions for (SP) in the absence of any constraint qualification.

Theorem 6.19 *Consider the sublinear function $\tilde{p}_0 : \mathbb{R}^n \to \mathbb{R}$ and separable sublinear functions $\tilde{p}_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$. Then the following are equivalent:*

(i) $x \in \mathbb{R}^n,\ \tilde{p}_i(x) \leq 0,\ i = 1, 2, \ldots, m \Rightarrow \tilde{p}_0(x) \geq 0$,

(ii) There exist $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, such that

$$\tilde{p}_0(x) + \sum_{i=1}^{m} \lambda_i \tilde{p}_i(x) \geq 0,\ \forall\ x \in \mathbb{R}^n.$$

Proof. Suppose that condition (i) holds. We claim that condition (ii) is also satisfied. On the contrary, assume that (ii) does not hold, which along with the fact that for a real-valued sublinear function $p : \mathbb{R}^n \to \mathbb{R}$, $p(0) = 0$ implies that for any $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, $\bar{x} = 0$ is not a point of minimizer of the unconstrained problem

$$\min\ \tilde{p}_0(x) + \sum_{i=1}^{m} \lambda_i \tilde{p}_i(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

As sublinear functions are a special class of convex functions, the sublinear programming problem (SP) is also a convex programming problem for which the KKT optimality conditions are necessary as well as sufficient for the point of minimizer. Therefore, the KKT optimality condition does not hold at $\bar{x} = 0$, that is,

$$0 \notin \partial(\tilde{p}_0 + \sum_{i=1}^{m} \lambda_i \tilde{p}_i)(0).$$

As *dom* $\tilde{p}_i = \mathbb{R}^n,\ i = 0, 1, \ldots, m$, by Theorem 2.69, $\tilde{p}_i,\ i = 0, 1, \ldots, m$, are continuous on $\mathbb{R}^n$. Applying the Sum Rule, Theorem 2.91, the above condition becomes

$$0 \notin \partial \tilde{p}_0(0) + \sum_{i=1}^{m} \lambda_i \partial \tilde{p}_i(0),$$

thereby implying $\partial \tilde{p}_0(0) \cap (-P) = \emptyset$, where

$$P = \{\sum_{i=1}^{m} \lambda_i \partial \tilde{p}_i(0) : \lambda_i \geq 0,\ i = 1, 2, \ldots, m\}.$$

As $\tilde{p}_i,\ i = 1, 2, \ldots, m$, are separable sublinear functions,

$$\tilde{p}_i(x) = \sum_{j=1}^{n} \tilde{p}_{ij}(x_j),\ i = 1, 2, \ldots, m,$$

where $\tilde{p}_{ij}$ are sublinear functions on $\mathbb{R}$. Thus,

$$P = \{\sum_{i=1}^{m} \lambda_i (\partial \tilde{p}_{i1}(0) \times \partial \tilde{p}_{i2}(0) \times \ldots \times \partial \tilde{p}_{in}(0)) : \lambda_i \geq 0,\ i = 1, 2, \ldots, m\}.$$

As $\tilde{p}_{ij} : \mathbb{R} \to \mathbb{R}$, by Proposition 2.83, $\partial \tilde{p}_{ij}$ is a nonempty convex and compact set in $\mathbb{R}$, that is,

$$\partial \tilde{p}_{ij}(0) = [l_{ij}, u_{ij}],\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$$

for some $l_{ij},\ u_{ij} \in \mathbb{R}$ with $l_{ij} \leq u_{ij}$. Therefore,

$$\begin{aligned} P &= \{\sum_{i=1}^{m} \lambda_i\ ([l_{i1}, u_{i1}] \times [l_{i2}, u_{i2}] \times \ldots \times [l_{in}, u_{in}]) : \lambda_i \geq 0,\ i = 1, 2, \ldots, m\} \\ &= cone\ co \bigcup_{i=1}^{m}\ ([l_{i1}, u_{i1}] \times [l_{i2}, u_{i2}] \times \ldots \times [l_{in}, u_{in}]) \\ &= cone\ co\ \{\bigcup_{i=1}^{m}\ (a_{i1}, a_{i2}, \ldots, a_{in}) : a_{ij} \in [l_{ij}, u_{ij}],\ j = 1, 2, \ldots, n\}. \end{aligned}$$

Note that $[l_{i1}, u_{i1}] \times [l_{i2}, u_{i2}] \times \ldots \times [l_{in}, u_{in}]$ forms a convex polytope in $\mathbb{R}^n$ with 2^n vertices denoted by

$$(v_{ir}) = (v^r_{i1}, v^r_{i2}, \ldots, v^r_{in}),\ i = 1, 2, \ldots, m,\ r = 1, 2, \ldots, 2^n,$$

where $v^r_{ij} \in \{l_{ij}, u_{ij}\}$. Also, any element in the polytope can be expressed as the convex combination of the vertices. Therefore,

$$(a_{i1}, a_{i2}, \ldots, a_{in}) = co\{(v_{i1}), (v_{i2}), \ldots, (v_{i2^n})\},$$

which implies that

$$P = cone\ co\ \bigcup_{i=1}^{m}\{(v_{i1}), (v_{i2}), \ldots, (v_{i2^n})\}.$$

Hence, P is a finitely generated convex cone and thus, by Proposition 2.44, is a polyhedral cone that is always closed.

As sublinear functions are convex, by Proposition 2.83, $\partial \tilde{p}_0(0)$ is a compact convex set and, from the above discussion, P is a closed convex cone. Therefore, by the Strict Separation Theorem, Theorem 2.26 (iii), there exists $\alpha \in \mathbb{R}^n$ with $\alpha \neq 0$ such that

$$\sup_{\xi \in \partial \tilde{s}_0(0)} \langle \alpha, \xi \rangle < \inf_{\xi \in -P} \langle \alpha, \xi \rangle = -\sup_{\xi \in P} \langle \alpha, \xi \rangle. \tag{6.51}$$

Consider

$$\begin{aligned}
\sup_{\xi \in P} \langle \alpha, \xi \rangle &= \sup \left\{ \langle \alpha, \xi \rangle : \xi \in \{\sum_{i=1}^{m} \lambda_i \partial \tilde{p}_i(0) : \lambda_i \geq 0,\ i = 1, 2, \ldots, m\} \right\} \\
&\geq \sup\{\langle \alpha, \xi \rangle : \xi \in \sum_{i=1}^{m} \lambda_i \partial \tilde{p}_i(0)\},\ \forall\, \lambda \in \mathbb{R}^m_+ \\
&= \sum_{i=1}^{m} \lambda_i \tilde{p}_i(\alpha),\ \forall\, \lambda \in \mathbb{R}^m_+.
\end{aligned}$$

From the preceding relation and condition (6.51),

$$\tilde{p}_0(\alpha) = \sup_{\xi \in \partial \tilde{p}_0(0)} \langle \alpha, \xi \rangle < -\sup_{\xi \in P} \langle \alpha, \xi \rangle \leq -\sum_{i=1}^{m} \lambda_i \tilde{p}_i(\alpha),\ \forall\, \lambda \in \mathbb{R}^m_+, \tag{6.52}$$

which implies

$$\sum_{i=1}^{m} \lambda_i \tilde{p}_i(\alpha) < -\tilde{p}_0(\alpha),\ \forall\, \lambda \in \mathbb{R}^m_+.$$

This inequality holds for every $\lambda \in \mathbb{R}^m_+$ if $\tilde{s}_i(\alpha) \leq 0,\ i = 1, 2, \ldots, m$. Otherwise, if for some $i \in \{1, 2, \ldots, m\}$, $\tilde{s}_i(\alpha) > 0$, then choosing the corresponding $\lambda_i \to +\infty$, we arrive at a contradiction. Also, as P is a closed convex cone, from (6.52),

$$\tilde{p}_0(\alpha) < -\sup_{\xi \in P} \langle \alpha, \xi \rangle \leq 0.$$

Therefore, for $\alpha \in \mathbb{R}^n$,

$$\tilde{p}_0(\alpha) < 0 \quad \text{and} \quad \tilde{p}_i(\alpha) \leq 0,\ i = 1, 2, \ldots, m,$$

which contradicts (i). Thus condition (ii) is satisfied.

Conversely, suppose that condition (ii) holds, which implies for some $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$,

$$-\sum_{i=1}^{m} \lambda_i \tilde{p}_i(x) \leq \tilde{p}_0(x),\ \forall\, x \in \mathbb{R}^n.$$

If for some $x \in \mathbb{R}^n$, $\tilde{p}_i(x) \leq 0$, $i = 1, 2, \ldots, m$, from the above inequality $\tilde{p}_0(x) \geq 0$, thereby establishing condition (i) and hence the desired result. $\square$

We end this chapter by deriving the constraint qualification free optimality condition for the sublinear programming problem (SP) from Jeyakumar and Li [69].

Theorem 6.20 *Consider the sublinear programming problem (SP). Then $\bar{x}$ is a minimizer of (SP) if and only if there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, such that*

$$0 \in \partial p_0(0) + \sum_{i=1}^{m} \lambda_i \partial p_i(0) \quad \text{and} \quad p_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i b_i = 0.$$

Proof. Observe that $\bar{x}$ is a minimizer of (SP) if and only if

$$p_i(x) - b_i \leq 0, \; i = 1, 2, \ldots, m \quad \Longrightarrow \quad p_0(x) - p_0(\bar{x}) \geq 0. \tag{6.53}$$

But Theorem 6.19 cannot be applied directly to the above system as the theorem is for the system involving sublinear functions, whereas here neither $p_i(x) - b_i$, $i = 1, 2, \ldots, m$, nor $p_0(x) - p_0(\bar{x})$ is positively homogeneous and hence not sublinear functions. So define $\tilde{p}_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 0, 1, \ldots, m$, as

$$\tilde{p}_0(x, t) = p_0(x) - t p_0(\bar{x}) \quad \text{and} \quad \tilde{p}_i(x, t) = p_i(x) - t b_i, \; i = 1, 2, \ldots, m.$$

Because p_i, $i = 1, 2, \ldots, m$, are separable sublinear functions on $\mathbb{R}^n$, $\tilde{p}_i$, $i = 1, 2, \ldots, m$, are also separable sublinear functions along with the sublinearity of $\tilde{p}_0$ on $\mathbb{R}^n \times \mathbb{R}$. Now consider the system

$$\tilde{p}_i(x, t) \leq 0, \; i = 1, 2, \ldots, m \quad \Longrightarrow \quad \tilde{p}_0(x, t) \geq 0. \tag{6.54}$$

This system is in the desired form needed for the application of Farkas' Lemma, Theorem 6.19. To establish the result, we will first establish the equivalence between the systems (6.53) and (6.54).

Suppose that the system (6.53) holds. We claim that (6.54) is also satisfied. On the contrary, assume that the system (6.54) does not hold, which implies there exists $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\tilde{p}_0(\tilde{x}, \tilde{t}) < 0 \quad \text{and} \quad \tilde{p}_i(\tilde{x}, \tilde{t}) \leq 0, \; i = 1, 2, \ldots, m.$$

For $\tilde{t} > 0$, by positive homogeneity of the sublinear function and the construction of $\tilde{p}_i$, $i = 0, 1, \ldots, m$,

$$\begin{aligned} p_0(\tilde{x}/\tilde{t}) - p_0(\bar{x}) &= \tilde{p}_0(\tilde{x}/\tilde{t}, 1) < 0, \\ p_i(\tilde{x}/\tilde{t}) - b_i &= \tilde{p}_i(\tilde{x}/\tilde{t}, 1) \leq 0, \quad i = 1, 2, \ldots, m, \end{aligned}$$

thereby contradicting (6.53).

Now, in particular, taking $\tilde{t} = 0$,

$$p_0(\tilde{x}) = \tilde{p}_0(\tilde{x}, 0) < 0 \quad \text{and} \quad p_i(\tilde{x}) = \tilde{p}_i(\tilde{x}, 0) \leq 0, \ i = 1, 2, \ldots, m.$$

For $t > 0$, consider $\bar{x} + t\tilde{x} \in \mathbb{R}^n$. Therefore, by the feasibility of $\bar{x}$ for (SP) and the above condition,

$$\begin{array}{c} p_0(\bar{x} + t\tilde{x}) - p_0(\bar{x}) \leq tp_0(\tilde{x}) < 0, \\ p_i(\bar{x} + t\tilde{x}) - b_i \leq p_i(\bar{x}) - b_i + tp_i(\tilde{x}) \leq 0, \quad i = 1, 2, \ldots, m, \end{array}$$

which is again a contradiction of the system (6.53).

If $\tilde{t} < 0$, then by construction of $\tilde{p}_i, \ i = 0, 1, \ldots, m$,

$$p_0(\tilde{x}) - \tilde{t}p_0(\bar{x}) < 0 \quad \text{and} \quad p_i(\tilde{x}) - \tilde{t}b_i \leq 0, \ i = 1, 2, \ldots, m.$$

Consider $\tilde{x} + (-\tilde{t} + 1)\bar{x} \in \mathbb{R}^n$. By the sublinearity of $p_i, \ i = 0, 1, \ldots, m$,

$$\begin{array}{rcl} p_0(\tilde{x} + (-\tilde{t} + 1)\bar{x}) & \leq & p_0(\tilde{x}) + (-\tilde{t} + 1)p_0(\bar{x}) \\ & \leq & \tilde{t}p_0(\bar{x}) + (-\tilde{t} + 1)p_0(\bar{x}) = p_0(\bar{x}), \end{array}$$

and

$$\begin{array}{rcl} p_i(\tilde{x} + (-\tilde{t} + 1)\bar{x}) & \leq & p_i(\tilde{x}) + (-\tilde{t} + 1)p_i(\bar{x}) \\ & \leq & \tilde{t}b_i + (-\tilde{t} + 1)b_i = b_i, \ i = 1, 2, \ldots, m, \end{array}$$

which contradicts (6.53). Thus from all three cases, it is obvious that our assumption is wrong and hence the system (6.54) holds.

Conversely, taking $t = 1$ in system (6.54) yields (6.53). Hence, both systems (6.53) and (6.54) are equivalent. Applying Farkas' Lemma, Theorem 6.19, for the sublinear systems to (6.54), there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, such that

$$\tilde{p}_0(x, t) + \sum_{i=1}^{m} \lambda_i \tilde{p}_i(x, t) \geq 0, \ \forall \ (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

which implies $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ is a point of minimizer of the unconstrained problem

$$\min \ \tilde{p}_0(x, t) + \sum_{i=1}^{m} \lambda_i \tilde{p}_i(x, t) \quad \text{subject to} \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

By the KKT optimality condition for the unconstrained problem, Theorem 2.89,

$$\begin{array}{rcl} (0, 0) & \in & \partial(\tilde{p}_0 + \sum_{i=1}^{m} \lambda_i \tilde{p}_i)(0, 0) \\ & = & \partial(p_0 + \sum_{i=1}^{m} \lambda_i p_i)(0) \times \nabla(tp_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i t b_i)(0), \end{array}$$

where the subdifferential is with respect to x and the gradient with respect to t. Therefore, a componentwise comparison leads to

$$0 \in \partial(p_0 + \lambda_i p_i)(0) \quad \text{and} \quad p_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i b_i = 0.$$

As *dom* $p_i = \mathbb{R}^n$, $i = 0, 1, \ldots, m$, by Theorem 2.69, p_i, $i = 0, 1, \ldots, m$, are continuous on $\mathbb{R}^n$. Thus, by the Sum Rule, Theorem 2.91, the first relation yields

$$0 \in \partial p_0(0) + \sum_{i=1}^{m} \lambda_i \partial p_i(0),$$

thereby establishing the desired result. $\square$

Chapter 7

Sequential Optimality Conditions

7.1 Introduction

In this chapter we are going to look into a completely different approach to develop optimality conditions in convex programming. These optimality conditions, called *sequential optimality conditions*, can hold without any qualification and thus both from a theoretical as well as practical point of view this is of great interest. To the best of our knowledge, this approach was initiated by Thibault [108]; Jeyakumar, Rubinov, Glover, and Ishizuka [70]; and Jeyakumar, Lee, and Dinh [68]. Unlike the approach of direction sets in Chapter 6, in the sequential approach one needs calculus rules for subdifferentials and ε-subdifferentials, namely the Sum Rule and the Chain Rule. As the name itself suggests, the sequential optimality conditions are established as a sequence of subdifferentials at neighborhood points as in the work of Thibault [108] or sequence of ε-subdifferentials at the exact point as in the study of Jeyakumar and collaborators [68, 70]. Thibault [108] used the approach of sequential subdifferential calculus rules while Jeyakumar and collaborators [68, 70] used the approach of *epigraphs of conjugate functions* to study the sequential optimality conditions extensively. In both these approaches, the convex programming problem involved cone constraints and abstract constraints. But keeping in sync with the convex programming problem (CP) studied in this book, we consider the feasible set C involving convex inequalities. The reader must have realized the central role of the Slater constraint qualification in the study of optimality and duality in optimization. However, as we have seen, the Slater constraint qualification can fail even for very simple problems. The failure of the Slater constraint qualification was overcome by the development of the so-called *closed cone constraint qualification.* It is a geometric qualification that uses the Fenchel conjugate of the constraint function. We will study this qualification condition in detail.

7.2 Sequential Optimality: Thibault's Approach

We first discuss the approach due to Thibault [108]. As already mentioned, he makes use of sequential subdifferential rules in his work. As one will observe, the Sum Rule and the Chain Rule are expressed in terms of the sequence of subdifferentials at neighborhood points. We present below the Sum Rule from Thibault [108] which involves the application of the Sum Rule given by Hiriart-Urruty and Phelps [64].

Theorem 7.1 *(Sequential Sum Rule) Consider two proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$. Then for any $\bar{x} \in dom\ \phi_1 \cap dom\ \phi_2$,*

$$\partial(\phi_1 + \phi_2)(\bar{x}) = \limsup_{x_i \to^{\phi_i - \langle , \rangle} \bar{x}} \{\partial\phi_1(x_1) + \partial\phi_2(x_2)\},$$

where $\limsup_{x_i \to^{\phi_i - \langle , \rangle} \bar{x}} \{\partial\phi_1(x_1) + \partial\phi_2(x_2)\}$ denotes the set of all limits $\lim_{k\to\infty}(\xi_1^k + \xi_2^k)$ for which there exists $x_i^k \to \bar{x}$, $i = 1, 2$ such that $\xi_i^k \in \partial\phi_i(x_i^k)$, $i = 1, 2$, and

$$\phi_i(x_i^k) - \langle \xi_i^k, x_i^k - \bar{x}\rangle \to \phi_i(\bar{x}),\ i = 1, 2. \tag{7.1}$$

Proof. Suppose that $\xi \in \partial(\phi_1 + \phi_2)(\bar{x})$. By Theorem 2.120,

$$\xi \in \bigcap_{k\in\mathbb{N}} cl\{\partial_{1/k}\phi_1(\bar{x}) + \partial_{1/k}\phi_2(\bar{x})\},$$

which implies for every $k \in \mathbb{N}$,

$$\xi \in cl\ \{\partial_{1/k}\phi_1(\bar{x}) + \partial_{1/k}\phi_2(\bar{x})\}.$$

From Definition 2.12 of the closure of a set, for every $k \in \mathbb{N}$,

$$\xi \in \partial_{1/k}\phi_1(\bar{x}) + \partial_{1/k}\phi_2(\bar{x}) + \frac{1}{k}\mathbb{B}.$$

Therefore, there exists $\xi_i'^k \in \partial_{1/k}\phi_i(\bar{x}),\ i = 1, 2$, and $b^k \in \mathbb{B}$ such that

$$\xi = \xi_1'^k + \xi_2'^k + \frac{1}{k}b^k. \tag{7.2}$$

Applying the modified version of the Brøndsted–Rockafellar Theorem, Theorem 2.114, there exist $x_i^k \in \mathbb{R}^n$ and $\xi_i^k \in \partial\phi_i(x_i^k)$ such that for $i = 1, 2$,

$$\|x_i^k - \bar{x}\| \leq \frac{1}{\sqrt{k}},\ \ \|\xi_i^k - \xi_i'^k\| \leq \frac{1}{\sqrt{k}},\ \ |\phi_i(x_i^k) - \langle \xi_i^k, x_i^k - \bar{x}\rangle - \phi_i(\bar{x})| \leq \frac{2}{k}, \tag{7.3}$$

which implies $\xi_i'^k = \xi_i^k + 1/\sqrt{k}\ b_i^k$ for some $b_i^k \in \mathbb{B}$ for $i = 1, 2$. Therefore, the condition (7.2) becomes

$$\xi = \xi_1^k + \xi_2^k + (\frac{1}{k}b^k + \frac{1}{\sqrt{k}}b_1^k + \frac{1}{\sqrt{k}}b_2^k),$$

that is,

$$\xi = \lim_{k\to\infty}(\xi_1^k + \xi_2^k),$$

which along with (7.3) yields the desired inclusion.

Conversely, suppose that

$$\xi \in \limsup_{x_i \to^{\phi_i - \langle,\rangle} \bar{x}} \{\partial\phi_1(x_1) + \partial\phi_2(x_2)\},$$

which implies for $i = 1, 2$, there exist $x_i^k \to \bar{x}$, $\xi_i^k \in \partial\phi_i(x_i^k)$ satisfying $\phi_i(x_i^k) - \langle \xi_i^k, x_i^k - \bar{x}\rangle \to \phi_i(\bar{x})$ and

$$\xi = \lim_{k\to\infty}(\xi_1^k + \xi_2^k).$$

As $\xi_i^k \in \partial\phi_i(x_i^k),\ i = 1, 2,$

$$\langle \xi_i^k, x - x_i^k\rangle \leq \phi_i(x) - \phi_i(x_i^k),\ \forall\ x \in \mathbb{R}^n.$$

Also, for every $x \in \mathbb{R}^n$,

$$\begin{array}{rcl} \langle \xi_i^k, x - \bar{x}\rangle & = & \langle \xi_i^k, x - x_i^k\rangle + \langle \xi_i^k, x_i^k - \bar{x}\rangle \\ & \leq & \phi_i(x) - \phi_i(x_i^k) + \langle \xi_i^k, x_i^k - \bar{x}\rangle, \quad i = 1, 2, \end{array}$$

thereby yielding

$$\begin{aligned} \langle \xi_1^k + \xi_2^k, x - \bar{x}\rangle \leq \phi_1(x) + \phi_2(x) - \phi_1(x_1^k) - \phi_2(x_2^k) \\ + \langle \xi_1^k, x_1^k - \bar{x}\rangle + \langle \xi_2^k, x_2^k - \bar{x}\rangle \end{aligned}$$

for every $x \in \mathbb{R}^n$. Taking the limit as $k \to +\infty$ and using the condition (7.1), the above inequality reduces to

$$\langle \xi, x - \bar{x}\rangle \leq (\phi_1 + \phi_2)(x) - (\phi_1 + \phi_2)(\bar{x}),\ \forall\ x \in \mathbb{R}^n,$$

which implies $\xi \in \partial(\phi_1 + \phi_2)(\bar{x})$, thereby establishing the requisite result. $\square$

Using a very different assumption, the Moreau–Rockafellar Sum Rule, Theorem 2.91, was obtained by Thibault [108].

Corollary 7.2 *Consider two proper lsc convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$. If*

$$0 \in core(dom\ \phi_1 - dom\ \phi_2),$$

then for every $\bar{x} \in dom\ \phi_1 \cap dom\ \phi_2$,

$$\partial(\phi_1 + \phi_2)(\bar{x}) = \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}).$$

Proof. By Definition 2.77 of the subdifferentials, it is easy to observe that the inclusion

$$\partial(\phi_1 + \phi_2)(\bar{x}) \supset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}) \tag{7.4}$$

always holds true.

To prove the result, we will show the reverse inclusion in relation (7.4). Consider $\xi \in \partial(\phi_1 + \phi_2)(\bar{x})$. Then by Theorem 7.1, for $i = 1, 2$, there exist $x_i^k \to \bar{x}$ and $\xi_i^k \in \partial\phi_i(x_i^k)$ such that

$$\xi = \lim_{k\to\infty} (\xi_1^k + \xi_2^k) \quad \text{and} \quad \gamma_i^k = \phi_i(x_i^k) - \langle \xi_i^k, x_i^k - \bar{x}\rangle \to \phi_i(\bar{x}). \tag{7.5}$$

Denote $\xi^k = \xi_1^k + \xi_2^k$. As $0 \in core(dom\ \phi_1 - dom\ \phi_2)$, by Definition 2.17, for any $y \in \mathbb{R}^n$ and $y \neq 0$, there exist $\alpha > 0$ and $x_i \in dom\ \phi_i,\ i = 1, 2$, such that $\alpha y = x_1 - x_2$. By the convexity of $\phi_i,\ i = 1, 2$ along with (7.5),

$$\begin{array}{rcl}
\langle \xi_1^k, \alpha y\rangle & = & \langle \xi_1^k, x_1 - x_1^k\rangle + \langle \xi_1^k, x_1^k - \bar{x}\rangle + \langle \xi_1^k, \bar{x} - x_2\rangle \\
& \leq & \phi_1(x_1) - \phi(x_1^k) + \langle \xi_1^k, x_1^k - \bar{x}\rangle + \langle \xi_1^k, \bar{x} - x_2\rangle \\
& = & \phi_1(x_1) - \gamma_1^k + \langle \xi^k, \bar{x} - x_2\rangle + \langle \xi_2^k, x_2 - x_2^k\rangle + \langle \xi_2^k, x_2^k - \bar{x}\rangle \\
& \leq & \phi_1(x_1) - \gamma_1^k + \langle \xi^k, \bar{x} - x_2\rangle + \phi_2(x_2) - \phi_2(x_2^k) + \langle \xi_2^k, x_2^k - \bar{x}\rangle \\
& = & (\phi_1(x_1) - \gamma_1^k) + (\phi_2(x_2) - \gamma_2^k) + \langle \xi^k, \bar{x} - x_2\rangle.
\end{array}$$

As the limit $k \to +\infty$, using the conditions (7.5),

$$\begin{aligned}
&(\phi_1(x_1) - \gamma_1^k) + (\phi_2(x_2) - \gamma_2^k) + \langle \xi^k, \bar{x} - x_2\rangle \\
&\qquad\qquad \to (\phi_1(x_1) - \phi_1(\bar{x})) + (\phi_2(x_2) - \phi_2(\bar{x})) + \langle \xi, \bar{x} - x_2\rangle.
\end{aligned}$$

Therefore,

$$\langle \xi_1^k, y\rangle \leq \frac{M_y}{\alpha},\ \forall\ k \in \mathbb{N},$$

that is, $\{\langle \xi_1^k, y\rangle\}$ is bounded above by $\dfrac{M_y}{\alpha}$ which is independent of k. Similarly, the sequence $\{\langle \xi_1^k, -y\rangle\}$ is bounded above. In particular, taking $y = e_i$, $i = 1, 2, \ldots, n$, where e_i is a vector in $\mathbb{R}^n$ with i-th component 1 and all other zeroes,

$$\|\xi_1^k\|_\infty = \max_{i=1,2,\ldots,n} |\langle \xi_1^k, e_i\rangle| \leq \max_{i=1,2,\ldots,n} |M_i|.$$

Thus, $\{\xi_1^k\}$ is a bounded sequence. As $\xi_1^k + \xi_2^k \to \xi$, $\{\xi_2^k\}$ is also a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, the sequences $\{\xi_i^k\},\ i = 1, 2$, have a convergent subsequence. Without loss of generality, assume that $\xi_i^k \to \xi_i,\ i = 1, 2$, such that $\xi_1 + \xi_2 = \xi$. By Theorem 2.84, $\xi_i \in \partial\phi_i(\bar{x}),\ i = 1, 2$, thereby yielding

$$\partial(\phi_1 + \phi_2)(\bar{x}) \subset \partial\phi_1(\bar{x}) + \partial\phi_2(\bar{x}),$$

which along with (7.4) leads to the desired result. □

Consider the convex optimization problem

$$\min\ f(x) \quad \text{subject to} \quad x \in C, \tag{CP}$$

where $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a proper lsc convex function and $C \subset \mathbb{R}^n$ is a closed convex set. We shall now provide the sequential optimality condition for (CP) as an application to Theorem 7.1.

Theorem 7.3 *Consider the convex optimization problem* (CP) *where* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *is an extended-valued proper lsc convex function. Then* $\bar{x}$ *is a point of minimizer of* (CP) *if and only if there exist* $x_i^k \to \bar{x}$, $i = 1, 2$, *with* $\xi_1^k \in \partial f(x_1^k)$ *and* $\xi_2^k \in N_C(x_2^k)$ *such that*

$$\xi_1^k + \xi_2^k \to 0, \quad f(x_1^k) - \langle \xi_1^k, x_1^k - \bar{x} \rangle \to f(\bar{x}) \quad \text{and} \quad \langle \xi_2^k, x_2^k - \bar{x} \rangle \to 0.$$

Proof. Observe that (CP) is equivalent to the unconstrained problem

$$\min\ (f + \delta_C)(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

By the optimality condition for the unconstrained programming problem, Theorem 2.89, $\bar{x}$ is a minimum to (CP) if and only if

$$0 \in \partial(f + \delta_C)(\bar{x}).$$

Applying Theorem 7.1, there exist sequence $\{x_i^k\} \subset \mathbb{R}^n$ with $x_i^k \to \bar{x}$, $i = 1, 2$, $\xi_1^k \in \partial f(x_1^k)$ and $\xi_2^k \in \delta_C(x_2^k) = N_C(x_2^k)$ satisfying

$$f(x_1^k) - \langle \xi_1^k, x_1^k - \bar{x} \rangle \to f(\bar{x}) \quad \text{and} \quad \langle \xi_2^k, x_2^k - \bar{x} \rangle \to 0$$

such that

$$\lim_{k \to \infty} (\xi_1^k + \xi_2^k) = 0,$$

thereby yielding a sequential optimality condition. □

It is important to note that the conditions on the problem data of (CP) was minimal. The importance of the Sequential Sum Rule becomes clear because under the assumptions in (CP), it is not obvious whether the qualification conditions needed to apply the exact Sum Rule holds or not.

For the convex programming problem (CP) with C given by (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\},$$

it was discussed in Chapter 3 how the normal cones could be explicitly expressed in terms of the subdifferentials of the constraint functions g_i, $i = 1, 2, \ldots, m$, in presence of the Slater constraint qualification. But if the Slater constraint qualification is not satisfied, then how would one explicitly compute the normal cone. For that we first present the sequential Chain Rule from Thibault [108] in a finite dimensional setting, a corollary to which plays a pivotal role in deriving the sequential optimality conditions, when C is explicitly given by convex inequalities. Note that in the following we will consider a vector-valued convex function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$. This means that each component function of Φ is a real-valued convex function on $\mathbb{R}^n$. Equivalently, Φ is *convex* if for every $x_1, x_2 \in \mathbb{R}^n$ and for every $\lambda \in [0,1]$,

$$(1-\lambda)\Phi(x_1) + \lambda\Phi(x_2) - \Phi((1-\lambda)x_1 + \lambda x_2) \in \mathbb{R}^m_+. \tag{7.6}$$

The *epigraph* of Φ, *epi* Φ, is defined as

$$epi\ \Phi = \{(x,\mu) \in \mathbb{R}^n \times \mathbb{R}^m : \mu \in \Phi(x) + \mathbb{R}^m_+\}.$$

A function $\phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ is said to be *nondecreasing* on a set $F \subset \mathbb{R}^m$ if for every $y_1, y_2 \in F$,

$$\phi(y_1) \leq \phi(y_2) \quad \text{whenever} \quad y_2 - y_1 \in \mathbb{R}^m_+.$$

Consider a vector-valued convex function Φ and let ϕ be nondecreasing convex function on $\Phi(\mathbb{R}^n) + \mathbb{R}^m_+$. By the convexity of Φ, for every $x_1, x_2 \in \mathbb{R}^n$ and for every $\lambda \in [0,1]$, the condition (7.6) leads to

$$(1-\lambda)\Phi(x_1) + \lambda\Phi(x_2) \in \Phi((1-\lambda)x_1 + \lambda x_2) + \mathbb{R}^m_+ \subset \Phi(\mathbb{R}^n) + \mathbb{R}^m_+.$$

Also,

$$\Phi((1-\lambda)x_1 + \lambda x_2) \in \Phi(\mathbb{R}^n) \subset \Phi(\mathbb{R}^n) + \mathbb{R}^m_+.$$

As ϕ is a nondecreasing function on $\Phi(\mathbb{R}^n) + \mathbb{R}^m_+$, by the convexity of Φ, (7.6) implies that

$$\phi(\Phi((1-\lambda)x_1 + \lambda x_2)) \leq \phi((1-\lambda)\Phi(x_1) + \lambda\Phi(x_2)).$$

By the convexity of ϕ, for every $x_1, x_2 \in \mathbb{R}^n$,

$$\phi(\Phi((1-\lambda)x_1 + \lambda x_2)) \leq (1-\lambda)\phi(\Phi(x_1)) + \lambda\phi(\Phi(x_2)),\ \forall\ \lambda \in [0,1],$$

that is, for every $\lambda \in [0,1]$,

$$(\phi \circ \Phi)((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)(\phi \circ \Phi)(x_1) + \lambda(\phi \circ \Phi)(x_2).$$

Hence, $(\phi \circ \Phi)$ is a convex function.

Below we present the Sequential Chain Rule from Thibault [108].

Theorem 7.4 *(Sequential Chain Rule) Consider a vector-valued convex function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ and a proper lsc convex function $\phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ that is nondecreasing over $\Phi(\mathbb{R}^n) + \mathbb{R}^m_+$. Then for $\bar{y} = \Phi(\bar{x}) \in dom\ \phi$, $\xi \in \partial(\phi \circ \Phi)(\bar{x})$ if and only if there exist $x_k \to \bar{x}$, $y_k \to \bar{y}$, $\xi_k \to \xi$, $\tau_k \to 0$, $y'_k \in \Phi(x_k) + \mathbb{R}^m_+$ with $y'_k \to \Phi(\bar{x})$ and $\eta_k \in \mathbb{R}^m_+$ such that*

$$\eta_k + \tau_k \in \partial\phi(y_k), \quad \xi_k \in \partial(\eta_k\Phi)(x_k), \quad \langle \eta_k, y'_k \rangle \to \langle \eta_k, \Phi(x_k) \rangle$$

and

$$\phi(y_k) - \langle \eta_k, y_k - \bar{y} \rangle \to \phi(\bar{y}) \quad \text{and} \quad \langle \eta_k, \Phi(x_k) - \bar{y} \rangle \to 0.$$

Proof. Define $\phi_1(x, y) = \phi(y)$ and $\phi_2(x, y) = \delta_{epi\ \Phi}(x, y)$. We claim that

$$\xi \in \partial(\phi \circ \Phi)(\bar{x}) \quad \text{if and only if} \quad (\xi, 0) \in \partial(\phi_1 + \phi_2)(\bar{x}, \bar{y}). \tag{7.7}$$

Suppose that $\xi \in \partial(\phi \circ \Phi)(\bar{x})$, which by Definition 2.77 of the subdifferential implies that

$$\phi(\Phi(x)) - \phi(\Phi(\bar{x})) = (\phi \circ \Phi)(x) - (\phi \circ \Phi)(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n.$$

Consider $(x, y) \in epi\ \Phi$, which implies $y - \Phi(x) \in \mathbb{R}^m_+$, that is, $y \in \Phi(x) + \mathbb{R}^m_+$. Because ϕ is nondecreasing over $\Phi(\mathbb{R}^n) + \mathbb{R}^m_+$, $\phi(y) \geq \phi(\Phi(x))$ for every $(x, y) \in epi\ \Phi$. Therefore, the above condition leads to

$$\phi(y) - \phi(\bar{y}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ (x, y) \in epi\ \Phi,$$

where $\bar{y} = \Phi(\bar{x})$. From the definition of ϕ_1 and ϕ_2, for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ the above condition leads to

$$\phi_1(x, y) + \phi_2(x, y) - \phi_1(\bar{x}, \bar{y}) - \phi_2(\bar{x}, \bar{y}) \geq \langle \xi, x - \bar{x} \rangle + \langle 0, y - \bar{y} \rangle,$$

thereby implying that $(\xi, 0) \in \partial(\phi_1 + \phi_2)(\bar{x}, \bar{y})$.

Conversely, suppose that $(\xi, 0) \in \partial(\phi_1 + \phi_2)(\bar{x}, \bar{y})$, which by the definition of subdifferential implies that for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$(\phi_1 + \phi_2)(x, y) - (\phi_1 + \phi_2)(\bar{x}, \bar{y}) \geq \langle \xi, x - \bar{x} \rangle + \langle 0, y - \bar{y} \rangle.$$

The above inequality holds in particular for every $(x, y) \in epi\ \Phi$. As $(x, \Phi(x)) \in epi\ \Phi$, the above inequality reduces to

$$\phi(\Phi(x)) - \phi(\Phi(\bar{x})) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\ x \in \mathbb{R}^n,$$

which implies that $\xi \in \partial(\phi \circ \Phi)(\bar{x})$, thereby establishing our claim (7.7).

Now by Theorem 7.1,

$$(0, \beta_k) + (\xi_k, \theta_k) \to (\xi, 0),$$

where

$$\begin{array}{ll}
\beta_k \in \partial\phi(y_k), & (\xi_k, \theta_k) \in \partial\delta_{epi\ \Phi}(x_k, y'_k), \\
y_k \to \bar{y}, & (x_k, y'_k) \to (\bar{x}, \bar{y}), \\
\multicolumn{2}{l}{\phi(y_k) - \langle \beta_k, y_k - \bar{y} \rangle \to \phi(\bar{y}),} \\
\multicolumn{2}{l}{\phi_2(x_k, y'_k) - \langle \theta_k, y'_k - \bar{y} \rangle - \langle \xi_k, x_k - \bar{x} \rangle \to \phi_2(\bar{x}, \bar{y}) = 0.}
\end{array}$$

Set $\theta_k = -\eta_k$ and define $\tau_k = \beta_k - \eta_k$. Observe that $\xi_k \to \xi$ and $\tau_k = \beta_k + \theta_k \to 0$. The preceding facts can thus be written as

$$(0, \eta_k + \tau_k) + (\xi_k, -\eta_k) \to (\xi, 0),$$

with

$$\begin{aligned}
&\eta_k + \tau_k \in \partial\phi(y_k), && (\xi_k, -\eta_k) \in \partial\delta_{epi\ \Phi}(x_k, y_k'), \\
&y_k \to \bar{y}, && (x_k, y_k') \to (\bar{x}, \bar{y}), \\
&\phi(y_k) - \langle \eta_k + \tau_k, y_k - \bar{y}\rangle \to \phi(\bar{y}), && \qquad (7.8) \\
&\phi_2(x_k, y_k') + \langle \eta_k, y_k' - \bar{y}\rangle - \langle \xi_k, x_k - \bar{x}\rangle \to \phi_2(\bar{x}, \bar{y}) = 0. && \qquad (7.9)
\end{aligned}$$

As $(x_k, y_k') \in epi\ \Phi$, $\phi_2(x_k, y_k') = 0$, which along with $\xi_k \to \xi$ and $x_k \to \bar{x}$ reduces (7.9) to

$$\langle \eta_k, y_k' - \bar{y}\rangle \to 0. \qquad (7.10)$$

Also, $(\xi_k, -\eta_k) \in \partial\delta_{epi\ \Phi}(x_k, y_k')$ implies that

$$\langle \xi_k, x - x_k\rangle - \langle \eta_k, y - y_k'\rangle \leq 0,\ \forall\ (x, y) \in epi\ \Phi. \qquad (7.11)$$

Observe that $(x_k, y_k') \in epi\ \Phi$, which implies that

$$y_k' - \Phi(x_k) \in \mathbb{R}^m_+.$$

Therefore, for any $y' \in \mathbb{R}^m_+$,

$$y' + y_k' - \Phi(x_k) \in \mathbb{R}^m_+,$$

that is, $(x_k, y' + y_k') \in epi\ \Phi$. In particular, taking $x = x_k$ and setting $y = y_k' + y'$ for any $y' \in \mathbb{R}^m_+$ in (7.11) yields

$$\langle \eta_k, y'\rangle \geq 0,\ \forall\ y' \in \mathbb{R}^m_+,$$

which implies that $\eta_k \in \mathbb{R}^m_+$. Taking $x = x_k$ and $y = \Phi(x_k)$ in (7.11), $\langle \eta_k, y_k' - \Phi(x_k)\rangle \leq 0$ which along with the facts that $\eta_k \in \mathbb{R}^m_+$ and $y_k' - \Phi(x_k) \in \mathbb{R}^m_+$ leads to $\langle \eta_k, y_k' - \Phi(x_k)\rangle = 0$. Therefore, (7.11) is equivalent to

$$\langle \xi_k, x - x_k\rangle \leq \langle \eta_k, y\rangle - \langle \eta_k, \Phi(x_k)\rangle,\ \forall\ (x, y) \in epi\ \Phi.$$

In particular, for $y = \Phi(x)$,

$$\langle \xi_k, x - x_k\rangle \leq (\eta_k\Phi)(x) - (\eta_k\Phi)(x_k),\ \forall\ x \in \mathbb{R}^n.$$

Observe that as $\eta_k \in \mathbb{R}^m_+$, $(\eta_k\Phi)$ is a convex function and thus the above inequality implies that (7.11) is equivalent to $\xi_k \in \partial(\eta_k\Phi)(x_k)$. Also from the condition $\langle \eta_k, y_k' - \Phi(x_k)\rangle = 0$, we have $\langle \eta_k, y_k'\rangle = \langle \eta_k, \Phi(x_k)\rangle$. Inserting this fact in (7.10) leads to

$$\langle \eta_k, \Phi(x_k) - \bar{y}\rangle \to 0.$$

Because $\tau_k \to 0$, (7.8) is equivalent to

$$\phi(y_k) - \langle \eta_k, y_k - \bar{y}\rangle \to \phi(\bar{y}),$$

thereby establishing the result. $\square$

Next we present the Sequential Chain Rule in a simpler form using the above theorem and a lemma for which one will require the notion of the Clarke subdifferential. Recall that at any $x \in \mathbb{R}^n$ the *Clarke generalized gradient* or *Clarke subdifferential* is given as

$$\partial^\circ \phi(x) = co\ \{\xi \in \mathbb{R}^n : \xi = \lim_{k \to \infty} \nabla \phi(x_k) \text{ where } x_k \to x,\ x_k \in \tilde{\mathcal{D}}\},$$

where $\tilde{\mathcal{D}}$ denotes the set of points at which ϕ is differentiable. The Sum Rule from Clarke [27] is as follows. Consider two locally Lipschitz functions $\phi_1, \phi_2 : \mathbb{R}^n \to \mathbb{R}$. Then for $\lambda_1,\ \lambda_2 \in \mathbb{R}$,

$$\partial^\circ(\lambda_1 \phi_1 + \lambda_2 \phi_2)(x) \subset \lambda_1 \partial^\circ \phi_1(x) + \lambda_2 \partial^\circ \phi_2(x).$$

For a convex function ϕ, the convex subdifferential and the Clarke subdifferential coincide, that is, $\partial \phi(\bar{x}) = \partial^\circ \phi(\bar{x})$. Now we present the lemma that plays an important role in obtaining the Sequential Chain Rule.

Lemma 7.5 *Consider a locally Lipschitz vector-valued function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$. Suppose that there exist $\{\lambda_k\} \subset \mathbb{R}^m$ and $\{x_k\} \subset \mathbb{R}^n$ with $\lambda_k \to 0$ and $x_k \to \bar{x}$ such that*

$$\omega_k \in \partial^\circ(\lambda_k \Phi)(x_k),\ \forall\ k \in \mathbb{N}.$$

Then $\omega_k \to 0$.

Proof. By the Clarke Sum Rule,

$$\omega_k \in \partial^\circ(\lambda_k \Phi)(x_k) \subset \sum_{i=1}^{m} \lambda_i^k \partial^\circ \phi_i(x_k),\ \forall\ k \in \mathbb{N},$$

where $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_m(x))$ and $\phi_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$, are locally Lipschitz functions. From the above condition, there exist $\omega_i^k \in \partial^\circ \phi_i(x_k)$, $i = 1, 2, \ldots, m$, such that

$$\omega_k = \sum_{i=1}^{m} \lambda_i^k \omega_i^k.$$

Therefore,

$$\|\omega_k\| = \| \sum_{i=1}^{m} \lambda_i^k \omega_i^k \| \leq \sum_{i=1}^{m} |\lambda_i^k|\ \|\omega_i^k\|.$$

Because the Clarke subdifferential $\partial^\circ \phi_i(x_k),\ i = 1, 2, \ldots, m$, are compact, $\{\omega_i^k\},\ i = 1, 2, \ldots, m$, are bounded sequences and hence by the Bolzano–Weierstrass Theorem, Proposition 1.3, have a convergent subsequence. Without loss of generality, assume that $\omega_i^k \to \omega_i,\ i = 1, 2, \ldots, m$. Because the

Clarke subdifferential as a set-valued map has a closed graph, $\omega_i \in \partial^\circ \phi_i(\bar{x})$, $i = 1, 2, \ldots, m$. Also, as $\lambda_k \to 0$, $|\lambda_i^k| \to 0$. Hence, by the compactness of the Clarke subdifferential, for $i = 1, 2, \ldots, m$, $\|\omega_i\| \leq M_i < +\infty$. Therefore,

$$\sum_{i=1}^{m} |\lambda_i^k| \, \|\omega_i^k\| \to \sum_{i=1}^{m} 0(M_i) = 0,$$

thus implying that $\omega_k \to 0$. □

Theorem 7.6 *(A simpler version of the Sequential Chain Rule) Consider a vector-valued convex function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ and a proper lsc convex function $\phi : \mathbb{R}^m \to \bar{\mathbb{R}}$ that is nondecreasing over $\Phi(\mathbb{R}^n) + \mathbb{R}^m_+$. Then for $\bar{y} = \Phi(\bar{x}) \in dom\ \phi$, $\xi \in \partial(\phi \circ \Phi)(\bar{x})$ if and only if there exist*

$$\eta_k \in \partial\phi(y_k) \quad \textit{and} \quad \xi_k \in \partial(\eta_k \Phi)(x_k)$$

satisfying

$$x_k \to \bar{x}, \quad y_k \to \Phi(\bar{x}), \quad \xi_k \to \xi,$$
$$\phi(y_k) - \langle \eta_k, y_k - \bar{y} \rangle \to \phi(\bar{y}) \quad \textit{and} \quad \langle \eta_k, \Phi(x_k) - \bar{y} \rangle \to 0.$$

Proof. Consider $\xi \in \partial(\phi \circ \Phi)(\bar{x})$. Suppose that x_k, y_k, y'_k, ξ_k, η_k, and τ_k are as in Theorem 7.4. Denote $\zeta_k = \eta_k + \tau_k$. Observe that for every $k \in \mathbb{N}$,

$$\xi_k \in \partial(\eta_k \Phi)(x_k) \quad \text{and} \quad \eta_k \Phi = \zeta_k \Phi + (\eta_k - \zeta_k)\Phi,$$

with $\eta_k - \zeta_k = -\tau_k \to 0$. Because Φ is convex and every component is locally Lipschitz, it is simple to show that Φ is also locally Lipschitz. As $x_k \to \bar{x}$, for sufficiently large k, $x_k \in \mathcal{N}(\bar{x})$ where $\mathcal{N}(\bar{x})$ is a neighborhood of $\bar{x}$ on which Φ satisfies the Lipschitz property with Lipschitz constant $\bar{L} > 0$. Hence, $(\eta_k - \zeta_k)\Phi$ is also locally Lipschitz over $\mathcal{N}(x_k)$ with Lipschitz constant $\bar{L}\|\eta_k - \zeta_k\| > 0$. This follows from the fact that for any $x, x' \in \mathcal{N}(x_k)$,

$$|(\eta_k - \zeta_k)\Phi(x) - (\eta_k - \zeta_k)\Phi(x')| = |\langle(\eta_k - \zeta_k), \Phi(x) - \Phi(x')\rangle|,$$

which by the Cauchy–Schwarz inequality, Proposition 1.1, implies that

$$\begin{aligned} |(\eta_k - \zeta_k)\Phi(x) - (\eta_k - \zeta_k)\Phi(x')| &\leq \|\eta_k - \zeta_k\| \, \|\Phi(x) - \Phi(x')\| \\ &\leq \bar{L}\|\eta_k - \zeta_k\| \, \|x - x'\|. \end{aligned}$$

From Theorem 7.4, $\eta_k \in \mathbb{R}^m_+$, which implies $(\eta_k \Phi)$ is convex. However, $(\zeta_k \Phi)$ and $((\eta_k - \zeta_k)\Phi)$ need not be convex. Thus, $\xi_k \in \partial(\eta_k \Phi)(x_k)$ implies

$$\xi_k \in \partial^\circ(\eta_k \Phi)(x_k) = \partial^\circ\{\zeta_k \Phi + (\eta_k - \zeta_k)\Phi\}(x_k).$$

By the Clarke Sum Rule,

$$\xi_k \in \partial^\circ(\zeta_k \Phi)(x_k) + \partial^\circ((\eta_k - \zeta_k)\Phi)(x_k),$$

which implies that there exist $\rho_k \in \partial^\circ(\zeta_k \Phi)(x_k)$ and $\varrho \in \partial^\circ((\eta_k - \zeta_k)\Phi)(x_k)$ such that $\xi_k = \rho_k + \varrho_k$. As $\eta_k - \zeta_k \to 0$ and $x_k \to \bar{x}$, by Lemma 7.5, $\varrho_k \to 0$. Setting $\varrho_k = -\beta_k$,

$$\rho_k = \xi_k + \beta_k \in \partial^\circ(\zeta_k \Phi)(x_k).$$

As the limit $k \to +\infty$, $\rho_k \to \xi$, $\zeta_k = \eta_k + \tau_k \in \partial\phi(y_k)$, $\langle \zeta_k, \Phi(x_k) - \bar{y}\rangle \to 0$, and

$$\phi(y_k) - \langle \zeta_k, y_k - \bar{y}\rangle \to \phi(\bar{y}),$$

thereby yielding the desired conditions.

Conversely, suppose that conditions are satisfied. As $\eta_k \in \partial\phi(y_k)$,

$$\phi(y) - \phi(y_k) \geq \langle \eta_k, y - y_k\rangle, \ \forall\ y \in \mathbb{R}^m,$$

which implies that

$$\phi(y) \geq \phi(y_k) - \langle \eta_k, y_k - \bar{y}\rangle + \langle \eta_k, y - \bar{y}\rangle, \ \forall\ y \in \mathbb{R}^m.$$

In particular, for $y = \Phi(x)$ for every $x \in \mathbb{R}^n$, the above inequality yields that for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
(\phi \circ \Phi)(x) &\geq \phi(y_k) - \langle \eta_k, y_k - \bar{y}\rangle + \langle \eta_k, \Phi(x) - \bar{y}\rangle \\
&= \phi(y_k) - \langle \eta_k, y_k - \bar{y}\rangle + \langle \eta_k, \Phi(x) - \Phi(x_k)\rangle + \langle \eta_k, \Phi(x_k) - \bar{y}\rangle \\
&= \phi(y_k) - \langle \eta_k, y_k - \bar{y}\rangle + (\eta_k \circ \Phi)(x) - (\eta \circ \Phi)(x_k) + \\
&\qquad\qquad \langle \eta_k, \Phi(x_k) - \bar{y}\rangle.
\end{aligned}$$

Because $\xi_k \in \partial(\eta_k \circ \Phi)(x_k)$, for every $x \in \mathbb{R}^n$,

$$(\phi \circ \Phi)(x) \geq \phi(y_k) - \langle \eta_k, y_k - \bar{y}\rangle + \langle \xi_k, x - x_k\rangle + \langle \eta_k, \Phi(x_k) - \bar{y}\rangle,$$

which as the limit $k \to +\infty$ reduces to

$$\begin{aligned}
(\phi \circ \Phi)(x) &\geq \phi(\bar{y}) + \langle \bar{\xi}, x - \bar{x}\rangle \\
&= (\phi \circ \Phi)(\bar{x}) + \langle \bar{\xi}, x - \bar{x}\rangle, \ \forall\ x \in \mathbb{R}^n.
\end{aligned}$$

Thus, $\xi \in \partial(\phi \circ \Phi)(\bar{x})$, thereby completing the result. □

Now we move on to establish the sequential optimality condition for (CP) with the real-valued convex objective function $f : \mathbb{R}^n \to \mathbb{R}$ obtained by Thibault [108] using the above theorem. To apply the result, we equivalently expressed the feasible set C as

$$C = \{x \in \mathbb{R}^n : G(x) \in -\mathbb{R}^m_+\},$$

where $G(x) = (g_1(x), g_2(x), \ldots, g_m(x))$. Observe that $G : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued convex function as $g_i,\ i = 1, 2, \ldots, m$, are convex. Now using the sequential subdifferential calculus rules, Theorems 7.1 and 7.6, we present the sequential optimality conditions for the constrained problem (CP).

Theorem 7.7 *Consider the convex programming problem problem (CP) with C given by (3.1). Then $\bar{x} \in C$ is a point of minimizer of (CP) if and only if there exist $x_k \to \bar{x}$, $y_k \to G(\bar{x})$, $\lambda_k \in \mathbb{R}^m_+$, $\xi \in \partial f(\bar{x})$, and $\xi_k \in \partial(\lambda_k G)(x_k)$ such that*

$$\begin{aligned} &\xi + \xi_k \to 0, && \langle \lambda_k, y_k \rangle = 0, \\ &\langle \lambda_k, y_k - G(\bar{x}) \rangle \to 0, && \langle \lambda_k, G(x_k) - G(\bar{x}) \rangle \to 0. \end{aligned}$$

Proof. Observe that the problem (CP) can be rewritten as the unconstrained problem

$$\min(f + (\delta_{-\mathbb{R}^m_+} \circ G))(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

By Theorem 2.89, $\bar{x}$ is a point of minimizer of (CP) if and only if

$$0 \in \partial(f + (\delta_{-\mathbb{R}^m_+} \circ G))(\bar{x}).$$

As $dom\ f = \mathbb{R}^n$, invoking the Sum Rule, Theorem 2.91,

$$0 \in \partial f(\bar{x}) + \partial(\delta_{-\mathbb{R}^m_+} \circ G)(\bar{x}),$$

which is equivalent to the existence of $\xi \in \partial f(\bar{x})$ and $\hat{\xi} \in \partial(\delta_{-\mathbb{R}^m_+} \circ G)(\bar{x})$ such that

$$\xi + \hat{\xi} = 0.$$

As G is a convex function and the indicator function $\delta_{-\mathbb{R}^m_+}$ is a proper lsc convex function nondecreasing over $G(\mathbb{R}^n) + \mathbb{R}^m_+$, thereby applying Theorem 7.6, $\hat{\xi} \in \partial(\delta_{-\mathbb{R}^m_+} \circ G)(\bar{x})$ if and only if there exist $x_k \to \bar{x}$, $y_k \to G(\bar{x})$, $\xi_k \to \hat{\xi}$ such that

$$\lambda_k \in \partial \delta_{-\mathbb{R}^m_+}(y_k) \quad \text{and} \quad \xi_k \in \partial(\lambda_k G)(x_k),$$

satisfying

$$\delta_{-\mathbb{R}^m_+}(y_k) - \langle \lambda_k, y_k - G(\bar{x}) \rangle \to (\delta_{-\mathbb{R}^m_+} \circ G)(\bar{x}) \ \text{ and } \ \langle \lambda_k, G(x_k) - G(\bar{x}) \rangle \to 0.$$

As $\lambda_k \in \partial \delta_{-\mathbb{R}^m_+}(y_k) = N_{-\mathbb{R}^m_+}(y_k)$, the sequence $\{y_k\} \subset -\mathbb{R}^m_+$ with

$$\langle \lambda_k, y - y_k \rangle \leq 0, \ \forall\ y \in -\mathbb{R}^m_+.$$

In particular, taking $y = 0$ and $y = 2y_k$ in the above inequality leads to

$$\langle \lambda_k, y_k \rangle = 0.$$

Thus,

$$\langle \lambda_k, y \rangle \leq 0, \ \forall\ y \in -\mathbb{R}^m_+,$$

which implies $\{\lambda_k\} \subset \mathbb{R}^m_+$. Using the fact that $\{y_k\} \subseteq -\mathbb{R}^m_+$, the condition

$$\delta_{-\mathbb{R}^m_+}(y_k) - \langle \lambda_k, y_k - G(\bar{x}) \rangle \to (\delta_{-\mathbb{R}^m_+} \circ G)(\bar{x})$$

reduces to

$$\langle \lambda_k, y_k - G(\bar{x}) \rangle \to 0,$$

thereby leading to the requisite result. $\square$

7.3 Fenchel Conjugates and Constraint Qualification

Observe that in the previous section, the sequential optimality conditions are obtained in terms of the subdifferentials that are calculated at some neighboring point rather than the exact point of minimum, as is the case in the standard KKT conditions. But this can be overcome by using the Brøndsted–Rockafellar Theorem, Theorem 2.114, thereby expressing the result in terms of the ε-subdifferentails at the exact point. To the best of our knowledge this was carried out by Jeyakumar, Rubinov, Glover, and Ishizuka [70] and Jeyakumar, Lee, and Dinh [68]. In their approach, the epigraph of the conjugate function of the objective function and the constraints play a central role in the characterization of the optimality for the convex programming problem (CP). The proof is based on the result in Jeyakumar, Rubinov, Glover, and Ishizuka [70].

Theorem 7.8 *Consider the convex programming problem (CP) with C given by (3.1). Then $\bar{x}$ is a point of minimizer of (CP) if and only if*

$$(0, -f(\bar{x})) \in epi\ f^* + cl \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*. \tag{7.12}$$

Proof. Recall that the feasible set C of the convex programming problem (CP) is given by (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\},$$

which can be equivalently expressed as

$$C = \{x \in \mathbb{R}^n : G(x) \in -\mathbb{R}^m_+\},$$

where $G : \mathbb{R}^n \to \mathbb{R}^m$ is defined as $G(x) = (g_1(x), g_2(x), \ldots, g_m(x))$. Because $g_i,\ i = 1, 2, \ldots, m$, are convex functions, G is also a convex function. $\bar{x}$ is a point of minimizer of (CP) if and only if

$$f(x) \geq f(\bar{x}),\ \forall\ x \in C,$$

that is,

$$\phi(x) + \delta_C(x) \geq 0,\ \forall\ x \in \mathbb{R}^n,$$

where $\phi(x) = f(x) - f(\bar{x})$. By Definition 2.101 of the conjugate function,

$$\begin{aligned} (\phi + \delta_C)^*(0) &= \sup_{x \in \mathbb{R}^n} \{\langle 0, x\rangle - (\phi + \delta_C)(x)\} \\ &= \sup_{x \in \mathbb{R}^n} -(\phi + \delta_C)(x) \leq 0. \end{aligned}$$

This shows that

$$(0,0) \in epi\ (\phi + \delta_C)^*,$$

which by the epigraph of the conjugate of the sum, Theorem 2.123, implies that

$$(0,0) \in cl\{epi\ \phi^* + epi\ \delta_C^*\}.$$

As $dom\ \phi = \mathbb{R}^n$, by Theorem 2.69, ϕ is continuous on $\mathbb{R}^n$. Hence, by Proposition 2.124, $epi\ \phi^* + epi\ \delta_C^*$ is closed, which reduces the above condition to

$$(0,0) \in epi\ \phi^* + epi\ \delta_C^*. \tag{7.13}$$

Consider

$$(\lambda G)(x) = \langle \lambda, G(x) \rangle.$$

For $x \in C$, $G(x) \in -\mathbb{R}^m_+$, which implies $(\lambda G)(x) \leq 0$ for every $\lambda \in \mathbb{R}^m_+$. Thus,

$$\sup_{\lambda \in \mathbb{R}^m_+} (\lambda G)(x) = 0. \tag{7.14}$$

If $x \notin C$, there exists some $i \in \{1, 2, \ldots, m\}$ such that $g_i(x) > 0$. Hence, it is simple to see that

$$\sup_{\lambda \in \mathbb{R}^m_+} (\lambda G)(x) = +\infty. \tag{7.15}$$

Combining (7.14) and (7.15),

$$\delta_C(x) = \sup_{\lambda \in \mathbb{R}^m_+} (\lambda G)(x).$$

Applying Theorem 2.123, relation (7.13) along with Proposition 2.103 yields

$$(0,0) \in epi\ f^* + (0, f(\bar{x})) + cl\ co \bigcup_{\lambda \in \mathbb{R}^m_+} epi(\lambda G)^*.$$

By Theorem 2.123, $\bigcup_{\lambda \in \mathbb{R}^m_+} epi(\lambda G)^*$ is a convex cone and thus, the above relation reduces to

$$\begin{aligned} (0,0) \quad &\in \quad epi\ f^* + (0, f(\bar{x})) + cl \bigcup_{\lambda \in \mathbb{R}^m_+} epi(\lambda G)^* \\ &= \quad epi\ f^* + (0, f(\bar{x})) + cl \bigcup_{\lambda \in \mathbb{R}^m_+} epi(\sum_{i=1}^m \lambda_i g_i)^*, \end{aligned}$$

thereby leading to the requisite condition (7.12).

Conversely, suppose that condition (7.12) holds, which implies there exist $\xi \in dom\ f^*$, $\alpha \geq 0$, $\alpha_k \geq 0$, $\{\lambda_k\} \subset \mathbb{R}^m_+$, $\{\xi_k\} \subset dom\ (\sum_{i=1}^m \lambda_i^k g_i)^*$, $i = 1, 2, \ldots, m$, such that

$$(0, -f(\bar{x})) = (\xi, f^*(\xi) + \alpha) + \lim_{k\to\infty} (\xi_k, \sum_{i=1}^m (\lambda_i^k g_i)^*(\xi_k) + \alpha_k).$$

Componentwise comparison leads to

$$0 = \xi + \lim_{k\to\infty} \xi_k, \tag{7.16}$$

$$-f(\bar{x}) = f^*(\xi) + \alpha + \lim_{k\to\infty} (\sum_{i=1}^m (\lambda_i^k g_i)^*(\xi_k) + \alpha_k). \tag{7.17}$$

By Definition 2.101 of the conjugate functions, the condition (7.17) implies that for every $x \in \mathbb{R}^n$,

$$\begin{aligned} f(\bar{x}) - f(x) &\leq -\langle \xi, x\rangle - \alpha - \lim_{k\to\infty} (\sum_{i=1}^m (\lambda_i^k g_i)^*(\xi_k) + \alpha_k) \\ &\leq -\langle \xi, x\rangle - \alpha - \lim_{k\to\infty} (\langle \xi_k, x\rangle - \sum_{i=1}^m \lambda_i^k g_i(x) + \alpha_k). \end{aligned}$$

In particular, taking $x \in C$, that is, $g_i(x) \leq 0$, $i = 1, 2, \ldots, m$, in the above inequality along with the nonnegativity of α, α_k, λ_i^k, $i = 1, 2, \ldots, m$, and the condition (7.16) yields

$$f(\bar{x}) \leq f(x), \ \forall\ x \in C.$$

Therefore, $\bar{x}$ is a point of minimizer of (CP), as desired. □

As one can express the epigraph of conjugate functions in terms of the ε-subdifferential of the function, Theorem 2.122, Jeyakumar et al. [70, 68] expressed the above theorem in terms of the ε-subdifferentials, thus obtaining the sequential optimality conditions presented below. We present the same using the condition (7.12) obtained in Theorem 7.8.

Theorem 7.9 *Consider the convex programming problem (CP) with C given by (3.1). Then $\bar{x}$ is a point of minimizer for (CP) if and only if there exist $\xi \in \partial f(\bar{x})$, $\varepsilon_i^k \geq 0$, $\lambda_i^k \geq 0$, $\xi_i^k \in \partial_{\varepsilon_i^k} g_i(\bar{x})$, $i = 1, 2, \ldots, m$, such that*

$$\xi + \sum_{i=1}^m \lambda_i^k \xi_i^k \to 0, \quad \sum_{i=1}^m \lambda_i^k g_i(\bar{x}) \to 0 \quad \textit{and} \quad \sum_{i=1}^m \lambda_i^k \varepsilon_i^k \downarrow 0 \quad \textit{as} \quad k \to +\infty.$$

Proof. Consider Theorem 7.8, according to which $\bar{x}$ is a point of minimizer of (CP) if and only if the containment (7.12) is satisfied. By Theorem 2.122, there exist $\xi \in \partial_\varepsilon f(\bar{x})$, $\lambda_i^k \geq 0$ and $\xi_k \in \partial_{\varepsilon_k}(\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})$, $i = 1, 2, \ldots, m$, with $\varepsilon,\ \varepsilon_k \geq 0$ such that

$$(0, -f(\bar{x})) = (\xi, \langle \xi, \bar{x}\rangle + \varepsilon - f(\bar{x})) + \lim_{k\to\infty} (\xi_k, \langle \xi_k, \bar{x}\rangle + \varepsilon_k - (\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})).$$

Componentwise comparison leads to

$$\begin{aligned} 0 &= \xi + \lim_{k\to\infty} \xi_k, \\ -\varepsilon &= \langle \xi, \bar{x}\rangle + \lim_{k\to\infty} (\langle \xi_k, \bar{x}\rangle + \varepsilon_k - (\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})), \end{aligned}$$

which together imply that

$$-\varepsilon = \lim_{k\to\infty} (\varepsilon_k - (\sum_{i=1}^m \lambda_i^k g_i)(\bar{x})).$$

This equation along with the nonnegativity of $\varepsilon,\ \varepsilon_k$, and $\lambda_i^k,\ i = 1, 2, \ldots, m$, implies

$$\varepsilon = 0, \quad \varepsilon_k \downarrow 0 \quad \text{and} \quad \sum_{i=1}^m \lambda_i^k g_i(\bar{x}) \to 0. \tag{7.18}$$

As $dom\ g_i = \mathbb{R}^n,\ i = 1, 2, \ldots, m$, by Theorem 2.115, there exist $\varepsilon_i^k \geq 0$, $i = 1, 2, \ldots, m$, such that

$$\xi_k \in \sum_{i=1}^m \partial_{\varepsilon_i^k}(\lambda_i^k g_i)(\bar{x}) \quad \text{and} \quad \varepsilon_k = \sum_{i=1}^m \varepsilon_i^k.$$

Define $\bar{I}_k = \{i \in \{1, 2, \ldots, m\} : \lambda_i^k > 0\}$. By Theorem 2.117,

$$\partial_{\varepsilon_i^k}(\lambda_i^k g_i)(\bar{x}) = \lambda_i^k \partial_{\bar{\varepsilon}_i^k} g_i(\bar{x}),\ \forall\ i \in \bar{I}_k,$$

where $\bar{\varepsilon}_i^k = \dfrac{\varepsilon_i^k}{\lambda_i^k} \geq 0$. Therefore,

$$\xi_k \in \sum_{i\in\bar{I}_k} \lambda_i^k \partial_{\bar{\varepsilon}_i^k} g_i(\bar{x}) + \sum_{i\notin\bar{I}_k} \partial_{\varepsilon_i^k}(\lambda_i^k g_i)(\bar{x}). \tag{7.19}$$

As discussed in Chapter 2, the ε-subdifferential of zero function is zero, that is, $\partial_\varepsilon 0(x) = \{0\}$. Thus,

$$\partial_{\varepsilon_i^k}(\lambda_i^k g_i)(\bar{x}) = 0 = \lambda_i^k \partial_{\varepsilon_i^k} g_i(\bar{x}),\ \forall\ i \notin \bar{I}_k.$$

The above relation along with the condition (7.19) yields that

$$\xi_k \in \sum_{i\in\bar{I}_k} \lambda_i^k \partial_{\bar{\varepsilon}_i^k} g_i(\bar{x}) + \sum_{i\notin\bar{I}_k} \lambda_i^k \partial_{\varepsilon_i^k} g_i(\bar{x}). \tag{7.20}$$

Also,

$$\varepsilon_k = \sum_{i \in \bar{I}_k} \lambda_i^k \bar{\varepsilon}_i^k + \sum_{i \notin \bar{I}_k} \lambda_i^k \varepsilon_i^k,$$

which along with (7.20) leads to the desired sequential optimality conditions.

Conversely, suppose that the sequential optimality conditions hold. From Definitions 2.77 and 2.109 of subdifferentials and ε-subdifferentials,

$$\begin{array}{rcl} f(x) - f(\bar{x}) & \geq & \langle \xi, x - \bar{x} \rangle, \\ g_i(x) - g_i(\bar{x}) & \geq & \langle \xi_i^k, x - \bar{x} \rangle - \varepsilon_i^k, \ i = 1, 2, \ldots, m, \end{array}$$

respectively. The above inequalities along with the sequential optimality conditions imply that

$$f(x) - f(\bar{x}) + \sum_{i=1}^{m} \lambda_i^k g_i(x) \geq 0, \ \forall \ x \in \mathbb{R}^n,$$

where $\{\lambda_i^k\} \subset \mathbb{R}_+, \ i = 1, 2, \ldots, m$. In particular, taking $x \in C$, that is, $g_i(x) \leq 0, \ i = 1, 2, \ldots, m$, which along with the condition on $\{\lambda_k\}$ reduces the above inequality to

$$f(x) \geq f(\bar{x}), \ \forall \ x \in C,$$

thereby establishing the optimality of $\bar{x}$ for (CP). □

Observe that not only is the optimality condition sequential, but one obtains a sequential complementary slackness condition. Note that we are working in a simple scenario with a convex inequality system. This helps in expressing the condition (7.12) derived in Theorem 7.8 in a more relaxed form. By applying Theorem 2.123, the condition becomes

$$(0, -f(\bar{x})) \in epi \ f^* + cl \bigcup_{\lambda \in \mathbb{R}_+^m} cl(\sum_{i=1}^{m} epi \ (\lambda_i g_i)^*). \tag{7.21}$$

By the closure properties of the arbitrary union of sets, the condition (7.21) leads to

$$(0, -f(\bar{x})) \in epi \ f^* + cl \bigcup_{\lambda \in \mathbb{R}_+^m} \sum_{i=1}^{m} epi \ (\lambda_i g_i)^*. \tag{7.22}$$

Define $\bar{I}_\lambda = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$. Again by Theorem 2.123,

$$epi \ (\lambda_i g_i)^* = \lambda_i \ epi \ g_i^*, \ \forall \ i \in \bar{I}_\lambda.$$

For $i \notin \bar{I}_\lambda$ with $\lambda_i = 0$,

$$(\lambda_i g_i)^*(\xi) = 0^*(\xi) = \left\{ \begin{array}{cl} 0, & \xi = 0, \\ +\infty, & \text{otherwise}, \end{array} \right.$$

which implies that

$$epi\ (\lambda_i g_i)^* = \{0\} \times \mathbb{R}_+,\ \forall\ i \notin \bar{I}_\lambda.$$

Using the preceding conditions, the relation (7.22) becomes

$$\begin{aligned}(0, -f(\bar{x})) \quad &\in \quad epi\ f^* + cl \bigcup_{\lambda \in \mathbb{R}^m_+} (\sum_{i \in \bar{I}_\lambda} \lambda_i epi\ g_i^* + \sum_{i \notin \bar{I}_\lambda} \{0\} \times \mathbb{R}_+) \\ &= \quad epi\ f^* + cl \bigcup_{\lambda \in \mathbb{R}^m_+} (\sum_{i \in \bar{I}_\lambda} \lambda_i epi\ g_i^* + \{0\} \times \mathbb{R}_+). \qquad (7.23)\end{aligned}$$

Now consider $(\xi, \alpha) \in \sum_{i \in \bar{I}_\lambda} \lambda_i epi\ g_i^*$, which implies that for $i \in \bar{I}_\lambda$ there exist $(\xi_i, \alpha_i) \in epi\ g_i^*$ such that

$$(\xi, \alpha) = \sum_{i \in \bar{I}_\lambda} \lambda_i (\xi_i, \alpha_i).$$

Therefore, for any element $(0, \bar{\alpha}) \in \{0\} \times \mathbb{R}_+$,

$$(\xi, \alpha + \bar{\alpha}) = \sum_{i \in \bar{I}_\lambda} \lambda_i (\xi_i, \alpha_i + \bar{\alpha}/\lambda_i),$$

where $\dfrac{\bar{\alpha}}{\lambda_i} \geq 0$. As $(\xi_i, \alpha_i) \in epi\ g_i^*$,

$$g_i^*(\xi_i) \leq \alpha_i \leq \alpha_i + \frac{\bar{\alpha}}{\lambda_i},\ \forall\ i \in \bar{I}_\lambda,$$

which implies that $(\xi_i, \alpha_i + \bar{\alpha}/\lambda_i) \in epi\ g_i^*$. Hence $(\xi, \alpha + \bar{\alpha}) \in \sum_{i \in \bar{I}_\lambda} \lambda_i epi\ g_i^*$ for every $\bar{\alpha} \geq 0$. Therefore, (7.23) reduces to

$$(0, -f(\bar{x})) \in epi\ f^* + cl \bigcup_{\lambda \in \mathbb{R}^m_+} \sum_{i=1}^m \lambda_i epi\ g_i^*.$$

It is quite simple to see that

$$cl \bigcup_{\lambda \in \mathbb{R}^m_+} \sum_{i=1}^m \lambda_i epi\ g_i^* = cl\ cone\ co \bigcup_{i=1}^m epi\ g_i^*.$$

We leave this as an exercise for the reader. Hence,

$$(0, -f(\bar{x})) \in epi\ f^* + cl\ cone\ co \bigcup_{i=1}^m epi\ g_i^*. \qquad (7.24)$$

The condition (7.24) implies that there exist $\xi \in dom\ f^*$, $\alpha \geq 0$, $\xi_i^k \in dom\ g_i^*$, $\alpha_i^k,\ \lambda_i^k \geq 0,\ i = 1, 2, \ldots, m$, such that

$$(0, -f(\bar{x})) = (\xi, f^*(\xi) + \alpha) + \lim_{k \to \infty} \sum_{i=1}^m \lambda_i^k (\xi_i^k, g_i^*(\xi_i^k) + \alpha_i^k).$$

Componentwise comparison leads to

$$\begin{aligned} 0 &= \xi + \lim_{k\to\infty} \sum_{i=1}^{m} \lambda_i^k \xi_i^k, &(7.25)\\ -f(\bar{x}) &= f^*(\xi) + \alpha + \lim_{k\to\infty} \sum_{i=1}^{m} \lambda_i^k (g_i^*(\xi_i^k) + \alpha_i^k). &(7.26) \end{aligned}$$

By Definition 2.101 of the conjugate functions, the condition (7.26) implies that

$$\begin{aligned} f(\bar{x}) - f(x) &\leq -\langle \xi, x\rangle - \alpha - \lim_{k\to\infty} \sum_{i=1}^{m} \lambda_i^k (g_i^*(\xi_i^k) + \alpha_i^k) \\ &\leq -\langle \xi, x\rangle - \alpha - \lim_{k\to\infty} \sum_{i=1}^{m} \lambda_i^k (\langle \xi_i^k, x\rangle - g_i(x) + \alpha_i^k). \end{aligned}$$

In particular, taking $x \in C$ along with the nonnegativity of α, α_i^k, and λ_i^k, $i = 1, 2, \ldots, m$, and the condition (7.25) yields

$$f(\bar{x}) \leq f(x), \ \forall\, x \in C.$$

Therefore, $\bar{x}$ is a point of minimizer of (CP) under the relation (7.24). This discussion can be stated as the following result.

Theorem 7.10 *Consider the convex programming problem (CP) with C given by (3.1). Then $\bar{x}$ is a point of minimizer of (CP) if and only if*

$$(0, -f(\bar{x})) \in epi\ f^* + cl\ cone\ co \bigcup_{i=1}^{m} epi\ g_i^*. \tag{7.27}$$

Using the above result, we present an alternate proof to the sequential optimality conditions, Theorem 7.9.

Alternate proof of Theorem 7.9. According to the Theorem 7.10, $\bar{x}$ is a point of minimizer of (CP) if and only if the containment (7.27) is satisfied. By Theorem 2.122, there exist $\xi \in \partial_\varepsilon f(\bar{x})$, $\xi_i^k \in \partial_{\varepsilon_i^k} g_i(\bar{x})$ and $\lambda_i^k \geq 0$, $i = 1, 2, \ldots, m$, with ε, $\varepsilon_i^k \geq 0$, $i = 1, 2, \ldots, m$, such that

$$(0, -f(\bar{x})) = (\xi, \langle \xi, \bar{x}\rangle + \varepsilon - f(\bar{x})) + \lim_{k\to\infty} \sum_{i=1}^{m} \lambda_i^k (\xi_i^k, \langle \xi_i^k, \bar{x}\rangle + \varepsilon_i^k - g_i(\bar{x})).$$

Componentwise comparison leads to

$$\begin{aligned} 0 &= \xi + \lim_{k\to\infty} \sum_{i=1}^{m} \lambda_i^k \xi_i^k, \\ -\varepsilon &= \langle \xi, \bar{x}\rangle + \lim_{k\to\infty} \sum_{i=1}^{m} \lambda_i^k (\langle \xi_i^k, \bar{x}\rangle + \varepsilon_i^k - g_i(\bar{x})), \end{aligned}$$

which together imply that

$$-\varepsilon = \lim_{k \to \infty} \sum_{i=1}^{m} \lambda_i^k (\varepsilon_i^k - g_i(\bar{x})).$$

This equation along with the nonnegativity of ε, ε_i^k and λ_i^k, $i = 1, 2, \ldots, m$, implies $\varepsilon = 0$, $\sum_{i=1}^{m} \lambda_i^k g_i(\bar{x}) \to 0$ and $\sum_{i=1}^{m} \lambda_i^k \varepsilon_i^k \downarrow 0$ as $k \to +\infty$, thereby establishing the sequential optimality conditions. The converse can be verified as in Theorem 7.9. $\square$

As already discussed in the previous chapters, if one assumes certain constraint qualifications, then the standard KKT conditions can be established. If we observe the necessary and sufficient condition given in Theorem 7.8 carefully, we will observe that the term *cl* $\bigcup_{\lambda \in \mathbb{R}^m_+}$ *epi* $(\sum_{i=1}^{m} \lambda_i g_i)^*$ prevents us from further manipulation. On the other hand, one might feel that the route to the KKT optimality conditions lies in further manipulation of the condition (7.12). Further, observe that we arrived at the condition (7.12) without any constraint qualification. However, in order to derive the KKT optimality conditions, one needs some additional qualification conditions on the constraints. Thus from (7.12) it is natural to consider that the set

$$\bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* \quad \text{is closed.}$$

This is usually known as the *closed cone constraint qualification* or the *Farkas–Minkowski (FM) constraint qualification.* One may also take the more relaxed constraint qualification based on condition (7.27), that is,

$$cone\ (co \bigcup_{i=1}^{m} epi\ g_i^*) \quad \text{is closed.}$$

We will call the above constraint qualification as the *relaxed FM constraint qualification.* Below we derive the standard KKT conditions under either of the two constraint qualification.

Theorem 7.11 *Consider the convex programming problem* (CP) *with* C *given by (3.1). Assume that either the FM constraint qualification holds or the relaxed FM constraint qualification holds. Then* $\bar{x}$ *is a point of minimizer of* (CP) *if and only if there exist* $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, *such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) \quad \textit{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

Proof. From Theorem 7.8, we know that $\bar{x}$ is a point of minimizer of (CP) if and only if the relation (7.12) holds. As the FM constraint qualification is satisfied, (7.12) reduces to

$$(0, -f(\bar{x})) \in epi\ f^* + \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^m \lambda_i g_i)^*.$$

By the ε-subdifferential characterization of the epigraph of the conjugate function, Theorem 2.122, there exist $\xi \in \partial_\varepsilon f(\bar{x})$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, and $\xi' \in \partial_{\varepsilon'}(\sum_{i=1}^m \lambda_i g_i)(\bar{x})$ with $\varepsilon,\ \varepsilon' \geq 0$ such that

$$(0, -f(\bar{x})) = (\xi, \langle \xi, \bar{x} \rangle + \varepsilon - f(\bar{x})) + (\xi', \langle \xi', \bar{x} \rangle + \varepsilon' - (\sum_{i=1}^m \lambda_i g_i)(\bar{x})).$$

Componentwise comparison leads to

$$0 = \xi + \xi', \tag{7.28}$$

$$-f(\bar{x}) = \langle \xi, \bar{x} \rangle + \varepsilon - f(\bar{x}) + \langle \xi', \bar{x} \rangle + \varepsilon' - (\sum_{i=1}^m \lambda_i g_i)(\bar{x}). \tag{7.29}$$

By the feasibility of $\bar{x} \in C$ along with the nonnegativity of $\varepsilon,\ \varepsilon'$, and λ_i, $i = 1, 2, \ldots, m$, the condition (7.29) leads to

$$\varepsilon = 0, \quad \varepsilon' = 0, \quad \text{and} \quad \sum_{i=1}^m \lambda_i g_i(\bar{x}) = 0.$$

Because $\varepsilon = 0$ and $\varepsilon' = 0$, the condition (7.28) lead to the fact that

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^m \lambda_i \partial g_i(\bar{x}). \tag{7.30}$$

Further, $\sum_{i=1}^m \lambda_i g_i(\bar{x}) = 0$ implies that

$$\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m. \tag{7.31}$$

The conditions (7.30) and (7.31) together yield the KKT optimality conditions.

Now if the relaxed constraint qualification is satisfied, (7.27) reduces to

$$(0, -f(\bar{x})) \in epi\ f^* + cone\ co \bigcup_{i=1}^m epi\ g_i^*.$$

By the ε-subdifferential characterization of the epigraph of the conjugate function, Theorem 2.122, there exist $\xi \in \partial_\varepsilon f(\bar{x})$, $\xi_i \in \partial_{\varepsilon_i} g_i(\bar{x})$ and $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, with $\varepsilon,\ \varepsilon_i \geq 0$, $i = 1, 2, \ldots, m$, such that

$$(0, -f(\bar{x})) = (\xi, \langle \xi, \bar{x} \rangle + \varepsilon - f(\bar{x})) + \sum_{i=1}^m \lambda_i (\xi_i, \langle \xi_i, \bar{x} \rangle + \varepsilon_i - g_i(\bar{x})).$$

Componentwise comparison leads to

$$0 = \xi + \sum_{i=1}^{m} \lambda_i \xi_i, \tag{7.32}$$

$$-f(\bar{x}) = \langle \xi, \bar{x} \rangle + \varepsilon - f(\bar{x}) + \sum_{i=1}^{m} \lambda_i(\langle \xi_i, \bar{x} \rangle + \varepsilon_i - g_i(\bar{x})). \tag{7.33}$$

By the feasibility of $\bar{x} \in C$ along with the nonnegativity of ε, ε_i, and λ_i, $i = 1, 2, \ldots, m$, the condition (7.33) leads to

$$\varepsilon = 0, \quad \sum_{i=1}^{m} \lambda_i \varepsilon_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = 0.$$

Let us assume that $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$ is nonempty. Then corresponding to any $i \in \bar{I}$, $\varepsilon_i = 0$ and $g_i(\bar{x}) = 0$, imply that

$$\xi \in \partial f(\bar{x}), \quad \xi_i \in \partial g_i(\bar{x}) \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0, \ i \in \bar{I}. \tag{7.34}$$

Therefore, from (7.32) and (7.34),

$$0 = \xi + \sum_{i \in \bar{I}} \lambda_i \xi_i \in \partial f(\bar{x}) + \sum_{i \in \bar{I}} \lambda_i \partial g_i(\bar{x}).$$

For $i \notin \bar{I}$, choose $\varepsilon_i = 0$, the above condition leads to

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}),$$

along with the complementary slackness condition $\lambda_i g_i(\bar{x}) = 0$, $i = 1, 2, \ldots, m$ and thereby establishing the standard KKT optimality conditions. The reader should try to see how to arrive at the KKT optimality conditions when $\bar{I}$ is empty. The sufficiency can be worked out using Definition 2.77 of subdifferentials, as done in Chapter 3. □

The proof of the KKT optimality conditions under the FM constraint qualification was given by Jeyakumar, Lee, and Dinh [68] and that using the relaxed FM condition is based on Jeyakumar [67]. It has been shown by Jeyakumar, Rubinov, Glover, and Ishizuka [70] that under the Slater constraint qualification, the FM constraint qualification holds. We present the result below proving the same.

Proposition 7.12 *Consider the set C given by (3.1). Assume that the Slater constraint qualification holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$. Then the FM constraint qualification is satisfied, that is,*

$$\bigcup_{\lambda \in \mathbb{R}^m_+} \textit{epi } (\sum_{i=1}^{m} \lambda_i g_i)^* \quad \textit{is closed.}$$

Proof. Observe that defining $G = (g_1, g_2, \ldots, g_m)$,

$$\bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^m \lambda_i g_i)^* = \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\lambda G)^*.$$

Suppose that

$$(\xi_k, \alpha_k) \to (\xi, \alpha) \in cl \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\lambda G)^*$$

with $(\lambda_k G)^*(\xi_k) \leq \alpha_k$ for some $\lambda_k \in \mathbb{R}^m_+$. As $int\ \mathbb{R}^m_+$ is nonempty, one can always find a compact convex set $\mathcal{R} \subset \mathbb{R}^m_+$ such that $0 \notin \mathcal{R}$ and $cone\ \mathcal{R} = \mathbb{R}^m_+$. Thus, $\lambda_k = \gamma_k b_k$, where $\gamma_k \geq 0$ and $b_k \in \mathcal{R}$. Assume that $\gamma_k \geq 0$ for every k and $b_k \to b \in \mathcal{R}$ by the compactness of $\mathcal{R}$. We consider the following cases.

(i) $\gamma_k \to \gamma > 0$: Consider

$$\begin{aligned}(\lambda_k G)^*(\xi_k) \leq \alpha_k \quad &\iff \quad (\gamma_k b_k G)^* \leq \alpha_k \\ &\iff \quad (b_k G)^*(\xi_k/\gamma_k) \leq \alpha_k/\gamma_k.\end{aligned}$$

Because $b_k G \to bG$, $\xi_k/\gamma_k \to \xi/\gamma$ and $\alpha_k/\gamma_k \to \alpha/\gamma$,

$$(bG)^*(\xi/\gamma) \leq \liminf_{k\to\infty} (b_k G)^*(\xi_k/\gamma_k) \leq \alpha/\gamma.$$

Therefore, $(\xi/\gamma, \alpha/\gamma) \in epi\ (bG)^*$ and hence $(\xi, \alpha) \in \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\lambda G)^*$.

(ii) $\gamma_k \to +\infty$: Then $\xi_k/\gamma_k \to 0$ and $\alpha_k/\gamma_k \to 0$. Therefore,

$$(bG)^*(0) \leq \liminf_{k\to\infty} (b_k G)^*(\xi_k/\gamma_k) \leq 0,$$

which implies

$$-\inf_{x \in \mathbb{R}^n} (bG)(x) = \sup_{x \in \mathbb{R}^n} (-(bG)(x)) \leq 0,$$

that is, $(bG)(x) \geq 0$ for every $x \in \mathbb{R}^n$. But by the Slater constraint qualification, $G(\hat{x}) \in -int\ \mathbb{R}^m_+$ and $b \neq 0$. Therefore, $(bG)(\hat{x}) < 0$, which is a contradiction.

(iii) $\gamma_k \to 0$: This implies that $\lambda_k \to 0$ and thus $(\lambda_k G) \to 0$. Therefore,

$$0^*(\xi) \leq \liminf_{k\to\infty} (\lambda_k G)^*(\xi_k) \leq \alpha.$$

Observe that

$$0^*(\xi') = \begin{cases} 0, & \xi' = 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

which leads to $\xi = 0$ and $\alpha \geq 0$. Thus,

$$(0, \alpha) \in epi\ (0G)^* \subset \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\lambda G)^*.$$

Therefore, the closed cone constraint qualification is satisfied. $\square$

Next we present some examples to show that the FM constraint qualification is weaker in comparison to the Slater constraint qualification. Consider $C = \{x \in \mathbb{R} : g(x) \leq 0\}$, where

$$g(x) = \begin{cases} x^2, & x \leq 0, \\ x, & x \geq 0. \end{cases}$$

Observe that $C = \{0\}$ and hence the Slater constraint qualification is not satisfied. Also $T_C(0) = \{0\}$ while

$$\begin{aligned} S(0) &= \{d \in \mathbb{R} : g'(0, d) \leq 0\} \\ &= \{d \in \mathbb{R} : d \leq 0\}, \end{aligned}$$

which implies that the Abadie constraint qualification is also not satisfied. For $\xi \in \mathbb{R}$,

$$g^*(\xi) = \begin{cases} 0, & 0 \leq \xi \leq 1, \\ \xi^2/4, & \xi \leq 0. \end{cases}$$

Observe that as only one constraint is involved,

$$\bigcup_{\lambda \geq 0} epi\ (\lambda g)^* = \bigcup_{\lambda \geq 0} \lambda\ epi\ g^* = cone\ epi\ g^*.$$

Therefore, the FM constraint qualification reduces to the set *cone epi* g^* being closed, which is same as the relaxed FM constraint qualification. Here,

$$cone\ epi\ g^* = \{(\xi, \alpha) \in \mathbb{R}^2 : \xi \leq 0,\ \alpha > 0\} \cup \{(\xi, \alpha) \in \mathbb{R}^2 : \xi \geq 0,\ \alpha \geq 0\}$$

is not closed and hence the FM constraint qualification is not satisfied.

Now suppose that in the previous example,

$$g(x) = \begin{cases} -2x, & x \leq 0, \\ x, & x \geq 0. \end{cases}$$

Again, $C = \{0\}$ and the Slater constraint qualification does not hold. But unlike the above example, $T_C(0) = \{0\} = S(0)$ which implies that the Abadie constraint qualification is satisfied. For $\xi \in \mathbb{R}$,

$$g^*(\xi) = 0, \quad -2 \leq \xi \leq 1.$$

Observe that the set

$$cone\ epi\ g^* = \{(\xi, \alpha) \in \mathbb{R}^2 : \xi \in \mathbb{R},\ \alpha \geq 0\}$$

is closed. Thus, the FM constraint qualification also holds.

Observe that in the above examples, either both the Abadie constraint qualification and the FM qualification are satisfied or neither holds. Now let us consider an example from Jeyakumar, Lee, and Dinh [70] showing that the FM constraint qualification is weaker than the Abadie constraint qualification as well. Consider a convex function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$g(x_1, x_2) = \sqrt{x_1^2 + x_2^2} - x_2.$$

Here $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0,\ x_2 \geq 0\}$. Observe that the Slater constraint qualification does not hold as for any $(x_1, x_2) \in C$, $g(x_1, x_2) = 0$. For $(0, x_2),\ x_2 > 0$, g is differentiable at $(0, x_2)$ and hence

$$S(0, x_2) = \mathbb{R}^2 \quad \text{while} \quad T_C(0, x_2) = \{(0, x_2) : x_2 \in \mathbb{R}\}.$$

Thus the Abadie constraint qualification is also not satisfied. Now, for any $(\xi_1, \xi_2) \in \mathbb{R}^2$,

$$g^*(\xi_1, \xi_2) = \begin{cases} 0, & \xi_1 = \xi_2 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore,

$$\textit{cone epi } g^* = \{(0, 0)\} \times \mathbb{R}_+,$$

which is closed. Hence, the FM constraint qualification holds, thereby showing that it is a weaker constraint qualification with respect to the Slater and Abadie constraint qualifications.

Until now we considered the convex programming problem (CP) with a real-valued objective function f. This fact played an important role in the derivation of Theorem 7.8 as the continuity of f on $\mathbb{R}^n$ along with Proposition 2.124 leads to the closedness of

$$\textit{epi } f^* + \textit{epi } \delta_C^*.$$

But if $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a proper lsc convex function and C involves inequality constraints and additionally an abstract constraint, that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m,\ x \in X\} \tag{7.35}$$

where $X \subset \mathbb{R}^n$ is a closed convex set, then one has to impose an additional condition along with the closed cone constraint qualification to establish the KKT optimality condition, namely the *CC qualification condition*, that is,

$$\textit{epi } f^* + \bigcup_{\lambda \in \mathbb{R}^m_+} \textit{epi } (\sum_{i=1}^{m} \lambda_i g_i)^* + \textit{epi } \delta_X^* \quad \text{is closed.}$$

Next we present the KKT optimality condition in the presence of the CC qualification condition from Dinh, Nghia, and Vallet [34]. A similar result was established by Burachik and Jeyakumar [20] under the assumption of CC as well as FM constraint qualification.

Theorem 7.13 *Consider the convex programming problem (CP) where $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a proper lsc convex function and the feasible set C is given by (7.35). Assume that the CC qualification condition is satisfied. Then $\bar{x} \in dom\ f \cap C$ is a point of minimizer of (CP) if and only if there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}) \quad \textit{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

Proof. Suppose that $\bar{x} \in dom\ f \cap C$ is a point of minimizer of the problem (CP). Then working along the lines of Theorem 7.8, we have

$$(0, 0) \in cl\ \{epi\ \phi^* + epi\ \delta_C^*\},$$

where $\phi(x) = f(x) - f(\bar{x})$. Expressing $C = \overline{C} \cap X$ implies that $\delta_C = \delta_{\overline{C}} + \delta_X$, where

$$\overline{C} = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}.$$

From the proof of Theorem 7.8,

$$epi\ \delta_{\overline{C}}^* = cl \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*.$$

Therefore, by Theorem 2.123 and Propositions 2.102 and 2.15 (vi), the above condition becomes

$$\begin{aligned} (0,0) \quad &\in \quad cl\ \{epi\ \phi^* + cl\ (epi\ \delta_{\overline{C}}^* + epi\ \delta_X^*)\} \\ &\subset \quad cl\ \{epi\ \phi^* + cl\ (\bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* + epi\ \delta_X^*)\} \\ &\subset \quad cl\ \{epi\ \phi^* + \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* + epi\ \delta_X^*\}. \end{aligned}$$

By Propositions 2.103 and 2.15 (vi), the above yields

$$(0, -f(\bar{x})) \in cl\ \{epi\ f^* + \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* + epi\ \delta_X^*\},$$

which under the CC qualification condition reduces to

$$(0, -f(\bar{x})) \in epi\ f^* + \bigcup_{\lambda \in \mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* + epi\ \delta_X^*.$$

Applying Theorem 2.122, there exist $\xi_f \in \partial_{\varepsilon_f} f(\bar{x})$, $\xi_g \in \partial_{\varepsilon_g}(\sum_{i=1}^m \lambda_i g_i)(\bar{x})$, and $\xi_x \in \partial_{\varepsilon_x} \delta_X(\bar{x}) = N_{X,\varepsilon_x}(\bar{x})$ with $\varepsilon_f, \varepsilon_g, \varepsilon_x \geq 0$ such that

$$\begin{aligned} 0 &= \xi_f + \xi_g + \xi_x, \qquad (7.36)\\ -f(\bar{x}) &= (\langle \xi_f, \bar{x}\rangle - f(\bar{x}) + \varepsilon_f)\\ &\quad + (\langle \xi_g, \bar{x}\rangle - (\sum_{i=1}^{m} \lambda_i g_i)(\bar{x}) + \varepsilon_g) + \langle \xi_x, \bar{x}\rangle + \varepsilon_x. \qquad (7.37) \end{aligned}$$

Condition (7.36) leads to

$$0 \in \partial_{\varepsilon_f} f(\bar{x}) + \partial_{\varepsilon_g}(\sum_{i=1}^{m} \lambda_i g_i)(\bar{x}) + N_{X,\varepsilon_x}(\bar{x}). \qquad (7.38)$$

Condition (7.37) along with (7.36) and the nonnegativity conditions yields

$$\varepsilon_f + \varepsilon_g + \varepsilon_x - \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = 0.$$

From the above condition it is obvious that

$$\varepsilon_f = 0,\ \varepsilon_g = 0,\ \varepsilon_x = 0, \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

Therefore, the condition (7.38) reduces to

$$0 \in \partial f(\bar{x}) + \partial(\sum_{i=1}^{m} \lambda_i g_i)(\bar{x}) + N_X(\bar{x}).$$

As $dom\ g_i = \mathbb{R}^n$, $i = 1, 2, \ldots, m$, by Theorem 2.91, the above condition becomes

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + N_X(\bar{x}),$$

which along with the complementary slackness condition yields the desired optimality conditions.

Conversely, suppose that the optimality conditions hold. Therefore, there exist $\xi \in \partial f(\bar{x})$ and $\xi_i \in \partial g_i(\bar{x})$ such that

$$-\xi - \sum_{i=1}^{m} \lambda_i \xi_i \in N_X(\bar{x}),$$

that is,

$$\langle \xi + \sum_{i=1}^{m} \lambda_i \xi_i, x - \bar{x} \rangle \geq 0, \ \forall \ x \in X.$$

The convexity of f and g_i, $i = 1, 2, \ldots, m$, along with Definition 2.77 of the subdifferentials, imply that

$$f(x) - f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(x) - \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \geq 0, \ \forall \ x \in X.$$

In particular, for $x \in \bar{C}$, that is, $g_i(x) \leq 0$, $i = 1, 2, \ldots, m$, along with the complementary slackness condition, reduces the above condition to

$$f(x) \geq f(\bar{x}), \ \forall \ x \in C.$$

Thus, $\bar{x}$ is a point of minimizer of (CP). □

7.4 Applications to Bilevel Programming Problems

Consider the following bilevel problem:

$$\min f(x) \quad \text{subject to} \quad x \in C, \tag{BP}$$

where C is given as

$$C = \operatorname{argmin}\{\phi(x) : x \in \Theta\},$$

$f, \phi : \mathbb{R}^n \to \mathbb{R}$ are convex functions, and $\Theta \subset \mathbb{R}^n$ is a convex set. Thus it is clear that C is a convex set and hence the problem (BP) is a convex programming problem. As C is the solution set to a subproblem, which is again a convex optimization problem, here we call (BP) a *simple convex bilevel programming problem.* In particular, (BP) contains the standard differentiable convex optimization problem of the form

$$\min f(x) \quad \text{subject to} \quad g_i(x) \leq 0, \ i = 1, 2, \ldots, m, \quad Ax = b,$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are differentiable convex functions, A is an $l \times n$ matrix, and $b \in \mathbb{R}^l$. This problem can be posed as the problem (BP) by defining ϕ as

$$\phi(x) = ||Ax - b||^2 + \sum_{i=1}^{m} || \max\{0, g_i(x)\}||^2,$$

and the lower-level problem is to minimize ϕ over $\mathbb{R}^n$.

The bilevel programming problem (BP) can be equivalently expressed as a convex programming problem by assuming C to be nonempty and defining

$$\alpha = \inf_{x \in \Theta} \phi(x).$$

Then the reformulated problem is given by

$$\min \ f(x) \quad \text{subject to} \quad \phi(x) \leq \alpha, \ \ x \in \Theta. \tag{RP}$$

Observe that (RP) has the same form as the convex programming problem (CP) studied in the previous section. From the definition of α, it is easy to see that there does not exist any $\hat{x} \in \Theta$ such that $\phi(\hat{x}) < \alpha$, which implies that the Slater constraint qualification does not hold for (RP). We present the KKT optimality condition as a consequence of Theorem 7.13.

Theorem 7.14 *Consider the reformulated problem (RP). Assume that*

$$\{cone\ \{(0,1)\} \cup cone\ [(0,\alpha) + epi\ \phi^*]\} + epi\ \delta_\Theta^*$$

is closed. Then $\bar{x} \in \Theta$ is a point of minimizer of (RP) if and only if there is $\lambda \geq 0$ such that

$$0 \in \partial f(\bar{x}) + \lambda \partial \phi(\bar{x}) + N_\Theta(\bar{x}) \quad \textit{and} \quad \lambda(\phi(\bar{x}) - \alpha) = 0.$$

Proof. Observe that the problem (RP) is of the type considered in Theorem 7.13. We can invoke Theorem 7.13 if the CC qualification condition holds, that is,

$$epi\ f^* + \bigcup_{\mu \geq 0} epi(\mu(\phi(.) - \alpha))^* + epi\ \delta_\Theta^*$$

is closed. As $dom\ f = \mathbb{R}^n$, by Theorem 2.69, f is continuous on $\mathbb{R}^n$ and thus the CC qualification condition can be replaced by the FM constraint qualification, that is,

$$\bigcup_{\mu \geq 0} epi\ (\mu(\phi(.) - \alpha))^* + epi\ \delta_\Theta^* \tag{7.39}$$

is closed. For $\mu > 0$, by Proposition 2.103,

$$(\mu(\phi(.) - \alpha))^*(\xi) = \mu\alpha + (\mu\phi)^*(\xi),$$

which along with Theorem 2.123 leads to

$$\begin{aligned} epi\ (\mu(\phi(.) - \alpha))^* &= (0, \mu\alpha) + epi\ (\mu\phi)^* \\ &= \mu((0,\alpha) + epi\ \phi^*), \ \forall\ \mu > 0. \end{aligned} \tag{7.40}$$

For $\mu = 0$,

$$(\mu(\phi(.) - \alpha))^*(\xi) = 0^*(\xi) = \begin{cases} 0, & \xi = 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

which implies

$$epi\ (\mu(\phi(.) - \alpha))^* = 0 \times \mathbb{R}_+ = cone\ \{(0,1)\},\ \mu = 0. \tag{7.41}$$

Using (7.40) and (7.41), the condition (7.39) becomes

$$cone\{(0,1)\} \cup \{\bigcup_{\mu > 0} \mu((0,\alpha) + epi\ \phi^*)\} + epi\ \delta^*_\Theta.$$

Observe that $cone\{(0,1)\} \cup \{(0,0)\} = cone\{(0,1)\}$ and thus the above becomes

$$cone\{(0,1)\} \cup cone\ ((0,\alpha) + epi\ \phi^*) + epi\ \delta^*_\Theta. \tag{7.42}$$

By the hypothesis of the theorem, (7.42) is a closed set and thus the reformulated problem (RP) satisfies the FM constraint qualification. Now invoking Theorem 7.13, there exists $\lambda \geq 0$ such that

$$0 \in \partial f(\bar{x}) + \lambda \partial(\phi(.) - \alpha)(\bar{x}) + N_\Theta(\bar{x}) \quad \text{and} \quad \lambda(\phi(\bar{x}) - \alpha) = 0.$$

As $\partial(\phi(.) - \alpha)(\bar{x}) = \partial h(\bar{x})$, the optimality condition reduces to

$$0 \in \partial f(\bar{x}) + \lambda \partial \phi(\bar{x}) + N_\Theta(\bar{x}),$$

thereby establishing the desired result. The converse can be proved as in Chapter 3. □

For a better understanding of the above result, consider the bilevel programming problem where $f(x) = x^2 + 1$, $\Theta = [-1, 1]$, and $\phi(x) = \max\{0, x\}$. Observe that $C = [-1, 0]$ and $\alpha = 0$. Thus the reformulated problem is

$$\min\ x^2 + 1 \quad \text{subject to} \quad \max\ \{0, x\} \leq 0, \quad x \in [-1, 0].$$

For $\xi \in \mathbb{R}$,

$$\phi^*(\xi) = \begin{cases} +\infty, & \xi < 0 \text{ or } \xi > 1, \\ 0, & \xi \in [0,1], \end{cases}$$

which implies

$$epi\ \phi^* = \{(\xi, \gamma) \in \mathbb{R}^2 : \xi \in [0,1],\ \xi \geq 0\} = [0,1] \times \mathbb{R}_+$$

while $epi\ \delta^*_\Theta = epi\ |.|$. Therefore,

$$cone\ epi\ \phi^* + epi\ \delta^*_\Theta = \mathbb{R}^2_+ \cup \{(\xi, \gamma) \in \mathbb{R}^2 : \xi \leq 0,\ \gamma \geq -\xi\},$$

which is a closed set. Because $cone\{(0,1)\} \subset cone\ epi\ \phi^*$, the reformulated problem satisfies the qualification condition in Theorem 7.14. It is easy to see that $\bar{x} = 0$ is a solution of the bilevel problem with $N_\Theta(0) = \{0\}$, $\partial f(0) = \{0\}$, and $\partial \phi(0) = [0, 1]$. Thus the KKT optimality conditions of Theorem 7.14 are satisfied with $\lambda = 0$. Note that the Slater condition fails to hold for the reformulated problem.

We end this chapter by presenting the optimality conditions for the bilevel programming problem

$$\inf \ f(x) \quad \text{subject to} \quad x \in C, \tag{$BP1$}$$

where C is the solution set of the lower-level problem

$$\min \ \phi(x) \quad \text{subject to} \quad g_i(x) \leq 0, \ i = 1, 2, \ldots, m, \ x \in X.$$

Here, $\phi : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ is a proper, convex, lsc function and $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, 2, \ldots, m$, are convex functions, and $X \subset \mathbb{R}^n$ is a closed convex set. Define

$$\alpha = \inf\{\phi(x) : g_i(x) \leq 0, \ i = 1, 2, \ldots, m, \ x \in X\} < +\infty.$$

Without loss of generality, assume that $\alpha = 0$. This can be achieved by setting $\phi(x) = \phi(x) - \alpha$. Then the bilevel programming problem $(BP1)$ is equivalent to the following optimization problem:

$$\min f(x) \quad \text{subject to} \quad \phi(x) \leq 0, \ g_i(x) \leq 0, \ i = 1, 2, \ldots, m, \quad x \in X. \tag{$RP1$}$$

Below we present the result on optimality conditions for the bilevel programming problem $(BP1)$.

Theorem 7.15 *Consider the bilevel programming problem* $(BP1)$*. Assume that*

$$\{\bigcup_{\lambda \in \mathbb{R}^m_+} epi \ (\sum_{i=1}^m \lambda_i g_i)^*\} \cup \{\bigcup_{\lambda_0 > 0} \lambda_0 \ epi \ \phi^* + \bigcup_{\lambda \in \mathbb{R}^m_+} epi \ (\sum_{i=1}^m \lambda_i g_i)^*\} + epi \ \delta_X^*$$

is closed. Then $\bar{x} \in C$ *is a point of minimizer of* $(BP1)$ *if and only if there exist* $\lambda_0 \geq 0$ *and* $\lambda_i \geq 0, \ i = 1, 2, \ldots, m,$ *such that*

$$0 \in \partial f(\bar{x}) + \lambda_0 \partial \phi(\bar{x}) + \partial (\sum_{i=1}^m \lambda_i g_i)(\bar{x}) + N_X(\bar{x}),$$
$$\lambda_0 \phi(\bar{x}) = 0 \quad \textit{and} \quad \lambda_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m.$$

Proof. Observe that for any $\widetilde{\lambda} = (\lambda_0, \lambda) \in \mathbb{R}_+ \times \mathbb{R}^m_+$,

$$(\widetilde{\lambda} g)(x) = \lambda_0 \phi(x) + \sum_{i=1}^m \lambda_i g_i(x).$$

Therefore,

$$epi (\widetilde{\lambda} g)^* = cl\{epi \ (\lambda_0 \phi)^* + epi (\sum_{i=1}^m \lambda_i g_i)^*\}.$$

As *dom* $g_i = \mathbb{R}^n$, $i = 1, 2, \ldots, m$, *dom* $(\lambda_i g_i) = \mathbb{R}^n$, $i = 1, 2, \ldots, m$, which by Theorem 2.69 are continuous on $\mathbb{R}^n$. By Proposition 2.124,

$$epi(\widetilde{\lambda g})^* = epi\ (\lambda_0 \phi)^* + epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*. \tag{7.43}$$

Now consider the two cases, namely $\lambda_0 = 0$ and $\lambda_0 > 0$. For $\lambda_0 = 0$, $epi\ (\lambda_0 \phi)^* = cone\ \{(0,1)\}$. Thus, the condition (7.43) reduces to

$$epi(\widetilde{\lambda g})^* = cone\ \{(0,1)\} + epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*.$$

Observe that for $\mu \geq 0$,

$$\mu(0,1) + (\xi, \alpha) = (\xi, \alpha + \mu) \in epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*,$$

where $(\xi, \alpha) \in epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*$. Because $\mu \geq 0$ was arbitrary,

$$cone\ \{(0,1)\} + epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* \subset epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*.$$

Also, for any $(\xi, \alpha) \in epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*$,

$$(\xi, \alpha) = (0,0) + (\xi, \alpha) \in cone\ \{(0,1)\} + epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*.$$

As $(\xi, \alpha) \in epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*$ was arbitrary,

$$epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* \subset cone\ \{(0,1)\} + epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*,$$

thereby implying that

$$cone\ \{(0,1)\} + epi\ (\sum_{i=1}^{m} \lambda_i g_i)^* = epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*.$$

Thus, for $\lambda_0 = 0$,

$$epi(\widetilde{\lambda g})^* = epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*.$$

For the case when $\lambda_0 > 0$, the condition (7.43) becomes

$$epi(\widetilde{\lambda}g)^* = \lambda_0 epi\ \phi * + epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*.$$

Therefore,

$$\bigcup_{\widetilde{\lambda}\in\mathbb{R}^{1+m}_+} epi(\widetilde{\lambda}g)^* + epi\ \delta_X^* = \{\bigcup_{\lambda\in\mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*)\} \cup$$

$$\{\bigcup_{\lambda_0>0} epi\ \phi^* + \bigcup_{\lambda\in\mathbb{R}^m_+} epi\ (\sum_{i=1}^{m} \lambda_i g_i)^*\} + epi\ \delta_X^*.$$

By the given hypothesis, the set

$$\bigcup_{\widetilde{\lambda}\in\mathbb{R}^{1+m}_+} epi(\widetilde{\lambda}g)^* + epi\ \delta_X^*$$

is closed. Hence, the FM constraint qualification holds for the problem $(RP1)$. Because $dom\ f = \mathbb{R}^n$, by Theorem 2.69 f is continuous on $\mathbb{R}^n$, CC qualification condition holds for $(RP1)$. As the bilevel problem $(BP1)$ is equivalent to $(RP1)$, by Theorem 7.11, $\bar{x} \in C$ is a point of minimizer of $(BP1)$ if and only if there exists $\widetilde{\lambda} = (\lambda_0, \lambda) \in \mathbb{R}_+ \times \mathbb{R}^m_+$ such that

$$0 \in \partial f(\bar{x}) + \partial(\widetilde{\lambda}g)(\bar{x}) + N_X(\bar{x}) \quad \text{and} \quad (\widetilde{\lambda}g)(\bar{x}) = 0. \tag{7.44}$$

As ϕ is proper convex, $\lambda_0\phi$ is also proper convex. Therefore, $dom\ (\lambda_0\phi)$ is a nonempty convex set in $\mathbb{R}^n$. By Proposition 2.14 (i), $ri\ dom\ (\lambda_0\phi)$ is nonempty. Because $dom\ g = \mathbb{R}^n$, $dom\ (\lambda g) = \mathbb{R}^n$. Now invoking the Sum Rule, Theorem 2.91,

$$\partial(\widetilde{\lambda}g)(\bar{x}) = \lambda_0\partial\phi(\bar{x}) + \partial(\sum_{i=1}^{m} \lambda_i g_i)(\bar{x}).$$

Thus the optimality condition in (7.44) becomes

$$0 \in \partial f(\bar{x}) + \lambda_0\partial\phi(\bar{x}) + \partial(\sum_{i=1}^{m} \lambda_i g_i)(\bar{x}) + N_X(\bar{x}). \tag{7.45}$$

By the complementary slackness condition in (7.44),

$$(\widetilde{\lambda}g)(\bar{x}) = \lambda_0\phi(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = 0.$$

As $(\lambda_0, \lambda) \in \mathbb{R}_+ \times \mathbb{R}^m_+$, which along with the feasibility of $\bar{x}$ yields that

$$\lambda_0\phi(\bar{x}) = 0 \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

The above condition together with (7.45) leads to the requisite conditions. The converse can be proved as in Chapter 3. $\square$

Chapter 8

Representation of the Feasible Set and KKT Conditions

8.1 Introduction

Until now, we discussed the convex programming problem (CP) with the convex feasible set C given by (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2 \ldots, m\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are convex functions and its variations like $(CP1)$ and (CCP). But is the convexity of the functions forming the convex feasible set C important? For example, assume C as a subset of $\mathbb{R}^2$ given by

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1 x_2 \leq 0,\ x_1 \geq 0\}.$$

This set is convex even though $g(x_1, x_2) = 1 - x_1 x_2$ is a nonconvex function. As stated in Chapter 1, convex optimization basically means minimizing a convex function over a convex set with no emphasis on as to how the feasible set is obtained. Very recently (2010), Lasserre [74] published a very interesting paper discussing this aspect of convex feasibility for smooth convex optimization.

8.2 Smooth Case

In this section, we turn our attention to the case of smooth convex optimization studied by Lasserre [74]. From Chapter 2 we know that when a convex function is differentiable, then $\partial\phi(x) = \{\nabla\phi(x)\}$ and its gradient is also continuous; thus any differentiable convex function is smooth. So one can obtain the KKT optimality conditions at the point of minimizer from the subdifferential optimality conditions discussed in Chapter 3; that is, if $\bar{x}$ is the point of minimizer of (CP) with (C) given by (3.1), then there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0 \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

Observe that the KKT conditions for smooth convex optimization problems look absolutely the same as the KKT conditions for the usual smooth optimization problem. As discussed in earlier chapters, under certain constraint qualifications like the Slater constraint qualification, the above KKT conditions are necessary as well as sufficient.

Lasserre observed that the convex feasible set C of (CP) need not always be defined by convex inequality constraints as in the above example. The question that Lasserre answers is "*in such a scenario what conditions would make the KKT optimality conditions necessary as well as sufficient?*" So now the convex set C given by (3.1) is considered, with the only difference that $g_i,\ i = 1, 2, \ldots, m$, need not be convex even though they are assumed to be smooth. Lasserre showed that if the Slater constraint qualification and an additional nondegeneracy condition hold, then the KKT condition is both necessary and sufficient. Though Lasserre defined the notion of nondegeneracy for every point of the set C, we define it for a particular point and extend it to the feasible set C.

Definition 8.1 The *nondegeneracy condition* is said to hold at $\bar{x} \in C$ if

$$\nabla g_i(\bar{x}) \neq 0, \ \forall\ i \in I(\bar{x}),$$

where $I(\bar{x}) = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = 0\}$ denotes the active index set at $\bar{x}$. The set C is said to satisfy the *nondegeneracy condition* if it holds for every $\bar{x} \in C$.

The Slater constraint qualification along with the nondegeneracy condition gives the following interesting characterization of a convex set given by Lasserre [74].

Theorem 8.2 *Consider the set C given by (3.1) where $g_i,\ i = 1, 2, \ldots, m$, are smooth. Assume that the Slater constraint qualification is satisfied, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$, and the nondegeneracy condition holds for C. Then C is convex if and only if*

$$\langle \nabla g_i(x), y - x \rangle \leq 0, \ \forall\ x, y \in C, \ \forall\ i \in I(x). \tag{8.1}$$

Proof. Suppose that C is a convex set and consider $\bar{x} \in C$. Therefore, for any $y \in C$, for every $\lambda \in [0, 1]$, $\bar{x} + \lambda(y - \bar{x}) \in C$, that is,

$$g_i(\bar{x} + \lambda(y - \bar{x})) \leq 0, \ \forall\ i = 1, 2, \ldots, m.$$

Now for $i \in I(\bar{x})$,

$$\lim_{\lambda \downarrow 0} \frac{g_i(\bar{x} + \lambda(y - \bar{x})) - g_i(\bar{x})}{\lambda} \leq 0,$$

that is, for every $i \in I(\bar{x})$,

$$\langle \nabla g_i(\bar{x}), y - \bar{x} \rangle \leq 0, \ \forall\ y \in C.$$

Because $\bar{x} \in C$ is arbitrary, the above inequality holds for every $\bar{x} \in C$, thereby establishing the desired inequality.

Conversely, suppose that the condition (8.1) holds. Observe that C has an interior because the Slater constraint qualification holds. Further, (8.1) along with the nondegeneracy condition of the set C implies that each boundary point of C has a nontrivial supporting hyperplane. The supporting hyperplane is nontrivial due to the non-degeneracy condition on C. Using Proposition 2.29, C is a convex set. $\square$

Observe that the nondegeneracy condition of the set C is required only in the sufficiency part of the proof.

Till now in the book, we have mainly dealt with the Slater constraint qualification and some others namely, Abadie, pseudonormality and the FM constraint qualification. Another well known constraint qualification for the convex programming problem (CP) is the Mangasarian–Fromovitz constraint qualification. For (CP) with C given by (3.1) in the smooth scenario, *Mangasarian–Fromovitz constraint qualification* is said to hold at $\bar{x} \in \mathbb{R}^n$ if there exists $d \in \mathbb{R}^n$ such that

$$\langle \nabla g_i(\bar{x}), d \rangle < 0, \ \forall \, i \in I(\bar{x}).$$

One may observe that if this constraint qualification is satisfied for $\bar{x} \in C$, then $\nabla g_i(\bar{x}) \neq 0$, for every $i \in I(\bar{x})$, thereby ensuring that the nondegeneracy condition at $\bar{x}$. But the converse need not hold, that is the nondegeneracy condition need not imply the Mangasarian–Fromovitz constraint qualification. We verify this claim by the following example. Consider the set $C \subset \mathbb{R}^2$ given by

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1 x_2 \leq 0, \ x_1 + x_2 - 2 \leq 0, \ x_1 \geq 0\}.$$

Here, $g_1(x_1, x_2) = 1 - x_1 x_2$, $g_2(x_1, x_2) = x_1 + x_2 - 2$ and $g_3(x_1, x_2) = -x_1$. Note that $C = \{(1,1)\}$ and thus trivially is a convex set. At $\bar{x} = (1,1)$, the active index set is $I(\bar{x}) = \{1, 2\}$ and

$$\nabla g_1(\bar{x}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq 0 \quad \text{and} \quad \nabla g_2(\bar{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq 0,$$

which implies that the nondegeneracy condition is satisfied for $C = \{\bar{x}\}$. But observe that there exists no $(d_1, d_2) \in \mathbb{R}^2$ satisfying

$$-d_1 - d_2 < 0 \quad \text{and} \quad d_1 + d_2 < 0$$

simultaneously, thereby not satisfying the Mangasarian–Fromovitz constraint qualification at $\bar{x}$.

We end this section by presenting the result from Lasserre [74] establishing the necessary and sufficient optimality condition for a minimizer of (CP) over C with $g_i, \ i = 1, 2, \ldots, m$, nonconvex smooth functions. As one will observe from the result below, the nondegeneracy condition is required only for the necessary part at the given point and not the set as mentioned in the statement of the theorem in Lasserre [74]. Also in the converse part, we require the necessary part of Theorem 8.2, which is independent of the nondegeneracy condition.

Theorem 8.3 *Consider the problem (CP) where f is smooth and C is given by (3.1), where g_i, $i = 1, 2, \ldots, m$, are smooth but need not be convex. Assume that the Slater constraint qualification is satisfied and the nondegeneracy condition holds at $\bar{x} \in C$. Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, such that*

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0 \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m.$$

Proof. Let $\bar{x}$ be a point of minimizer of f over C. By the Fritz John optimality conditions, Theorem 5.1, there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, not all simultaneously zero such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0 \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m.$$

Suppose that $\lambda_0 = 0$, which implies for some $i \in \{1, 2, \ldots, m\}$, $\lambda_i > 0$. Therefore, the set

$$\bar{I} \quad = \quad \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$$

is nonempty. By the complementary slackness condition,

$$g_i(\bar{x}) = 0, \ \forall \ i \in \bar{I},$$

which implies $\bar{I} \subset I(\bar{x})$. By the optimality condition,

$$\sum_{i \in \bar{I}} \lambda_i \nabla g_i(\bar{x}) = 0,$$

which implies that

$$\sum_{i \in \bar{I}} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle = 0, \ \forall \ x \in C.$$

As the Slater constraint qualification is satisfied, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$ for every $i = 1, 2, \ldots, m$. As *dom* $g_i = \mathbb{R}^n$, $i = 1, 2, \ldots, m$, by Theorem 2.69, g_i, $i = 1, 2, \ldots, m$, are continuous on $\mathbb{R}^n$. Thus there exists $\delta > 0$ such that for every $x \in \mathbb{B}_\delta(\hat{x})$ and every $i = 1, 2, \ldots, m$, $g_i(x) < 0$, that is, $\mathbb{B}_\delta(\hat{x}) \subset$ *int* C. By the preceding equality,

$$\sum_{i \in \bar{I}} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle = 0, \ \forall \ x \in \mathbb{B}_\delta(\hat{x}). \tag{8.2}$$

As $\bar{I} \subset I(\bar{x})$, along with the convexity of C and Theorem 8.2, yields that for every $i \in \bar{I}$,

$$\langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \leq 0, \ \forall \ x \in C,$$

which along with the condition (8.2) implies that for $i \in \bar{I}$,

$$\langle \nabla g_i(\bar{x}), x - \bar{x} \rangle = 0, \ \forall \ x \in \mathbb{B}_\delta(\hat{x}). \tag{8.3}$$

Because $\hat{x} \in \mathbb{B}_\delta(\hat{x})$, the condition (8.3) reduces to

$$\langle \nabla g_i(\bar{x}), \hat{x} - \bar{x} \rangle = 0, \ \forall \ i \in \bar{I}. \tag{8.4}$$

For any $d \in \mathbb{R}^n$, consider the vector $\hat{x} + \lambda d$ such that for $\lambda > 0$ sufficiently small, $\hat{x} + \lambda d \in \mathbb{B}_\delta(\hat{x})$. Hence, by the condition (8.3), for each $i \in \bar{I}$,

$$\langle \nabla g_i(\bar{x}), \hat{x} + \lambda d - \bar{x} \rangle = 0,$$

which implies

$$\langle \nabla g_i(\bar{x}), \hat{x} - \bar{x} \rangle + \lambda \langle \nabla g_i(\bar{x}), d \rangle = 0.$$

By condition (8.4), for every $i \in \bar{I}$,

$$\langle \nabla g_i(\bar{x}), d \rangle = 0, \ \forall \ d \in \mathbb{R}^n.$$

Hence, $\nabla g_i(\bar{x}) = 0$ for every $i \in \bar{I} \subset I(\bar{x})$ and thereby contradicting the nondegeneracy condition at $\bar{x}$. Thus, $\lambda_0 \neq 0$. Dividing the Fritz John optimality condition by λ_0, the KKT optimality condition is established at $\bar{x}$ as

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0 \quad \text{and} \quad \bar{\lambda}_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m,$$

where $\bar{\lambda}_i = \dfrac{\lambda_i}{\lambda_0}, \ i = 1, 2, \ldots, m.$

Conversely, suppose that $\bar{x}$ satisfies the KKT optimality conditions. Assume that $\bar{x}$ is not a point of minimizer of (CP). Therefore, there exists $x \in C$ such that $f(x) < f(\bar{x})$, which along with the convexity of f implies that

$$0 > f(x) - f(\bar{x}) \geq \langle \nabla f(\bar{x}), x - \bar{x} \rangle.$$

Therefore, by the KKT optimality conditions,

$$0 > f(x) - f(\bar{x}) \geq -\sum_{i=1}^{m} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle. \tag{8.5}$$

If $\lambda_i = 0$ for every $i = 1, 2, \ldots, m$, we reach a contradiction. Now assume that $\bar{I} \neq \emptyset$. By Theorem 8.2, for every $i \in \bar{I} \subset I(\bar{x})$,

$$\langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall \ x \in C,$$

which implies that

$$\sum_{i=1}^{m} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle = \sum_{i \in \bar{I}} \lambda_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in C,$$

thereby contradicting the condition (8.5) and thus leading to the requisite result, that is, $\bar{x}$ is a point of minimizer of f over C. $\square$

8.3 Nonsmooth Case

Motivated by the above work of Lasserre [74], Dutta and Lalitha [40] extended the study to a nonsmooth scenario involving the locally Lipschitz function. But before we move on with the work done in this respect, we need some tools for nonsmooth Lipschitz functions.

Consider a locally Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$. The *Clarke directional derivative* of ϕ at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ is defined as

$$\phi^\circ(\bar{x}, d) = \lim_{x \to \bar{x},\ \lambda \downarrow 0} \frac{\phi(x + \lambda d) - \phi(x)}{\lambda}.$$

The Clarke directional derivative is a sublinear function of the direction d. In Section 3.6 we defined the Clarke subdifferential using the Rademacher Theorem. Here, we express the Clarke subdifferential of ϕ at $\bar{x}$ using the Clarke directional derivative defined above as

$$\partial^\circ \phi(\bar{x}) = \{\xi \in \mathbb{R}^n : \phi^\circ(\bar{x}, d) \geq \langle \xi, d \rangle, \ \forall\, d \in \mathbb{R}^n\}.$$

The function ϕ is said to be *regular* at $\bar{x}$ if for every $d \in \mathbb{R}^n$, the directional derivative $\phi'(\bar{x}, d)$ exists and

$$\phi^\circ(\bar{x}, d) = \phi'(\bar{x}, d), \ \forall\, d \in \mathbb{R}^n.$$

Every convex function is regular.

In the nonsmooth scenario, Dutta and Lalitha [40] considered the convex feasible set C of (CP) to be defined by inequality constraints involving nonsmooth locally Lipschitz functions that are regular. For example, consider

$$\phi(x) = \max\{\phi_1(x), \phi_2(x), \ldots, \phi_m(x)\},$$

where $\phi_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$, are smooth functions. Then ϕ is a locally Lipschitz regular function.

Now, similar to the nondegeneracy condition in the smooth case given by Lasserre [74], Dutta and Lalitha [40] defined the notion for nonsmooth locally Lipschitz scenario as follows.

Definition 8.4 Consider the set C given by (3.1) where each g_i, $i = 1, 2, \ldots, m$ is a locally Lipschitz function. The set C is said to satisfy the *nondegeneracy condition* at $\bar{x} \in C$ if

$$0 \notin \partial^\circ g_i(\bar{x}), \ \forall\, i \in I(\bar{x}).$$

If the condition holds for every $\bar{x} \in C$, the nondegeneracy condition is said to hold for the set C.

Before moving on to discuss the results obtained in this work, we present some examples from Dutta and Lalitha [40] to have a look at the above nondegeneracy condition.

Consider the set

$$C = \{x \in \mathbb{R} : g_0(x) \leq 0\},$$

where $g_0(x) = \max\{x^3, x\} - 1$. Hence $C = (-\infty, 1]$. At the boundary point $\bar{x} = 1$, $I(\bar{x}) = \{0\}$ where $\partial^\circ g_0(\bar{x}) = [1, 3]$, thereby satisfying the nondegeneracy condition. Now if we define the function $g_0(x) = \max\{x^3, x\}$, then $C = (-\infty, 0]$ with boundary point $\bar{x} = 0$ at which $\partial^\circ g_0(\bar{x}) = [0, 1]$. Thus, the nondegeneracy condition is not satisfied at $\bar{x}$. Observe that in both the cases g_0 is a regular function and the Slater constraint qualification is also satisfied. Yet in the second scenario the nondegeneracy condition is not satisfied.

But if the functions $g_i, \ i = 1, 2, \ldots, m$, involved are convex and the Slater constraint qualification holds for (CP), then the Mangasarian–Fromovitz constraint qualification for the nonsmooth case is satisfied at $\bar{x} \in C$, that is there exists $d \in \mathbb{R}^n$ such that

$$g_i'(\bar{x}, d) < 0, \ \forall\, i \in I(\bar{x}).$$

As the directional derivative is a support function to the subdifferential set, Theorem 2.79, the above condition is equivalent to

$$\langle \xi_i, d \rangle < 0, \ \forall\, \xi_i \in \partial g_i(\bar{x}), \ \forall\, i \in I(\bar{x}),$$

from which it is obvious that the nondegeneracy condition is ensured for the convex nonsmooth scenario.

Next we present the equivalent characterization of the convex set C under the nonsmooth scenario.

Theorem 8.5 *Consider the set C be given by (3.1) represented by nonsmooth locally Lipschitz inequality constraints where $g_i, \ i = 1, 2, \ldots, m$, are regular. Assume that the Slater constraint qualification holds and satisfies the nondegeneracy condition. Then C is convex if and only if*

$$g_i^\circ(x, y - x) \leq 0, \ \forall\, x, y \in C, \ \forall\, i \in I(x). \tag{8.6}$$

Proof. Consider the convex set C. Working along the lines of Theorem 8.2, for arbitrary but fixed $\bar{x} \in C$, for $\lambda \in (0,1)$,

$$\frac{g_i(\bar{x} + \lambda(y - \bar{x})) - g_i(\bar{x})}{\lambda} \leq 0, \ \forall\ i \in I(\bar{x}).$$

As the functions $g_i,\ i = 1, 2, \ldots, m$, are locally Lipschitz regular functions,

$$g_i^\circ(\bar{x}, y - \bar{x}) = g_i'(\bar{x}, y - \bar{x}) \leq 0, \ \forall\ y \in C, \ \forall\ i \in I(x),$$

thus leading to the requisite result.

Conversely, suppose that (8.6) holds. As the Slater constraint qualification holds, the set C has an interior. Now consider any boundary point $x \in C$. By the condition (8.6) along with the fact that the Clarke directional derivative is the support function of the Clarke subdifferential, then for every $y \in C$,

$$\langle \xi_i, y - x \rangle \leq g_i^\circ(x, y - x) \leq 0, \ \forall\ \xi_i \in \partial^\circ g_i(x), \ \forall\ i \in I(x).$$

As the nondegeneracy condition is satisfied, $\xi_i \neq 0$ for every $\xi_i \in \partial^\circ g_i(x)$ and every $i \in I(x)$, which implies that there is a nontrivial supporting hyperplane to C at x. Hence, by Proposition 2.29, C is a convex set, as desired. $\square$

Now we present the theorem establishing the necessary and sufficient optimality conditions for the class of problem (CP) dealt with in this section.

Theorem 8.6 *Consider the problem (CP) with C is given by (3.1), where $g_i,\ i = 1, 2, \ldots, m$, are locally Lipschitz regular functions. Assume that the Slater constraint qualification holds and the nondegeneracy condition is satisfied at $\bar{x} \in C$. Then $\bar{x}$ is a point of minimizer of (CP) if and only if there exist $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial^\circ g_i(\bar{x}) \quad \textit{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, \ldots, m.$$

Proof. Suppose that $\bar{x}$ is a point of minimizer of f over C. We know by Theorem 2.72 that a convex function f is locally Lipschitz. Then by the optimality conditions for locally Lipschitz functions at $\bar{x}$, there exist $\lambda_i \geq 0$, $i = 0, 1, \ldots, m$, not all simultaneously zero, such that

$$0 \in \lambda_0 \partial^\circ f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial^\circ g_i(\bar{x}) \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m.$$

Because f is convex, $\partial^\circ f(\bar{x}) = \partial f(\bar{x})$. Therefore, the optimality condition can be rewritten as

$$0 \in \lambda_0 \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial^\circ g_i(\bar{x}).$$

We claim that $\lambda_0 \neq 0$. On the contrary, suppose that $\lambda_0 = 0$. As λ_i, $i = 0, 1, \ldots, m$, are not all zeroes, the set $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$ is nonempty. Then the above optimality condition reduces to

$$0 \in \sum_{i \in \bar{I}} \lambda_i \partial^\circ g_i(\bar{x}),$$

which implies there exist $\xi_i \in \partial^\circ g_i(\bar{x})$, $i \in \bar{I}$ such that

$$0 = \sum_{i \in \bar{I}} \lambda_i \xi_i.$$

From the definition of the Clarke subdifferential, the above condition leads to

$$\sum_{i \in \bar{I}} \lambda_i g_i^\circ(\bar{x}, d) \geq \sum_{i \in \bar{I}} \lambda_i \langle \xi_i, d \rangle = 0, \ \forall\, d \in \mathbb{R}^n. \tag{8.7}$$

As the Slater constraint qualification is satisfied, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$ for every $i = 1, \ldots m$. Also, as g_i, $i = 1, 2, \ldots, m$, are locally Lipschitz, and hence continuous. Thus there exists $\delta > 0$ such that for every $x \in \mathbb{B}_\delta(\hat{x})$, $g_i(x) < 0$, $i = 1, \ldots, m$. In condition (8.7), in particular, taking $d = x - \bar{x}$ where $x \in \mathbb{B}_\delta(\hat{x}) \subset C$,

$$\sum_{i \in \bar{I}} \lambda_i g_i^\circ(\bar{x}, x - \bar{x}) \geq 0, \ \forall\, x \in \mathbb{B}_\delta(\hat{x}). \tag{8.8}$$

By the complementary slackness condition, $g_i(\bar{x}) = 0$ for every $i \in \bar{I}$, that is, $\bar{I} \subset I(\bar{x})$. Therefore, by Theorem 8.5, as C is a convex set, we have

$$g_i^\circ(\bar{x}, x - \bar{x}) \leq 0, \ \forall\, x \in \mathbb{B}_\delta(\hat{x}), \ \forall\, i \in \bar{I},$$

which along with the condition (8.8) implies that for every $i \in \bar{I}$,

$$g_i^\circ(\bar{x}, x - \bar{x}) = 0, \ \forall\, x \in \mathbb{B}_\delta(\hat{x}). \tag{8.9}$$

In particular, for $\hat{x} \in \mathbb{B}_\delta(\hat{x})$, the above condition reduces to

$$g_i^\circ(\bar{x}, \hat{x} - \bar{x}) = 0, \ \forall\, i \in \bar{I}. \tag{8.10}$$

Consider any $v \in \mathbb{R}^n$ and choose $\lambda > 0$ sufficiently small such that $\hat{x} + \lambda v \in \mathbb{B}_\delta(\hat{x})$. Hence, from the condition (8.9), for every $i \in \bar{I}$,

$$g_i^\circ(\bar{x}, \hat{x} + \lambda v - \bar{x}) = 0, \ \forall\, v \in \mathbb{R}^n.$$

As the Clarke generalized directional derivative is sublinear in the direction, for every $i \in \bar{I}$, the above condition becomes

$$g_i^\circ(\bar{x}, \hat{x} - \bar{x}) + \lambda g_i^\circ(\bar{x}, v) \geq 0, \ \forall\, v \in \mathbb{R}^n,$$

which by (8.10) leads to

$$g_i^\circ(\bar{x}, v) \geq 0, \ \forall\, v \in \mathbb{R}^n, \ \forall\, i \in \bar{I}.$$

From the definition of the Clarke subdifferential, $0 \in \partial^\circ g_i(\bar{x})$ for every $i \in \bar{I}$, thereby contradicting the nondegeneracy condition. Therefore $\lambda_0 \neq 0$ and dividing the optimality condition throughout by λ_0 reduces it to

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial^\circ g_i(\bar{x}) \quad \text{and} \quad \bar{\lambda}_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m,$$

where $\bar{\lambda}_i = \dfrac{\lambda_i}{\lambda_0}$, $i = 1, 2, \ldots, m$ leading to the requisite result.

Conversely, suppose that the conditions hold at $\bar{x}$. On the contrary, assume that $\bar{x}$ is not a point of minimizer of f over C. Thus, there exists $x \in C$ such that $f(x) < f(\bar{x})$, which along with the convexity of f,

$$0 > f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\, \xi \in \partial f(\bar{x}). \tag{8.11}$$

Using the optimality conditions at $\bar{x}$, there exists $\xi_0 \in \partial f(\bar{x})$ and $\xi_i \in \partial^\circ g_i(\bar{x})$, $i = 1, 2, \ldots, m$, such that

$$0 = \xi_0 + \sum_{i=1}^{m} \lambda_i \xi_i.$$

The above condition along with (8.11) leads to

$$0 > -\sum_{i=1}^{m} \lambda_i \langle \xi_i, x - \bar{x} \rangle,$$

which by the definition of Clarke subdifferential along with Theorem 8.5 yields

$$0 > -\sum_{i=1}^{m} \lambda_i g_i^\circ(\bar{x}, x - \bar{x}) \geq 0,$$

thereby leading to a contradiction. Therefore, $\bar{x}$ is a point of minimizer of (CP). $\square$

We end this chapter with an example from Dutta and Lalitha [40] to illustrate that in the absence of the nondegeneracy condition, even though the Slater constraint qualification and the regularity of the constraint functions hold, the KKT optimality condition need not be satisfied.

Consider the problem

$$\min \ f(x) \quad \text{subject to} \quad g_1(x) \leq 0, \ g_2(x) \leq 0$$

where

$$f(x) = -x, \quad g_1(x) = x^3 \text{ and } g_2(x) = \begin{cases} -x-1, & x \leq 0, \\ -1, & x > 0. \end{cases}$$

Then the feasible set is $C = [-1, 0]$ and the point of minimizer is $\bar{x} = 0$. Also, C does not satisfy the nondegeneracy condition but the Slater constraint qualification holds along with the constraint functions being regular. Observe that $\partial f(\bar{x}) = \{-1\}$, $\partial^\circ g_1(\bar{x}) = \{0\}$ and $\partial^\circ g_2(\bar{x}) = \partial g_2(\bar{x}) = [-1, 0]$, and thus the KKT optimality conditions are not satisfied. Now if in the above example one takes the objective function to be $f(x) = x$, then the point of minimizer is $\bar{x} = -1$ at which $\partial f(\bar{x}) = \{1\}$, $\partial^\circ g_1(\bar{x}) = \{3\}$, and $\partial^\circ g_2(\bar{x}) = \partial g_2(\bar{x}) = \{-1\}$. Observe that the KKT optimality conditions hold with $\lambda_1 = 0$ and $\lambda_2 = 1$.

Chapter 9

Weak Sharp Minima in Convex Optimization

9.1 Introduction

In the preceding chapters we studied the necessary and sufficient optimality conditions for $\bar{x} \in \mathbb{R}^n$ to be a point of minimizer for the convex optimization problem wherein a convex objective function f is minimized over a convex feasible set $C \subset \mathbb{R}^n$. From Theorem 2.90, if the objective function f is strictly convex, then the point of minimizer $\bar{x}$ is unique. The notion of unique minimizer was extended to the concept of *sharp minimum* or, equivalently, *strongly unique local minimum.* The ideas of sharp minimizer and strongly unique minimizer were introduced by Polyak [94, 95] and Cromme [29]. These notions played an important role in the approximation theory or the study of perturbation in optimization problems and also in the analysis of the convergence of algorithms [1, 26, 56]. Below we define the notion of sharp minimum.

Definition 9.1 A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ defined over a set $F \subset \mathbb{R}^n$ is said to be *sharp minima* at $\bar{x} \in F$ if there exists $\alpha > 0$ such that

$$\phi(x) - \phi(\bar{x}) \geq \alpha \, \|x - \bar{x}\|, \ \forall \ x \in F.$$

From the above definition it is obvious that a point of sharp minimizer is unique. This is one of the major drawbacks of the concept of sharp minimum as it rules out the most basic optimization problem, namely the *linear programming problem.* To overcome this difficulty, the notion of *weak sharp minimum* was introduced by Ferris [46]. We study this notion for the convex optimization problem

$$\min \ f(x) \quad \text{subject to} \quad x \in C, \tag{CP}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $C \subset \mathbb{R}^n$ is a closed convex set.

9.2 Weak Sharp Minima and Optimality

We begin this section by defining the weak sharp minimum for the convex optimization problem (CP) from Ferris [46].

Definition 9.2 Let $\mathcal{S} \subset \mathbb{R}^n$ denote the nonempty solution set of (CP). Then $\mathcal{S}$ is said to be the set of *weak sharp minimizer* on C if there exists $\alpha > 0$ such that

$$f(x) - f(proj_{\mathcal{S}}(x)) \geq \alpha \, \|x - proj_{\mathcal{S}}(x)\|, \; \forall \, x \in C.$$

Observe that for any $x \in C$, $proj_{\mathcal{S}}(x) \in \mathcal{S}$ and as $\mathcal{S}$ is the solution set,

$$f(\bar{x}) = \text{constant}, \; \forall \, \bar{x} \in \mathcal{S}.$$

Equivalently, $\mathcal{S}$ is the set of the weak sharp minimizer if there exists $\alpha > 0$ such that

$$f(x) - f(\bar{x}) \geq \alpha \, d_{\mathcal{S}}(x), \; \forall \, x \in C, \; \forall \, \bar{x} \in \mathcal{S}.$$

The equivalent definition was given in Burke and Ferris [25].

Before moving on with the results on equivalent conditions for weak sharp minimizers, we present some results from Aubin and Ekeland [4], Lucchetti [79], Luenberger [80], and Rockafellar [97], which act as a tool in proving the equivalence.

Proposition 9.3 *Consider nonempty closed convex set $F \subset \mathbb{R}^n$.*

(i) For every $x \in F$,

$$N_F(x) = \{v \in \mathbb{R}^n : \langle v, x \rangle = \sigma_F(v)\}.$$

(ii) For every $y \in \mathbb{R}^n$,

$$d_F(y) = \max_{v \in cl \, \mathbb{B}} (\langle v, y \rangle - \sigma_F(v)).$$

(iii) If F is a closed convex cone, then for every $y \in \mathbb{R}^n$,

$$d_F(y) = \sigma_{cl \, \mathbb{B} \cap F^\circ}(y).$$

(iv) For every $y \in \mathbb{R}^n$,

$$d_F(y) = \sup_{x \in F} d_{x + T_F(x)}(y).$$

(v) For every $x \in F$, the subdifferential of the distance function d_F is

$$\partial d_F(x) = cl \, \mathbb{B} \cap N_F(x)$$

and the directional derivative is

$$d'_F(x, v) = d_{T_F(x)}(v) = \sigma_{cl \, \mathbb{B} \cap N_F(x)}(v), \; \forall \, v \in \mathbb{R}^n.$$

Proof. (i) From Definition 2.36 of normal cone,

$$N_F(\bar{x}) = \{v \in \mathbb{R}^n : \langle v, x - \bar{x}\rangle \leq 0, \ \forall\, x \in F\}.$$

Observe that any $v \in N_F(\bar{x})$ along with the fact that $\bar{x} \in F$ satisfies the inequality

$$\langle v, \bar{x}\rangle \leq \sigma_F(v) \leq \langle v, \bar{x}\rangle,$$

that is, $\sigma_F(v) \leq \langle v, \bar{x}\rangle$. Thus

$$N_F(\bar{x}) = \{v \in \mathbb{R}^n : \langle v, \bar{x}\rangle = \sigma_F(v)\}.$$

(ii) By the definition of the distance function,

$$\begin{aligned} d_F(y) = \inf_{x \in F} \|y - x\| &= \inf_{x \in F} \sup_{v \in cl\ \mathbb{B}} \langle v, y - x\rangle \\ &= \sup_{v \in cl\ \mathbb{B}} \{\langle v, y\rangle + \inf_{x \in F}(-\langle v, x\rangle)\} \\ &= \sup_{v \in cl\ \mathbb{B}} \{\langle v, y\rangle - \sigma_F(v)\}. \end{aligned}$$

(iii) For a closed convex cone F, by Definition 2.30 of polar cone,

$$F^\circ = \{v \in \mathbb{R}^n : \langle v, x\rangle \leq 0, \ \forall\, x \in F\}.$$

Therefore,

$$\sigma_F(v) = \begin{cases} 0, & v \in F^\circ, \\ +\infty, & \text{otherwise}. \end{cases} \tag{9.1}$$

From (ii), which along with the above relation (9.1) yields that

$$d_F(y) = \sup_{v \in cl\ \mathbb{B}} \langle v, y\rangle, \text{ provided } v \in F^\circ,$$

which implies

$$d_F(y) = \sup_{v \in cl\ \mathbb{B} \cap F^\circ} \langle v, y\rangle = \sigma_{cl\ \mathbb{B} \cap F^\circ}(y),$$

as desired.

(iv) By Theorem 2.35, $T_F(x)$ is a closed convex cone and hence $x + T_F(x)$ is also a closed convex cone. Invoking (iii) along with Proposition 2.37 leads to

$$d_{x + T_F(x)}(y) = d_{T_F(x)}(y - x) = \sigma_{cl\ \mathbb{B} \cap N_F(x)}(y - x).$$

Therefore,

$$\sup_{x \in F} d_{x + T_F(x)}(y) = \sup_{x \in F} \sup_{v \in cl\ \mathbb{B} \cap N_F(x)} \langle v, y - x\rangle.$$

By (i) and (ii), the above condition reduces to

$$\sup_{x \in F} d_{x+T_F(x)}(y) = \sup_{v \in cl\ \mathbb{B}} \{\langle v, y\rangle - \sigma_F(v)\} = d_F(y),$$

thereby establishing the result.

(v) As an example to inf-convolution, Definition 2.54,

$$d_F(x) = (\|.\| \,\square\, \delta_F)(x),$$

which is exact at every $x \in \mathbb{R}^n$. Invoking the subdifferential inf-convolution rule at the point where the inf-convolution is exact, Theorem 2.98,

$$\partial d_F(x) = \partial\|.\|(y) \cap \partial\delta_F(x - y).$$

For $x \in int\ F$, taking $y = 0$,

$$\partial\|.\|(0) = cl\ \mathbb{B} \quad \text{while} \quad \partial\delta_F(x) = N_F(x) = \{0\}.$$

Thus, $\partial d_F(x) = \{0\}$ for $x \in int\ F$. For $x \in bdry\ F$, again taking $y = 0$,

$$\partial\|.\|(0) = cl\ \mathbb{B} \quad \text{while} \quad \partial\delta_F(x) = N_F(x),$$

and hence $\partial d_F(x) = \mathbb{B} \cap N_F(x)$. Therefore,

$$\partial d_F(x) = cl\ \mathbb{B} \cap N_F(x), \ \forall\ x \in F. \tag{9.2}$$

As $dom\ d_F = \mathbb{R}^n$, by Theorem 2.79 and the condition (9.2),

$$d'_F(x, v) = \sigma_{\partial d_F(x)}(v) = \sigma_{cl\ \mathbb{B} \cap N_F(x)}(v),$$

which by (iii) implies that

$$d'_F(x, v) = d_{T_F(x)}(v)$$

and hence the result. $\square$

As we know, the convex optimization problem can be equivalently expressed as the unconstrained problem

$$\min\ f_0(x) \quad \text{subject to} \quad x \in \mathbb{R}^n, \tag{CP_u}$$

where $f_0(x) = f(x) + \delta_C(x)$ is an lsc proper convex function. As (CP) and (CP_u) are equivalent, the solution set of both problems coincide, which implies that $\mathcal{S}$ is also the set of weak sharp minimizers of (CP_u). Before moving on to prove the main result on the characterization of the weak sharp minimizer, we present the results in terms of the objective function f_0 of (CP_u).

Lemma 9.4 *Consider the unconstrained convex optimization problem* (CP_u) *and the set of weak sharp minimizers* $\mathcal{S}$*. Let* $\alpha > 0$*. Then the following are equivalent:*

(i) $\alpha \, cl \, \mathbb{B} \cap N_{\mathcal{S}}(x) \subset \partial f_0(x)$ *for every* $x \in \mathcal{S}$,

(ii) $\alpha \, cl \, \mathbb{B} \cap \bigcup_{x \in \mathcal{S}} N_{\mathcal{S}}(x) \subset \bigcup_{x \in \mathcal{S}} \partial f_0(x)$.

Proof. It is easy to observe that (i) implies (ii). Conversely, suppose that (ii) holds. Consider $\bar{x} \in \mathcal{S}$ with $\xi \in \alpha \, cl \, \mathbb{B} \cap N_{\mathcal{S}}(\bar{x})$. As (ii) is satisfied, there exists $\bar{y} \in \mathcal{S}$ such that $\xi \in \partial f_0(\bar{y})$. By Definition 2.77 of subdifferential,

$$f_0(x) - f_0(\bar{y}) \geq \langle \xi, x - \bar{y} \rangle, \ \forall \, x \in \mathbb{R}^n. \tag{9.3}$$

In particular, for any $x \in \mathcal{S}$, $f_0(x) = f_0(\bar{y})$, thereby reducing the above inequality to

$$\langle \xi, x - \bar{y} \rangle \leq 0, \ \forall \, x \in \mathcal{S},$$

which implies $\xi \in N_{\mathcal{S}}(\bar{y})$. By assumption, $\xi \in N_{\mathcal{S}}(\bar{x})$. Thus, by Proposition 9.3 (i),

$$\langle \xi, \bar{x} \rangle = \sigma_{\mathcal{S}}(\xi) = \langle \xi, \bar{y} \rangle. \tag{9.4}$$

As $\bar{x} \in \mathcal{S}$, $f_0(\bar{x}) = f_0(\bar{y})$, which along the conditions (9.3) and (9.4) leads to

$$f_0(x) - f_0(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall \, x \in \mathbb{R}^n,$$

thereby implying that $\xi \in \partial f_0(\bar{x})$. Because $\bar{x} \in \mathcal{S}$ was arbitrary, (i) holds. $\square$

The above result was from Burke and Ferris [25]. The next result from Burke and Deng [22] provides a characterization for weak sharp minimizer in terms of f_0.

Theorem 9.5 *Consider the convex optimization problem* (CP) *and its equivalent unconstrained problem* (CP_u). *Let* $\alpha > 0$. *Then* $\mathcal{S}$ *is the set of weak sharp minimizers with modulus* α *if and only if*

$$f_0'(\bar{x}, v) \geq \alpha \, d_{T_{\mathcal{S}}(\bar{x})}(v), \ \forall \, \bar{x} \in \mathcal{S}, \ \forall \, v \in \mathbb{R}^n. \tag{9.5}$$

Proof. Suppose that $\mathcal{S}$ is the set of weak sharp minimizers with modulus $\alpha > 0$. Consider $\bar{x} \in \mathcal{S}$. Therefore, by Definition 9.2,

$$f(x) - f(\bar{x}) \geq \alpha \, d_{\mathcal{S}}(x), \ \forall \, x \in C.$$

As $\bar{x} \in \mathcal{S} \subset C$, $f_0(\bar{x}) = f(\bar{x})$. Also for $x \in C$, $f_0(x) = f(x)$. Therefore, the above inequality leads to

$$f_0(x) - f_0(\bar{x}) \geq \alpha \, d_{\mathcal{S}}(x), \ \forall \, x \in C. \tag{9.6}$$

For $x \notin C$, $f_0(x) = +\infty$. Thus,

$$f_0(x) - f_0(\bar{x}) \geq \alpha \, d_{\mathcal{S}}(x), \ \forall \, x \notin C \tag{9.7}$$

trivially. Combining (9.6) and (9.7) yields

$$f_0(x) - f_0(\bar{x}) \geq \alpha\, d_{\mathcal{S}}(x), \ \forall\, x \in \mathbb{R}^n.$$

In particular, taking $x = \bar{x} + \lambda v \in \mathbb{R}^n$ for $\lambda > 0$ and $v \in \mathbb{R}^n$ in the above condition leads to

$$f_0(\bar{x} + \lambda v) - f_0(\bar{x}) \geq \alpha\, d_{\mathcal{S}}(\bar{x} + \lambda v), \ \forall\, \lambda > 0, \ \forall\, v \in \mathbb{R}^n,$$

which implies that for every $\lambda > 0$,

$$\frac{f_0(\bar{x} + \lambda v) - f_0(\bar{x})}{\lambda} \geq \alpha \frac{d_{\mathcal{S}}(\bar{x} + \lambda v)}{\lambda}, \ \forall\, v \in \mathbb{R}^n. \tag{9.8}$$

Observe that

$$d_{\mathcal{S}}(\bar{x} + \lambda v) = \inf_{x \in \mathcal{S}} \|\bar{x} + \lambda v - x\| = \lambda \inf_{x \in \mathcal{S}} \|v - \frac{(x - \bar{x})}{\lambda}\|,$$

which by Definition 2.33 of tangent cone implies that

$$\frac{d_{\mathcal{S}}(\bar{x} + \lambda v)}{\lambda} \geq \inf_{y \in T_{\mathcal{S}}(\bar{x})} \|v - y\| = d_{T_{\mathcal{S}}(\bar{x})}(v). \tag{9.9}$$

Therefore, using (9.8) along with (9.9) leads to

$$\frac{f_0(\bar{x} + \lambda v) - f_0(\bar{x})}{\lambda} \geq \alpha d_{T_{\mathcal{S}}(\bar{x})}(v), \ \forall\, v \in \mathbb{R}^n.$$

Taking the limit as $\lambda \to 0$ in the above inequality reduces it to

$$f_0'(\bar{x}, v) \geq \alpha d_{T_{\mathcal{S}}(\bar{x})}(v), \ \forall\, v \in \mathbb{R}^n.$$

Because $\bar{x} \in \mathcal{S}$ was arbitrary, the above condition yields (9.5).

Conversely, suppose that the relation (9.5) is satisfied. Consider $x \in C$ and $\bar{x} \in \mathcal{S}$. Therefore,

$$f_0(x) - f_0(\bar{x}) \geq f_0'(\bar{x}, x - \bar{x}) \geq \alpha\, d_{T_{\mathcal{S}}(\bar{x})}(x - \bar{x}) = \alpha\, d_{\bar{x} + T_{\mathcal{S}}(\bar{x})}(x).$$

By Proposition 9.3 (iv), the above inequality leads to

$$f_0(x) - f_0(\bar{x}) \geq \alpha \sup_{\bar{x} \in \mathcal{S}} d_{\bar{x} + T_{\mathcal{S}}(\bar{x})}(x) = \alpha\, d_{\mathcal{S}}(x).$$

Because $x \in C$ and $\bar{x} \in \mathcal{S}$ were arbitrary, the above condition holds for every $x \in C$ and every $\bar{x} \in \mathcal{S}$, and hence $\mathcal{S}$ is the set of weak sharp minimizers. $\square$

We end this chapter by giving equivalent characterizations for the set of weak sharp minimizers, $\mathcal{S}$, for (CP) from Burke and Deng [22].

Theorem 9.6 *Consider the convex optimization problem (CP) and its equivalent unconstrained problem (CP_u). Let $\alpha > 0$. Then the following statements are equivalent:*

(i) $\mathcal{S}$ is the set of weak sharp minimizers for (CP) with modulus $\alpha > 0$.

(ii) For every $\bar{x} \in \mathcal{S}$ and $v \in T_C(\bar{x})$,

$$f'(\bar{x}, v) \geq \alpha \, d_{T_{\mathcal{S}}(\bar{x})}(v).$$

(iii) For every $\bar{x} \in \mathcal{S}$,

$$\alpha \, cl \, \mathbb{B} \cap N_{\mathcal{S}}(\bar{x}) \subset \partial f_0(\bar{x}).$$

(iv) The inclusion

$$\alpha \, cl \, \mathbb{B} \cap \bigcup_{\bar{x} \in \mathcal{S}} N_{\mathcal{S}}(\bar{x}) \subset \bigcup_{\bar{x} \in \mathcal{S}} \partial f_0(\bar{x})$$

holds.

(v) For every $\bar{x} \in \mathcal{S}$ and $v \in T_C(\bar{x}) \cap N_{\mathcal{S}}(\bar{x})$,

$$f'(\bar{x}, v) \geq \alpha \, \|v\|.$$

(vi) For every $\bar{x} \in \mathcal{S}$,

$$\alpha \, \mathbb{B} \subset \partial f(\bar{x}) + (T_C(\bar{x}) \cap N_{\mathcal{S}}(\bar{x}))^{\circ}.$$

(vii) For every $x \in C$,

$$f'(\bar{x}, x - \bar{x}) \geq \alpha \, d_{\mathcal{S}}(x),$$

where $\bar{x} \in proj_{\mathcal{S}}(x)$.

Proof. [(i) $\Longrightarrow$ (ii)] Because $\mathcal{S}$ is the set of weak sharp minimizers, by Theorem 9.5,

$$f_0'(x, v) \geq \alpha \, d_{T_{\mathcal{S}}(x)}(v), \; \forall \, x \in \mathcal{S}, \; \forall \, v \in \mathbb{R}^n. \tag{9.10}$$

The above condition holds in particular for $v \in T_C(x)$. As $f_0(x) = f(x) + \delta_C(x)$, which along with the fact that $f_0'(x, v) = f'(x, v)$ for every $x \in \mathcal{S}$ and $v \in T_C(x)$, and condition (9.10) yields

$$f'(x, v) \geq \alpha \, d_{T_{\mathcal{S}}(x)}(v), \; \forall \, x \in \mathcal{S}, \; \forall \, v \in T_C(x),$$

thereby establishing (ii).

[(ii) $\Longrightarrow$ (iii)] As $dom \; f = \mathbb{R}^n$, by Theorem 2.79 and the relation (ii),

$$\sigma_{\partial f(x)}(v) \geq \alpha \, d_{T_{\mathcal{S}}(x)}(v), \; \forall \, x \in \mathcal{S}, \; \forall \, v \in T_C(x).$$

By Theorem 2.35, $T_C(x)$ is a closed convex cone. Invoking Proposition 2.61 (v) along with Proposition 2.37 yields

$$\sigma_{\partial f(x)+N_C(x)}(v) \geq \alpha\, d_{T_{\mathcal{S}}(x)}(v), \ \forall\, x \in \mathcal{S}, \ \forall\, v \in \mathbb{R}^n.$$

By the fact that $N_C(x) = \partial \delta_C(x)$ and from the Sum Rule, Theorem 2.91, $\partial f(x) + N_C(x) \subset \partial(f + \delta_C)(x) = \partial f_0(x)$, which is always true along with Proposition 2.61 (i), the above inequality yields

$$\sigma_{\partial f_0(x)}(v) \geq \alpha\, d_{T_{\mathcal{S}}(x)}(v), \ \forall\, x \in \mathcal{S}, \ \forall\, v \in \mathbb{R}^n. \tag{9.11}$$

By Proposition 9.3 (v), for any $x \in \mathcal{S}$ and $v \in \mathbb{R}^n$,

$$\alpha\, d_{T_{\mathcal{S}}(x)}(v) = \alpha\, \sigma_{cl\ \mathbb{B} \cap N_{\mathcal{S}}(x)}(v) = \alpha \sup_{v^* \in cl\ \mathbb{B} \cap N_{\mathcal{S}}(x)} \langle v^*, v \rangle.$$

As $\alpha > 0$, the above condition becomes

$$\begin{aligned} \alpha\, d_{T_{\mathcal{S}}(x)}(v) &= \sup_{v^* \in cl\ \mathbb{B} \cap N_{\mathcal{S}}(x)} \langle \alpha\, v^*, v \rangle \\ &= \sup_{\alpha\, v^* \in \alpha\ cl\ \mathbb{B} \cap N_{\mathcal{S}}(x)} \langle \alpha\, v^*, v \rangle = \sigma_{\alpha\ cl\ \mathbb{B} \cap N_{\mathcal{S}}(x)}(v). \end{aligned} \tag{9.12}$$

Substituting the above relation in the inequality (9.11) leads to

$$\sigma_{\partial f_0(x)}(v) \geq \sigma_{\alpha\ cl\ \mathbb{B} \cap N_{\mathcal{S}}(x)}(v), \ \forall\, x \in \mathcal{S}, \ \forall\, v \in \mathbb{R}^n. \tag{9.13}$$

By Proposition 2.82, $\partial f_0(x)$ is a closed convex set which along with Proposition 2.61 (iv) and (ii) implies that

$$\alpha\ cl\ \mathbb{B} \cap N_{\mathcal{S}}(x) \subset \partial f_0(x), \ \forall\, x \in \mathcal{S},$$

thereby proving (iii).

[(iii) $\Longrightarrow$ (i)] By Proposition 2.61 (i), relation (9.13) holds which along with (9.12) implies that

$$\sigma_{\partial f_0(x)}(v) \geq \alpha\, d_{T_{\mathcal{S}}(x)}(v), \ \forall\, x \in \mathcal{S}, \ \forall\, v \in \mathbb{R}^n.$$

By Theorem 2.79, the above inequality leads to

$$f_0'(x, v) \geq \alpha\, d_{T_{\mathcal{S}}(x)}(v), \ \forall\, x \in \mathcal{S}, \ \forall\, v \in \mathbb{R}^n,$$

that is, (9.5) is satisfied. Therefore by Theorem 9.5, (i) holds.

[(iii) $\Longleftrightarrow$ (iv)] This holds by Lemma 9.4.

[(v) $\Longrightarrow$ (vi)] Because $dom\ f = \mathbb{R}^n$, by Theorem 2.79, the relation (v) becomes

$$\sigma_{\partial f(x)}(v) \geq \alpha \sup_{v^* \in cl\ \mathbb{B}} \langle v^*, v \rangle, \ \forall\, x \in \mathcal{S}, \ \forall\, v \in T_C(x) \cap N_{\mathcal{S}}(x).$$

As $\alpha > 0$, for every $x \in \mathcal{S}$ and every $v \in T_C(x) \cap N_{\mathcal{S}}(x)$, the above inequality is equivalent to

$$\sigma_{\partial f(x)}(v) \geq \sup_{\alpha\, v^* \in \alpha\, cl\, \mathbb{B}} \langle \alpha\, v^*, v \rangle = \sigma_{\alpha\, cl\, \mathbb{B}}(v).$$

Because $T_C(x) \cap N_{\mathcal{S}}(x)$ is a closed convex cone, by Proposition 2.61 (v), the above condition yields that for every $x \in \mathcal{S}$,

$$\alpha\, cl\, \mathbb{B} \subset cl\, \{\partial f(x) + (T_C(x) \cap N_{\mathcal{S}}(x))^\circ\}.$$

Invoking Proposition 2.15,

$$\begin{aligned} \alpha\, \mathbb{B} = int\, (\alpha\, cl\, \mathbb{B}) &\subset int\, \{\partial f(x) + (T_C(x) \cap N_{\mathcal{S}}(x))^\circ\} \\ &\subset \partial f(x) + (T_C(x) \cap N_{\mathcal{S}}(x))^\circ, \ \forall\, x \in \mathcal{S}, \end{aligned}$$

thereby leading to (vi).

[(vi) $\Longrightarrow$ (v)] Applying Proposition 2.61 (v) to condition (vi) leads to

$$\sigma_{\partial f(x)}(v) \geq \sigma_{\alpha\, \mathbb{B}}(v), \ \forall\, x \in \mathcal{S}, \ \forall\, v \in T_C(x) \cap N_{\mathcal{S}}(x).$$

As $dom\ f = \mathbb{R}^n$, by Theorem 2.79, the above inequality leads to

$$f'(x, v) \geq \alpha\, \|v\|, \ \forall\, x \in \mathcal{S}, \ \forall\, v \in T_C(x) \cap N_{\mathcal{S}}(x),$$

thereby establishing (v).

[(ii) $\Longrightarrow$ (v)] By Proposition 9.3 (vi), for every $x \in \mathcal{S}$,

$$d_{T_{\mathcal{S}}(x)}(v) = \sigma_{cl\, \mathbb{B} \cap N_{\mathcal{S}}(x)}(v).$$

For every $v \in N_{\mathcal{S}}(x)$,

$$d_{T_{\mathcal{S}}(x)}(v) = \|v\|. \tag{9.14}$$

Therefore, for every $x \in \mathcal{S}$ and every $v \in T_C(x) \cap N_{\mathcal{S}}(x)$, the relation (ii) along with (9.14) leads to

$$f'(x, v) \geq \alpha\, \|v\|,$$

thereby deriving (v).

[(v) $\Longrightarrow$ (vii)] Consider $x \in C$ and let $\bar{x} \in proj_{\mathcal{S}}(x)$. By Theorem 2.35, $x - \bar{x} \in T_C(\bar{x})$. As $\bar{x} \in proj_{\mathcal{S}}(x)$, by Proposition 2.52,

$$\langle x - \bar{x}, \bar{y} - \bar{x} \rangle \leq 0, \ \forall\, \bar{y} \in \mathcal{S},$$

which by Definition 2.36 of normal cone, $x - \bar{x} \in N_{\mathcal{S}}(\bar{x})$. Therefore,

$$x - \bar{x} \in T_C(\bar{x}) \cap N_{\mathcal{S}}(\bar{x}).$$

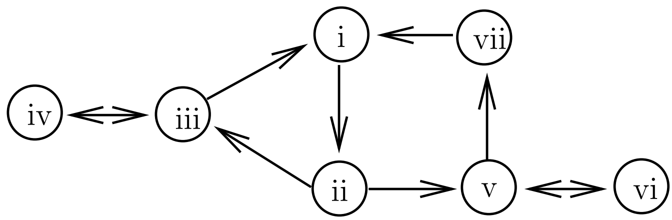

FIGURE 9.1: Pictorial representation of Theorem 9.6.

Now by relation (v),

$$f'(\bar{x}, x - \bar{x}) \geq \alpha \, \|x - \bar{x}\|.$$

As $\bar{x} \in proj_{\mathcal{S}}(x)$, $d_{\mathcal{S}}(x) = \|x - \bar{x}\|$. Thus the above inequality becomes

$$f'(\bar{x}, x - \bar{x}) \geq \alpha \, d_{\mathcal{S}}(x).$$

Because $x \in C$ and $\bar{x} \in proj_{\mathcal{S}}(x)$ were arbitrary, the inequality holds for every $x \in C$ and $\bar{x} \in proj_{\mathcal{S}}(x)$, thereby yielding the relation (vii).

[(vii) $\Longrightarrow$ (i)] As $dom\ f = \mathbb{R}^n$, by Theorem 2.79 along with Definition 2.77 of subdifferential and the relation (vii) leads to

$$f(x) - f(\bar{x}) \geq f'(\bar{x}, x - \bar{x}) \geq \alpha \, d_{\mathcal{S}}(x), \ \forall \ x \in C, \tag{9.15}$$

with $\bar{x} \in proj_{\mathcal{S}}(x)$. Because for any $\bar{y} \in \mathcal{S}$ with $\bar{y} \neq \bar{x}$, $f(\bar{y}) = f(\bar{x})$. Thus, (9.15) holds for every $x \in C$ and every $\bar{x} \in \mathcal{S}$, thereby leading to (i). $\square$

Figure 9.1 presents the pictorial representation of Theorem 9.6. We have devoted this chapter only to the theoretical aspect of weak sharp minimizers, though as mentioned in the beginning this notion plays an important role from the algorithmic point of view. For readers interested in its computational aspects, one may refer to Burke and Deng [23, 24] and Ferris [46].

Chapter 10

Approximate Optimality Conditions

10.1 Introduction

We have discussed the various aspects of studying optimality conditions for the convex programming problem (CP). Throughout, we concentrated on establishing the standard or the sequential optimality conditions at the exact point of minima. But it may not be always possible to find the point of minimizer. There may be cases where the infimum exists but is not attainable. For instance, consider

$$\min\ e^x \quad \text{subject to} \quad x \in \mathbb{R}.$$

As we know, the infimum for the above problem is zero but it is not attainable over the whole real line. Thus for scenarios we try to approximate the solution. In this example, for a given $\varepsilon > 0$, one can always find $\bar{x} \in \mathbb{R}$ such that $e^{\bar{x}} < \varepsilon$. This leads to the notion of *approximate solutions*, which play a crucial role in algorithmic study of optimization problems. Recall the convex optimization problem

$$\min f(x) \quad \text{subject to} \quad x \in C, \qquad (CP)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and C is a convex subset of $\mathbb{R}^n$.

Definition 10.1 Let $\varepsilon \geq 0$ be given. Then $\bar{x} \in C$ is said to be an *ε-solution* of (CP) or an *approximate up to ε* for (CP) if

$$f(\bar{x}) \leq f(x) + \varepsilon, \ \forall\, x \in C.$$

This is not the only way to study approximate solutions. In the literature, one finds the notions of various approximate solutions introduced over the years, such as quasi ε-solution, regular ε-solution, almost ε-solution [76], to name a few. We will define these solution concepts before moving on to study the approximate optimality conditions. The classes of quasi ε-solution and regular ε-solution are motivated by Ekeland's variational principle stated in Chapter 2.

Definition 10.2 Let $\varepsilon \geq 0$ be given. Then $\bar{x} \in C$ is said to be *quasi ε-solution* of (CP) if

$$f(\bar{x}) \leq f(x) + \sqrt{\varepsilon}\|x - \bar{x}\|, \ \forall \ x \in C.$$

A point $\bar{x} \in C$, which is an ε-solution as well as a quasi ε-solution of (CP), is known as the *regular ε-solution* of (CP).

The class of almost ε-solution, as the name itself suggests, seems to be an approximation to the ε-solution. Actually, it is the approximate solution concept associated with the perturbed problem. Before defining the almost ε-solution, recall the feasible set C given by (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, 2, \ldots, m\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, 2, \ldots, m$, are convex functions.

Definition 10.3 Let $\varepsilon \geq 0$ be given. The *ε-feasible set* of (CP) with the feasible set C given by (3.1) is defined as

$$C_\varepsilon = \{x \in \mathbb{R}^n : g_i(x) \leq \varepsilon, \ i = 1, 2, \ldots, m\}.$$

Then $\bar{x} \in \mathbb{R}^n$ is said to be an *almost ε-solution* of (CP) if

$$\bar{x} \in C_\varepsilon \quad \text{and} \quad f(\bar{x}) \leq f(x) + \varepsilon, \ \forall \ x \in C.$$

Observe that here the almost ε-solution need not be from the actual feasible set but should belong to the perturbed feasible set that is ε-feasible set.

Now we move on to discuss the approximate optimality conditions for the various classes of approximate solutions. In this chapter we concentrate on the ε-solution, quasi ε-solution, and almost ε-solution. We begin with the study of ε-solutions.

10.2 ε-Subdifferential Approach

Consider the unconstrained convex programming problem (CP_u)

$$\min \ f(x) \quad \text{subject to} \quad x \in \mathbb{R}^n. \tag{CP_u}$$

If $\bar{x} \in \mathbb{R}^n$ is an ε-solution, then by Definition 10.1,

$$f(x) - f(\bar{x}) \geq -\varepsilon, \ \forall \ x \in \mathbb{R}^n.$$

Using the definition of ε-subdifferential, Definition 2.109, $0 \in \partial_\varepsilon f(\bar{x})$. The converse can be established by directly applying the definition of ε-subdifferential. This has been stated as a result characterizing the ε-solution in Theorem 2.121 as follows.

Theorem 10.4 *Consider the unconstrained problem* (CP_u)*. Then* $\bar{x} \in \mathbb{R}^n$ *is an* ε*-solution of* (CP_u) *if and only if* $0 \in \partial_\varepsilon f(\bar{x})$*.*

As the convex programming problem (CP) can be reformulated as an unconstrained problem with the objective function f replaced by $(f + \delta_C)$, then from the above theorem one has that $\bar{x}$ is an ε-solution of (CP) if and only if

$$0 \in \partial_\varepsilon (f + \delta_C)(\bar{x}).$$

Observe that *dom* $f = \mathbb{R}^n$. If in addition, the Slater constraint qualification, that is, C has a nonempty relative interior holds, then by invoking the Sum Rule of ε-subdifferential, Theorem 2.115, along with the definition of ε-normal set, Definition 2.110, leads to

$$0 \in \partial_{\varepsilon_1} f(\bar{x}) + N_{C,\varepsilon_2}(\bar{x})$$

for some $\varepsilon_i \geq 0,\ i = 1, 2$, with $\varepsilon_1 + \varepsilon_2 = \varepsilon$. This may be stated as the following theorem.

Theorem 10.5 *Consider the convex optimization problem* (CP)*. Assume that the Slater constraint qualification holds, that is ri* C *is nonempty. Let* $\varepsilon \geq 0$ *be given. Then* $\bar{x} \in C$ *is an* ε*-solution of* (CP) *if and only if there exist* $\varepsilon_i \geq 0,\ i = 1, 2,$ *satisfying* $\varepsilon_1 + \varepsilon_2 = \varepsilon$ *such that*

$$0 \in \partial_{\varepsilon_1} f(\bar{x}) + N_{C,\varepsilon_2}(\bar{x}).$$

Note that for a nonempty convex set C, by Proposition 2.14 (i), *ri* C is nonempty and hence the Slater constraint qualification holds. From the above theorem it is obvious that to obtain the approximate optimality conditions in terms of the constraint functions $g_i,\ i = 1, 2, \ldots, m$, $N_{C,\varepsilon}(x)$ must be explicitly expressed in their terms. Below we present the result from Strodiot, Nguyen, and Heukemes [106], which acts as the tool in establishing the approximate optimality conditions. But before that, we define the *right scalar multiplication* from Rockafellar [97].

Definition 10.6 Let $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper convex function and $\lambda \geq 0$. The *right scalar multiplication*, $\phi\lambda$, is defined as

$$(\phi\lambda)(x) = \begin{cases} \lambda\phi(\lambda^{-1}x), & \lambda > 0, \\ \delta_{\{0\}}(x), & \lambda = 0. \end{cases}$$

A positively homogeneous convex function generated by ϕ, ψ, is defined as

$$\psi(x) = \inf\{(\phi\lambda)(x) : \lambda \geq 0\}.$$

Proposition 10.7 *Consider* $\varepsilon \geq 0$ *and* $g : \mathbb{R}^n \to \mathbb{R}$ *is a convex function. Let* $\bar{x} \in \bar{C} = \{x \in \mathbb{R}^n : g(x) \leq 0\}$*. Assume that the Slater constraint qualification holds, that is, there exist* $\hat{x} \in \mathbb{R}^n$ *such that* $g(\hat{x}) < 0$*. Then* $\xi \in N_{\bar{C},\varepsilon}(\bar{x})$ *if and only if there exists* $\lambda \geq 0$ *and* $\bar{\varepsilon} \geq 0$ *such that*

$$\bar{\varepsilon} \leq \lambda g(\bar{x}) + \varepsilon \qquad \textit{and} \qquad \xi \in \partial_{\bar{\varepsilon}}(\lambda g)(\bar{x}).$$

Proof. Using the definition of an ε-normal set, Definition 2.110,

$$\begin{aligned} N_{\bar{C},\varepsilon}(\bar{x}) &= \{\xi \in \mathbb{R}^n : \langle \xi, x - \bar{x} \rangle \leq \varepsilon, \ \forall \, x \in \bar{C}\} \\ &= \{\xi \in \mathbb{R}^n : \sigma_{\bar{C}}(\xi) \leq \langle \xi, \bar{x} \rangle + \varepsilon\}, \end{aligned}$$

where $\sigma_{\bar{C}}(\xi)$ denotes the support function to the set $\bar{C}$ at ξ. Observe that *dom* $g = \mathbb{R}^n$ and hence by Theorem 2.69 continuous over the whole of $\mathbb{R}^n$. Now invoking Theorem 13.5 from Rockafellar [97] (see also Remark 10.8), the support function $\sigma_{\bar{C}}$ is the closure of the positively homogenous function ϕ generated by g^*, which is defined as

$$\phi(\xi) = \inf_{\lambda \geq 0} (g^* \lambda)(\xi) = \inf_{\lambda \geq 0} \lambda g^*(\lambda^{-1} \xi) = \inf_{\lambda \geq 0} (\lambda g)^*(\xi).$$

Therefore,

$$\begin{aligned} N_{\bar{C},\varepsilon}(\bar{x}) &= \{\xi \in \mathbb{R}^n : \inf_{\lambda \geq 0} (\lambda g)^*(\xi) \leq \langle \xi, \bar{x} \rangle + \varepsilon\} \\ &= \{\xi \in \mathbb{R}^n : \text{there exists } \lambda \geq 0 \text{ such that } (\lambda g)^*(\xi) \leq \langle \xi, \bar{x} \rangle + \varepsilon\} \\ &= \{\xi \in \mathbb{R}^n : \text{there exists } \lambda \geq 0 \text{ such that} \\ &\qquad\qquad (\lambda g)^*(\xi) + (\lambda g)(\bar{x}) \leq \langle \xi, \bar{x} \rangle + \varepsilon + (\lambda g)(\bar{x})\} \\ &= \{\xi \in \mathbb{R}^n : \text{there exists } \lambda \geq 0 \text{ such that} \\ &\qquad (\lambda g)(x) - (\lambda g)(\bar{x}) \geq \\ &\qquad\qquad \langle \xi, x - \bar{x} \rangle - \varepsilon - (\lambda g)(\bar{x}), \ \forall \, x \in \mathbb{R}^n\}. \end{aligned}$$

From the above condition, there exists $\lambda \geq 0$ such that $\xi \in \partial_{\varepsilon + (\lambda g)(\bar{x})}(\lambda g)(\bar{x})$. As $\partial_{\varepsilon_1}\phi(x) \subset \partial_{\varepsilon_2}\phi(x)$ whenever $\varepsilon_1 \leq \varepsilon_2$, there exists an $\bar{\varepsilon}$ satisfying $0 \leq \bar{\varepsilon} \leq \varepsilon + (\lambda g)(\bar{x})$ such that $\xi \in \partial_{\bar{\varepsilon}}(\lambda g)(\bar{x})$. Therefore,

$$\begin{aligned} N_{\bar{C},\varepsilon}(\bar{x}) = \bigcup_{0 \leq \bar{\varepsilon} \leq \varepsilon + (\lambda g)(\bar{x})} &\{\xi \in \mathbb{R}^n : \text{there exists } \lambda \geq 0 \text{ such that} \\ &\qquad (\lambda g)^*(\xi) + (\lambda g)(\bar{x}) \leq \langle \xi, \bar{x} \rangle + \bar{\varepsilon}\} \\ = \bigcup_{0 \leq \bar{\varepsilon} \leq \varepsilon + (\lambda g)(\bar{x})} &\bigcup_{\lambda \geq 0} \partial_{\bar{\varepsilon}}(\lambda g)(\bar{x}), \end{aligned}$$

thereby leading to the desired result. □

Remark 10.8 We state Theorem 13.5 from Rockafellar [97].

> *Let $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper lsc convex function. The support function of the set $C = \{x \in \mathbb{R}^n : \phi(x) \leq 0\}$ is then cl ψ, where ψ is the positively homogeneous convex function generated by ϕ^*. Dually, the closure of the positively homogeneous convex function ψ generated by ϕ is the support function of the set $\{x^* \in \mathbb{R}^n : \phi^*(x^*) \leq 0\}$.*

For more details, readers are advised to refer to Rockafellar [97].

Next we present the approximate optimality conditions for the convex programming problem (CP).

Theorem 10.9 *Consider the convex programming problem (CP) with C given by (3.1). Assume that the Slater constraint qualification is satisfied. Let $\varepsilon \geq 0$. Then $\bar{x}$ is an ε-solution of (CP) if and only if there exist $\bar{\varepsilon}_0 \geq 0$, $\bar{\varepsilon}_i \geq 0$, and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$, such that*

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\varepsilon}_i} (\bar{\lambda}_i g_i)(\bar{x}) \quad \textit{and} \quad \sum_{i=0}^{m} \bar{\varepsilon}_i - \varepsilon \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq 0.$$

Proof. Observe that (CP) is equivalent to the unconstrained problem

$$\min \;(f + \sum_{i=1}^{m} \delta_{C_i})(x) \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

where $C_i = \{x \in \mathbb{R}^n : g_i(x) \leq 0\},\ i = 1, 2, \ldots, m$. By the Slater constraint qualification, there exist $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$ for every $i = 1, 2, \ldots, m$, which implies $ri\ C_i,\ i = 1, 2, \ldots, m$, is nonempty. Invoking Theorem 10.5, there exist $\varepsilon_i \geq 0,\ i = 0, 1, \ldots, m$, with $\varepsilon_0 + \sum_{i=1}^{m} \varepsilon_i = \varepsilon$ such that

$$0 \in \partial_{\varepsilon_0} f(\bar{x}) + \sum_{i=1}^{m} N_{C_i, \varepsilon_i}(\bar{x}).$$

Applying Proposition 10.7 to $C_i,\ i = 1, 2, \ldots, m$, there exist $\bar{\lambda}_i \geq 0$ and $\bar{\varepsilon}_i \geq 0$, $i = 1, 2, \ldots, m$, such that

$$\begin{aligned} &0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\varepsilon}_i} (\bar{\lambda}_i g)(\bar{x}) \\ \text{and} \quad &\bar{\varepsilon}_i - \varepsilon_i \leq \bar{\lambda}_i g_i(\bar{x}) \leq 0,\ i = 1, 2, \ldots, m, \end{aligned} \tag{10.1}$$

where $\bar{\varepsilon}_0 = \varepsilon_0$. Now summing (10.1) over $i = 1, 2, \ldots, m$, and using the condition $\varepsilon_0 + \sum_{i=1}^{m} \varepsilon_i = \varepsilon$ leads to

$$\sum_{i=0}^{m} \bar{\varepsilon}_i - \varepsilon \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq 0, \tag{10.2}$$

as desired.

Conversely, define $\varepsilon_i = \bar{\varepsilon}_i - \bar{\lambda}_i g_i(\bar{x}),\ i = 1, 2, \ldots, m$. Applying Proposition 10.7, $\xi_i \in \partial_{\bar{\varepsilon}_i} (\bar{\lambda}_i g_i)(\bar{x})$ is equivalent to $\xi_i \in N_{C_i, \varepsilon_i}(\bar{x})$ for $i = 1, 2, \ldots, m$. Also, from the condition (10.2),

$$\bar{\varepsilon}_0 + \sum_{i=1}^{m} \varepsilon_i + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) - \varepsilon \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq 0,$$

which implies $\bar{\varepsilon}_0+\sum_{i=1}^m \varepsilon_i \leq \varepsilon$. Define $\varepsilon_0 = \bar{\varepsilon}_0+\varepsilon_s$, where $\varepsilon_s = \varepsilon-\bar{\varepsilon}_0-\sum_{i=1}^m \varepsilon_i$. Observe that $\varepsilon_s \geq 0$. Therefore,

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} N_{C_i,\varepsilon_i}(\bar{x}) \subset \partial_{\varepsilon_0} f(\bar{x}) + \sum_{i=1}^{m} N_{C_i,\varepsilon_i}(\bar{x}),$$

where $\varepsilon_0 + \sum_{i=1}^m \varepsilon_i = \varepsilon$. By Theorem 10.5, $\bar{x}$ is an ε-solution of (CP). □

Observe that in the above approximate optimality conditions instead of the complementary slackness conditions, we have an *ε-complementary slackness condition.* Also, we derived the approximate optimality conditions in terms of the ε-subdifferentials of the objective function as well as the constraint functions at the ε-solution of (CP) by equivalent characterization of ε-normal set in terms of the ε-subdifferentials of the constraint functions $g_i,\ i = 1, 2, \ldots, m$.

10.3 Max-Function Approach

As discussed in the Section 3.5, another approach that is well known in establishing the standard KKT optimality conditions is the *max-function approach.* Applying a similar approach for an ε-solution, $\bar{x}$, of (CP) we introduce an unconstrained minimization problem

$$\min F(x) \quad \text{subject to} \quad x \in \mathbb{R}^n, \qquad (CP_{max})$$

where $F(x) = \max\{f(x)-f(\bar{x})+\varepsilon, g_1(x), \ldots, g_m(x)\}$. Using this max-function, an alternative proof is provided to derive the approximate optimality conditions. But before that we present a result to study the relation between the ε-solution of (CP) and those of the unconstrained problem (CP_{max}).

Theorem 10.10 *Consider the convex programming problem (CP) with C given by (3.1). If $\bar{x}$ is an $\bar{\varepsilon}$-solution of (CP), then $\bar{x}$ is an ε-solution of the unconstrained problem (CP_{max}) for every $\varepsilon \geq \bar{\varepsilon}$. Conversely, if $\bar{x}$ is an ε-solution of (CP_{max}), then it is an almost 2ε-solution of (CP).*

Proof. Because $\bar{x}$ is an $\bar{\varepsilon}$-solution of (CP), $\bar{x} \in C$ with

$$f(\bar{x}) \leq f(x) + \bar{\varepsilon},\ \forall\, x \in C. \qquad (10.3)$$

Observe that $F(\bar{x}) = \bar{\varepsilon}$. To show that for every $\varepsilon \geq \bar{\varepsilon}$, $\bar{x} \in \mathbb{R}^n$ is an ε-solution for (CP_{max}), it is sufficient to establish that

$$F(\bar{x}) \leq F(x) + \bar{\varepsilon},\ \forall\, x \in \mathbb{R}^n,$$

which is equivalent to proving that $F(x) \geq 0$ for every $x \in \mathbb{R}^n$.

For $x \in C$, $g_i(x) \leq 0,\ i = 1, 2, \ldots, m$, while condition (10.3) ensures that

$f(x) - f(\bar{x}) + \bar{\varepsilon} \geq 0$. Therefore, $F(x) \geq 0$ for every $x \in C$. If $x \notin C$, then for some $i \in \{1, 2, \ldots, m\}$, $g_i(x) > 0$ and thus, $F(x) > 0$ for every $x \notin C$. Hence, $\bar{x}$ is an ε-solution of (CP_{max}).

Conversely, as $\bar{x}$ is an ε-solution of (CP_{max}),

$$F(\bar{x}) \leq F(x) + \varepsilon, \ \forall \ x \in \mathbb{R}^n.$$

Therefore,

$$0 < \varepsilon = \max\{\varepsilon, g_1(\bar{x}), g_2(\bar{x}), \ldots, g_m(\bar{x})\} \leq F(x) + \varepsilon, \ \forall \ x \in \mathbb{R}^n.$$

The above condition yields

$$F(x) > 0 \quad \text{and} \quad g_i(\bar{x}) \leq F(x) + \varepsilon, \ i = 1, 2, \ldots, m, \ \forall \ x \in \mathbb{R}^n.$$

From the first condition, in particular for $x \in C$,

$$f(\bar{x}) \leq f(x) + \varepsilon \leq f(x) + 2\varepsilon$$

while in the second condition, taking $x = \bar{x}$ leads to

$$g_i(\bar{x}) \leq 2\varepsilon, \ i = 1, 2, \ldots, m,$$

thereby implying that $\bar{x}$ is an almost 2ε-solution of (CP). □

Theorem 10.11 *Consider the convex programming problem (CP) with C defined by (3.1). Assume that the Slater constraint qualification is satisfied and let $\varepsilon \geq 0$. Then $\bar{x}$ is an ε-solution of (CP) if and only if there exist $\bar{\varepsilon}_0 \geq 0$, $\bar{\varepsilon}_i \geq 0$, and $\bar{\lambda}_i \geq 0, \ i = 1, \ldots, m$, such that*

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x}) \qquad \text{and} \qquad \sum_{i=0}^{m} \bar{\varepsilon}_i - \varepsilon = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq 0.$$

Proof. As $\bar{x}$ is an ε-solution of (CP), then by Theorem 10.10, $\bar{x}$ is also an ε-solution of the unconstrained minimization problem (CP_{max}). By the approximate optimality condition, Theorem 10.4, for the unconstrained problem,

$$0 \in \partial_\varepsilon F(\bar{x}).$$

By the ε-subdifferential Max-Function Rule, Remark 2.119, there exist $\varepsilon_i \geq 0$, $\lambda_i \geq 0, \ i = 0, 1, \ldots, m$, with $\sum_{i=1}^{m} \lambda_i = 1$ and $\xi_0 \in \partial_{\varepsilon_0}(\lambda_0 f)(\bar{x})$ provided $\lambda_0 > 0$ and $\xi_i \in \partial_{\varepsilon_i}(\lambda_i g_i)(\bar{x})$ for those $i \in \{1, 2, \ldots, m\}$ satisfying $\lambda_i > 0$ such that

$$0 = \xi_0 + \sum_{i \in \bar{I}} \xi_i \quad \text{and} \quad \sum_{i=0}^{m} \varepsilon_i + F(\bar{x}) - \lambda_0 \varepsilon - \sum_{i \in \bar{I}} \lambda_i g_i(\bar{x}) = \varepsilon, \tag{10.4}$$

where $\bar{I} = \{i \in \{1, 2, \ldots, m\} : \lambda_i > 0\}$. Now if $\lambda_0 = 0$, again invoking the ε-subdifferential Max-Function Rule, Remark 2.119, there exists some $i \in \{1, 2, \ldots, m\}$ such that $\lambda_i > 0$, which implies $\bar{I}$ is nonempty. Thus, corresponding to $i \in \bar{I}$, there exist $\xi_i \in \partial_{\varepsilon_i}(\lambda_i g_i)(\bar{x})$ such that

$$0 = \sum_{i \in \bar{I}} \xi_i \quad \text{and} \quad \sum_{i=0}^{m} \varepsilon_i + F(\bar{x}) - \sum_{i \in \bar{I}} \lambda_i g_i(\bar{x}) = \varepsilon. \tag{10.5}$$

As $F(\bar{x}) = \varepsilon$, the second equality condition reduces to

$$\sum_{i=0}^{m} \varepsilon_i = \sum_{i \in \bar{I}} \lambda_i g_i(\bar{x}). \tag{10.6}$$

By the definition of ε-subdifferentiability, Definition 2.109,

$$\lambda_i g_i(x) \geq \lambda_i g_i(\bar{x}) + \langle \xi_i, x - \bar{x} \rangle - \varepsilon_i, \ \forall\, x \in \mathbb{R}^n, \ i \in \bar{I}.$$

Therefore, the above inequality along with (10.5) and the nonnegativity of $\varepsilon_i, \ i = 0, 1, \ldots, m$, leads to

$$\sum_{i=1}^{m} \lambda_i g_i(x) = \sum_{i \in \bar{I}} \lambda_i g_i(x) \geq \sum_{i \in \bar{I}} \lambda_i g_i(\bar{x}) - \sum_{i=0}^{m} \varepsilon_i,$$

which by the condition (10.6) yields

$$\sum_{i=1}^{m} \lambda_i g_i(x) \geq 0, \ \forall\, x \in \mathbb{R}^n. \tag{10.7}$$

As the Slater constraint qualification holds, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0, \ i = 1, 2, \ldots, m$. Thus,

$$\sum_{i=1}^{m} \lambda_i g_i(\hat{x}) < 0,$$

thereby contradicting the inequality (10.7). Therefore, $\lambda_0 \neq 0$. Now dividing both relations of (10.4) throughout by $\lambda_0 > 0$, along with $F(\bar{x}) = \varepsilon$ and Theorem 2.117, leads to

$$0 \in \bar{\xi}_0 + \sum_{i \in \bar{I}} \bar{\xi}_i \quad \text{and} \quad \sum_{i=0}^{m} \bar{\varepsilon}_i - \varepsilon = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq 0,$$

where $\bar{\xi}_0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x})$, $\bar{\xi}_i \in \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x}), \ i \in \bar{I}$, $\bar{\varepsilon}_i = \dfrac{\varepsilon_i}{\lambda_0}, \ i = 0, 1, \ldots, m$, and $\bar{\lambda}_i = \dfrac{\lambda_i}{\lambda_0}, \ i \in \bar{I}$. Corresponding to $i \notin \bar{I}$, $\bar{\lambda}_i = 0$ with $\bar{\xi}_i = 0 \in \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x})$, thereby leading to the approximate optimality condition

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x})$$

along with the ε-complementary slackness condition. The converse can be worked along the lines of Theorem 10.9 taking $\varepsilon_s = 0$. $\square$

Note that in the ε-complementary slackness condition of Theorem 10.9, we had inequality whereas in the above theorem it is in the form of an equation. Actually, this condition can also be treated as an inequality if for condition (10.4) we consider $F(\bar{x}) = \max\{\varepsilon, 0\} \geq \varepsilon$ instead of $F(\bar{x}) = \varepsilon$.

10.4 ε-Saddle Point Approach

While studying the optimality conditions for the convex programming problem (CP), we have already devoted a chapter on saddle point theory. Now to derive the approximate optimality conditions, we make use of the *ε-saddle point* approach. Recall the *Lagrangian function* $L : \mathbb{R}^n \times \mathbb{R}^m_+ \to \mathbb{R}$ associated with the convex programming problem (CP) with C given by (3.1), that is, involving convex inequalities, introduced in Chapter 4, is given by

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x).$$

Definition 10.12 A point $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m_+$ is said to be an *ε-saddle point* of (CP) if

$$L(\bar{x}, \lambda) - \varepsilon \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) + \varepsilon, \ \forall\, x \in \mathbb{R}^n, \ \forall\, \lambda \in \mathbb{R}^m_+.$$

Below we present a saddle point result established by Dutta [37].

Theorem 10.13 *Consider the convex programming problem (CP) with C given by (3.1). Let $\varepsilon \geq 0$ be given and $\bar{x}$ be an ε-solution of (CP). Assume that the Slater constraint qualification holds. Then there exist $\bar{\lambda}_i \geq 0$, $i = 1, 2, \ldots, m$, such that $(\bar{x}, \bar{\lambda})$ is an ε-saddle point of (CP) and $\varepsilon + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \geq 0$.*

Proof. As $\bar{x}$ is an ε-solution of (CP), the following system

$$\begin{aligned} &f(x) - f(\bar{x}) + \varepsilon < 0, \\ &g_i(x) < 0, \ i = 1, 2, \ldots, m, \end{aligned}$$

has no solution $x \in \mathbb{R}^n$. Define the set

$$\Lambda = \{(y, z) \in \mathbb{R} \times \mathbb{R}^m : f(x) - f(\bar{x}) + \varepsilon < y, \ g_i(x) < z_i, \ i = 1, 2, \ldots, m\}.$$

The reader is urged to verify that Λ is an open convex set. Observe that $(0, 0) \notin \Lambda$. Therefore, by the Separation Theorem, Theorem 2.26 (ii), there exists $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$ with $(\lambda_0, \lambda) \neq (0, 0)$ such that

$$\lambda_0(f(x) - f(\bar{x}) + \varepsilon) + \sum_{i=1}^{m} \lambda_i g_i(x) \geq 0, \ \forall \ x \in \mathbb{R}^n. \tag{10.8}$$

Working along the lines of proof of Theorem 4.2, it can be proved that $(\lambda_0, \lambda) \in \mathbb{R}_+ \times \mathbb{R}^m_+$.

We claim that $\lambda_0 \neq 0$. On the contrary, suppose that $\lambda_0 = 0$. By the Slater constraint qualification, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$ which implies

$$\sum_{i=1}^{m} \lambda_i g_i(\hat{x}) < 0,$$

thereby contradicting (10.8). Therefore, $\lambda_0 \neq 0$ and thus the condition (10.8) can be expressed as

$$f(x) - f(\bar{x}) + \varepsilon + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) \geq 0, \ \forall \ x \in \mathbb{R}^n, \tag{10.9}$$

where $\bar{\lambda}_i = \dfrac{\lambda_i}{\lambda_0}$ for $i = 1, 2, \ldots, m$. In particular, taking $x = \bar{x}$, the above inequality reduces to

$$\varepsilon + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \geq 0. \tag{10.10}$$

As $g_i(\bar{x}) \leq 0, \ i = 1, 2, \ldots, m$, which along with (10.9) leads to

$$f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) + \varepsilon, \ \forall \ x \in \mathbb{R}^n,$$

which implies

$$L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) + \varepsilon, \ \forall \ x \in \mathbb{R}^n. \tag{10.11}$$

For any $\lambda_i \geq 0, \ i = 1, 2, \ldots, m$, the feasibility of $\bar{x}$ along with the nonnegativity of ε and (10.10) leads to

$$f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) - \varepsilon \leq f(\bar{x}) - \varepsilon \leq f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}),$$

that is,

$$L(\bar{x}, \lambda) - \varepsilon \leq L(\bar{x}, \bar{\lambda}), \ \forall \ \lambda \in \mathbb{R}^m_+.$$

The above inequality along with (10.11) implies that $(\bar{x}, \bar{\lambda})$ is an ε-saddle point of (CP), which satisfies (10.10), thereby yielding the desired result. □

Using this ε-saddle point result, we establish the approximate optimality conditions. But unlike Theorems 10.9 and 10.11, the result below is only necessary with a *relaxed ε-complementary slackness condition.*

Theorem 10.14 *Consider the convex programming problem* (CP) *with* C *given by (3.1). Let* $\varepsilon \geq 0$ *be given and* $\bar{x}$ *be an* ε*-solution of* (CP)*. Assume that the Slater constraint qualification holds. Then there exist* $\bar{\varepsilon}_0 \geq 0$, $\bar{\varepsilon}_i \geq 0$, *and* $\bar{\lambda}_i \geq 0$, $i = 1, 2, \ldots, m$, *with* $\bar{\varepsilon}_0 + \sum_{i=1}^{m} \bar{\varepsilon}_i = \varepsilon$ *such that*

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x}) \quad \textit{and} \quad \varepsilon + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \geq 0.$$

Proof. By the previous theorem, there exist $\bar{\lambda}_i \geq 0,\ i = 1, 2, \ldots, m$, such that

$$L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) + \varepsilon, \ \forall\, x \in \mathbb{R}^n$$

along with $\varepsilon + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \geq 0$. By Definition 10.1 of ε-solution, the above inequality implies that $\bar{x}$ is an ε-solution of the unconstrained problem

$$\inf\ f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

By Theorem 10.4, the approximate optimality condition is

$$0 \in \partial_{\varepsilon}(f + \sum_{i=1}^{m} \bar{\lambda}_i g_i)(\bar{x}). \tag{10.12}$$

As *dom* $f = \mathbb{R}^n$ and *dom* $g_i = \mathbb{R}^n,\ i = 1, 2, \ldots, m$, applying the Sum Rule of ε-subdifferential, Theorem 2.115, there exist $\bar{\varepsilon}_i \geq 0,\ i = 0, 1, \ldots, m$, satisfying $\bar{\varepsilon}_0 + \sum_{i=1}^{m} \bar{\varepsilon}_i = \varepsilon$ such that (10.12) becomes

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x}),$$

thereby establishing the result. □

Observe that the conditions obtained in Theorem 10.14 are only necessary and not sufficient. The approach used in Theorems 10.9 and 10.11 for the sufficiency part cannot be invoked here. But if instead of the relaxed ε-complementary slackness condition, one has the standard complementary slackness, which is equivalent to

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) = 0,$$

then working along the lines of Theorem 10.9 the sufficiency can also be established. The result below shows that the optimality conditions derived in the above theorem imply toward the 2ε-solution of (CP) instead of the ε-solution.

Theorem 10.15 *Consider the convex programming problem* (CP) *with* C *given by (3.1). Let* $\varepsilon \geq 0$ *be given. Assume that the approximate optimality condition and the relaxed* ε*-complementary slackness condition of Theorem 10.14 hold for* $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m_+$ *and* $\varepsilon_i \geq 0,\ i = 0, 1, \ldots, m$*, satisfying* $\varepsilon_0 + \sum_{i=1}^m \varepsilon_i = \varepsilon$*. Then* $\bar{x}$ *is a* 2ε*-solution of* (CP)*.*

Proof. From the approximate optimality condition of Theorem 10.14, there exist $\bar{\lambda}_i \geq 0,\ i = 1, 2, \ldots, m$, and $\varepsilon_i \geq 0,\ i = 0, 1, \ldots, m$, with $\varepsilon_0 + \sum_{i=1}^m \varepsilon_i = \varepsilon$, $\xi_0 \in \partial_{\varepsilon_0} f(\bar{x})$, and $\xi_i \in \partial_{\varepsilon_i}(\bar{\lambda}_i g_i)(\bar{x}),\ i = 1, 2, \ldots, m$ such that

$$0 = \xi_0 + \sum_{i=1}^m \xi_i. \tag{10.13}$$

By Definition 2.109 of the ε-subdifferential,

$$\begin{array}{rcl} f(x) - f(\bar{x}) & \geq & \langle \xi_0, x - \bar{x} \rangle - \varepsilon_0, \\ \bar{\lambda}_i g_i(x) - \bar{\lambda}_i g_i(\bar{x}) & \geq & \langle \xi_i, x - \bar{x} \rangle - \varepsilon_i,\ i = 1, 2, \ldots, m. \end{array}$$

Summing the above inequalities along with the condition (10.13) leads to

$$f(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x) \geq f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) - (\varepsilon_0 + \sum_{i=1}^m \varepsilon_i).$$

For any x feasible for (CP), $g_i(x) \leq 0,\ i = 1, 2, \ldots, m$, which along with the relaxed ε-complementary slackness condition and the fact that $\varepsilon_0 + \sum_{i=1}^m \varepsilon_i = \varepsilon$ implies that

$$f(x) \geq f(\bar{x}) - 2\varepsilon,\ \forall\, x \in C.$$

Thus, $\bar{x}$ is a 2ε-solution of (CP). □

From Definition 10.12, $(\bar{x}, \bar{\lambda})$ is an ε-saddle point of (CP) if

$$L(\bar{x}, \lambda) - \varepsilon \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) + \varepsilon,\ \forall\, x \in \mathbb{R}^n,\ \forall\, \lambda \in \mathbb{R}^m_+.$$

With respect to the ε-solution, we will call $\bar{x}$ an *ε-minimum solution* of $L(., \bar{\lambda})$ and similarly, call $\bar{\lambda}$ an *ε-maximum solution* of $L(\bar{x}, .)$.

We end this section by presenting a result relating the ε-solutions of the saddle point to the almost ε-solution of (CP) that was derived by Dutta [37].

Theorem 10.16 *Consider the convex programming problem* (CP) *with* C *given by (3.1). Let* $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m_+$ *be such that* $\bar{x}$ *is an* ε_1*-minimum solution of* $L(., \bar{\lambda})$ *and* $\bar{\lambda}$ *is an* ε_2*-maximum solution of* $L(\bar{x}, .)$*. Then* $\bar{x}$ *is an almost* $(\varepsilon_1 + \varepsilon_2)$*-solution of* (CP)*.*

Proof. Because $\bar{\lambda} \in \mathbb{R}^m_+$ is an ε_2-maximum solution of $L(\bar{x}, \lambda)$ over $\mathbb{R}^m_+$,

$$L(\bar{x}, \lambda) - \varepsilon_2 \leq L(\bar{x}, \bar{\lambda}),\ \forall\, \lambda \in \mathbb{R}^m_+.$$

As $L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$, the above inequality reduces to

$$\sum_{i=1}^{m} (\lambda_i - \bar{\lambda}_i) g_i(\bar{x}) \leq \varepsilon_2, \ \forall \ \lambda_i \geq 0, \ i = 1, 2, \ldots, m. \tag{10.14}$$

We claim that $\bar{x} \in C_{\varepsilon_2} = \{x \in \mathbb{R}^n : g_i(x) \leq \varepsilon_2, \ i = 1, 2, \ldots, m\}$. On the contrary, suppose that $\bar{x} \notin C_{\varepsilon_2}$, which implies that the system

$$g_i(\bar{x}) - \varepsilon_2 \leq 0, \ i = 1, 2, \ldots, m$$

does not hold. Equivalently, the above condition implies that

$$(g_1(\bar{x}) - \varepsilon_2, g_2(\bar{x}) - \varepsilon_2, \ldots, g_m(\bar{x}) - \varepsilon_2) \notin \mathbb{R}^m_-.$$

As $\mathbb{R}^m_-$ is a closed convex set, by the Strict Separation Theorem, Theorem 2.26 (iii), there exists $\gamma \in \mathbb{R}^m$ with $\gamma \neq 0$ such that

$$\sum_{i=1}^{m} \gamma_i g_i(\bar{x}) - \sum_{i=1}^{m} \gamma_i \varepsilon_2 > 0 \geq \sum_{i=1}^{m} \gamma_i y_i, \ \forall \ y \in \mathbb{R}^m_-. \tag{10.15}$$

We claim that $\gamma \in \mathbb{R}^m_+$. On the contrary, assume that $\gamma \notin \mathbb{R}^m_+$, which implies for some $i \in \{1, 2, \ldots, m\}$, $\gamma_i < 0$. As the inequality (10.15) holds for every $y \in \mathbb{R}^m_-$, taking the corresponding $y_i \to -\infty$ leads to a contradiction. Hence, $\gamma \in \mathbb{R}^m_+$.

Because $\gamma \neq 0$, it can be so chosen satisfying $\sum_{i=1}^{m} \gamma_i = 1$. Therefore, the strict inequality condition in (10.15) reduces to

$$\sum_{i=1}^{m} \gamma_i g_i(\bar{x}) > \varepsilon_2. \tag{10.16}$$

As $\bar{\lambda} \in \mathbb{R}^m_+$ and $\gamma \in \mathbb{R}^m_+$, $\bar{\lambda} + \gamma \in \mathbb{R}^m_+$. Therefore, taking $\lambda = \bar{\lambda} + \gamma$ in (10.14) leads to

$$\sum_{i=1}^{m} \gamma_i g_i(\bar{x}) \leq \varepsilon_2,$$

which contradicts (10.16). Thus, $\bar{x} \in C_{\varepsilon_2} \subset C_{\varepsilon_1 + \varepsilon_2}$, where

$$C_{\varepsilon_1 + \varepsilon_2} = \{x \in \mathbb{R}^n : g_i(x) \leq \varepsilon_1 + \varepsilon_2, \ i = 1, 2, \ldots, m\}.$$

As $\bar{x}$ is an ε_1-minimum solution of $L(x, \bar{\lambda})$ over $\mathbb{R}^n$,

$$L(\bar{x}, \bar{\lambda}) \leq L(x, \bar{\lambda}) + \varepsilon_1, \ \forall \ x \in \mathbb{R}^n,$$

which implies

$$f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) + \varepsilon_1, \ \forall \ x \in \mathbb{R}^n.$$

For any x feasible to (CP), $g_i(x) \leq 0$, $i = 1, 2, \ldots, m$, which implies $\bar{\lambda}_i g_i(x) \leq 0$, $i = 1, 2, \ldots, m$. Taking $\lambda_i = 0$, $i = 1, 2, \ldots, m$, in (10.14), $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \geq -\varepsilon_2$. Thus, the preceding inequality reduces to

$$f(\bar{x}) \leq f(x) + \varepsilon_1 + \varepsilon_2, \ \forall \ x \in C.$$

Therefore, $\bar{x}$ is an almost $(\varepsilon_1 + \varepsilon_2)$-solution of (CP). □

10.5 Exact Penalization Approach

We have discussed different approaches like the ε-subdifferential approach, max-function approach, and saddle point approach to study the approximate optimality conditions. Another approach to deal with the relationship between the different classes of approximate solutions is the penalty function approach by Loridan [76]. In the work of Loridan that appeared in 1982, he dealt with the notion of regular and almost regular approximate solutions. But here we will concentrate more on ε-solutions and almost ε-solutions for which we move on to study the work done by Loridan and Morgan [77]. This approach helps in dealing with the stability analysis with respect to the perturbed problem, thereby relating the ε-solutions of the perturbed problem and almost ε-solutions of (CP).

We consider the *exact penalty function*

$$f_\rho(x) = f(x) + \sum_{i=1}^{m} \rho_i \max\{0, g_i(x)\},$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$, with $\rho_i > 0$, $i = 1, 2, \ldots, m$ and the following unconstrained problem

$$\min \ f_\rho(x) \quad \text{subject to} \quad x \in \mathbb{R}^n, \qquad (CP)_\rho$$

is associated with it. The convergence of the ε-solutions of the sequence of problems $(CP)_\rho$ under certain assumptions leads to an ε-solution of the problem (CP). So before moving on to establish the convergence result, we present a result relating the ε-solution of $(CP)_\rho$ with the almost ε-solution of (CP).

Theorem 10.17 *Assume that f is bounded below on $\mathbb{R}^n$. Then there exists $\rho_\varepsilon = \dfrac{(\alpha + \varepsilon)}{\varepsilon}$ where $\alpha = \inf_{x \in C} f(x) - \inf_{x \in \mathbb{R}^n} f(x)$ such that whenever $\rho_i \geq \rho_\varepsilon$, $i = 1, 2, \ldots, m$, every ε-solution of $(CP)_\rho$ is an almost ε-solution of (CP).*

Proof. Suppose that x_ρ is an ε-solution for $(CP)_\rho$. Then

$$f_\rho(x_\rho) \leq f_\rho(x) + \varepsilon, \ \forall \ x \in \mathbb{R}^n. \qquad (10.17)$$

Observe that for $x \in C$, as $g_i(x) \leq 0,\ i = 1, 2, \ldots, m,\ f_\rho(x) = f(x)$. This along with the condition (10.17) and the definition of f_ρ implies

$$f(x_\rho) \leq f_\rho(x_\rho) \leq f(x) + \varepsilon,\ \forall\, x \in C. \tag{10.18}$$

Again, from the definition of f_ρ along with (10.18),

$$\inf_{x\in\mathbb{R}^n} f(x) + \sum_{i=1}^{m} \rho_i \max\{0, g_i(x_\rho)\} \leq f_\rho(x_\rho) \leq \inf_{x\in C} f(x) + \varepsilon,$$

which implies

$$\sum_{i=1}^{m} \rho_i \max\{0, g_i(x_\rho)\} \leq \alpha + \varepsilon. \tag{10.19}$$

Now consider $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ such that $\rho_i \geq \rho_\varepsilon = \dfrac{(\alpha+\varepsilon)}{\varepsilon}$ for every $i = 1, 2, \ldots, m$. Therefore, for the ε-solution x_ρ of $(CP)_\rho$, the condition (10.19) leads to

$$g_i(x_\rho) \leq \max\{0, g_i(x_\rho)\} \leq \sum_{i=1}^{m} \max\{0, g_i(x_\rho)\} \leq \varepsilon,\ \forall\, i = 1, 2, \ldots, m,$$

which implies $x_\rho \in C_\varepsilon$. This along with (10.18) yields that x_ρ is an almost ε-solution of (CP). $\square$

In the above theorem, it was shown that the ε-solutions of the penalized problem $(CP)_\rho$ are almost ε-solutions of (CP). But we are more interested in deriving an ε-solution rather than an almost ε-solution of (CP). The next result paves a way in this direction by obtaining an ε-solution of (CP) from the ε-solutions of the sequence of problems $\{(CP)_{\rho_k}\}_k$, where $\rho_k = (\rho_1^k, \rho_2^k, \ldots, \rho_m^k)$.

Theorem 10.18 *Assume that f is bounded below on $\mathbb{R}^n$ and satisfies the coercivity condition*

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

Let $\{\rho_k\}_k$ be a sequence such that $\lim_{k\to+\infty} \rho_i^k = +\infty$ for every $i = 1, 2, \ldots, m$ and x_{ρ_k} be the ε-solution of $(CP)_{\rho_k}$. Then every convergent sequence of $\{x_{\rho_k}\}$ has a limit point that is an ε-solution of (CP).

Proof. As $\{x_{\rho_k}\}$ is the ε-solution of $(CP)_{\rho_k}$, by Theorem 10.17, $\{x_{\rho_k}\}$ is an almost ε-solution of (CP) and thus satisfies

$$f(x_{\rho_k}) \leq f(x) + \varepsilon,\ \forall\, x \in C.$$

Because $f(x_{\rho_k})$ is bounded above for every k, therefore by the given hypothesis

$\{x_{\rho_k}\}$ is a bounded sequence and thus by the Bolzano–Weierstrass Theorem, Proposition 1.3, has a convergent subsequence. Without loss of generality, assume that $x_{\rho_k} \to x_\rho$. As *dom* $f = \mathbb{R}^n$, by Theorem 2.69, f is continuous on $\mathbb{R}^n$. Thus, taking the limit as $k \to +\infty$, the above inequality leads to

$$f(x_\rho) \leq f(x) + \varepsilon, \ \forall \ x \in C. \tag{10.20}$$

Using the condition (10.19) in the proof of Theorem 10.17

$$g_i(x_{\rho_k}) \leq (\alpha + \varepsilon)/\rho_i^k.$$

Again, by Theorem 2.69, g_i, $i = 1, 2, \ldots, m$, is continuous on *dom* $g_i = \mathbb{R}^n$, $i = 1, 2, \ldots, m$. Therefore, taking the limit as $k \to +\infty$, the above inequality leads to

$$g_i(x_\rho) \leq 0, \ \forall \ i = 1, 2, \ldots, m.$$

Thus, $x_\rho \in C$ along with the condition (10.20) implies that x_ρ is an ε-solution of (CP). $\square$

From the above discussions it is obvious that an ε-solution of $(CP)_\rho$ need not be an ε-solution of (CP) when $x_\rho \notin C$. But in case $x_\rho \in C$, it may be considered as an ε-solution of (CP). The result below tries to find an ε-solution for (CP) by using an ε-solution of $(CP)_\rho$ under the Slater constraint qualification. Even though the result is from Loridan and Morgan [77] but the proof is based on the work by Zangwill [114] on penalty functions. Here we present the detailed proof for a better understanding.

Theorem 10.19 *Consider the convex programming problem (CP) with C given by (3.1). Assume that f is bounded below on $\mathbb{R}^n$ and the Slater constraint qualification is satisfied, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$. Define $\beta = \inf_{x \in C} f(x) - f(\hat{x})$ and $\gamma = \max_{i=1,\ldots,m} g_i(\hat{x}) < 0$. Let $\rho_0 = (\beta - 1)/\gamma > 0$. For $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ with $\rho_i \geq \rho_0$, $i = 1, 2, \ldots, m$, let $x_\rho \notin C$ be an ε-solution for $(CP)_\rho$. Let $\bar{x}$ be the unique point on the line segment joining x_ρ and $\hat{x}$ lying on the boundary of C. Then $\bar{x}$ is an ε-solution of (CP).*

Proof. Because $\bar{x}$ is a unique point on the line segment joining x_ρ and $\hat{x}$ lying on the boundary, the active index set $I(\bar{x}) = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) = 0\}$ is nonempty. Define a convex auxiliary function as

$$\mathcal{F}(x) = f(x) + \rho_0 \sum_{i \in I(\bar{x})} g_i(x).$$

Observe that for $i \in I(\bar{x})$, $g_i(\bar{x}) = 0$ while for $i \notin I(\bar{x})$, $g_i(\bar{x}) < 0$. Therefore,

$$\mathcal{F}(\bar{x}) = f(\bar{x}) = f_{\rho_0}(\bar{x}). \tag{10.21}$$

As $\bar{x}$ lies on the line segment joining x_ρ and $\hat{x}$, there exists $\lambda \in (0,1)$ such that $\bar{x} = \lambda x_\rho + (1-\lambda)\hat{x}$. Then by the convexity of $g_i,\ i = 1,2,\ldots,m$,

$$g_i(\bar{x}) \leq \lambda g_i(x_\rho) + (1-\lambda) g_i(\hat{x}),\ i = 1,2,\ldots,m.$$

For $i \in I(\bar{x})$, $g_i(\bar{x}) = 0$, which along with the Slater constraint qualification reduces the above inequality to

$$0 < -(1-\lambda) g_i(\hat{x}) \leq \lambda g_i(x_\rho),\ \forall\, i \in I(\bar{x}).$$

Therefore, for $i \in I(\bar{x})$, $g_i(x_\rho) > 0$, which implies

$$\sum_{i \in I(\bar{x})} g_i(x_\rho) = \sum_{i \in I(\bar{x})} \max\{0, g_i(x_\rho)\} \leq \sum_{i=1}^{m} \max\{0, g_i(x_\rho)\},$$

thereby leading to the fact that

$$\mathcal{F}(x_\rho) \leq f_{\rho_0}(x_\rho). \tag{10.22}$$

To prove the result, it is sufficient to show that $\mathcal{F}(\bar{x}) < \mathcal{F}(x_\rho)$. But first we will show that $\mathcal{F}(\hat{x}) < \mathcal{F}(\bar{x})$. Consider

$$\mathcal{F}(\hat{x}) = f(\hat{x}) + \rho_0 \sum_{i \in I(\bar{x})} g_i(\hat{x}).$$

Because $g_i(\hat{x}) < 0,\ i = 1,2,\ldots,m$, $\sum_{i \in I(\bar{x})} g_i(\hat{x}) \leq \max_{i=1,\ldots,m} g_i(\hat{x})$, which by the given hypothesis implies

$$\mathcal{F}(\hat{x}) \leq f(\hat{x}) + \rho_0 \gamma = \inf_{x \in C} f(x) - 1 < f(\bar{x}) = \mathcal{F}(\bar{x}). \tag{10.23}$$

The convexity of $\mathcal{F}$ along with (10.23) leads to

$$\mathcal{F}(\bar{x}) < \lambda \mathcal{F}(x_\rho) + (1-\lambda)\mathcal{F}(\bar{x}),$$

which implies $\mathcal{F}(\bar{x}) < \mathcal{F}(x_\rho)$. Therefore, by (10.21) and (10.22), $f(\bar{x}) < f_{\rho_0}(x_\rho)$. By the definition of f_ρ, $f_{\rho_0}(x) \leq f_\rho(x)$ for every $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ with $\rho_i \geq \rho_0,\ i = 1,2,\ldots,m$, which along with the fact that x_ρ is an ε-solution of $(CP)_\rho$ implies

$$f(\bar{x}) < f_\rho(x_\rho) \leq f_\rho(x) + \varepsilon,\ \forall\, x \in \mathbb{R}^n.$$

For $x \in C$, $f_\rho(x) = f(x)$, which reduces the above condition to

$$f(\bar{x}) \leq f(x) + \varepsilon,\ \forall\, x \in C,$$

thereby implying that $\bar{x}$ is an ε-solution of (CP). $\square$

For a better understanding of the above result, let us consider the following example. Consider

$$\inf e^{x} \quad \text{subject to} \quad x \leq 0.$$

Obviously the Slater constraint qualification holds. Consider $\hat{x} = -1$ and then $\rho_0 = e^{-1} + 1$. For $\varepsilon = 2$, $x_\rho = 1/2 > 0$ is an ε-solution for every $\rho \geq \rho_0$. Here, $\bar{x} = 0 \in [-1, 1/2]$ is an ε-solution for the constrained problem.

Observe that one requires the fact that x_ρ is an ε-solution of $(CP)_\rho$ only to establish that $\bar{x}$ is an ε-solution of (CP). So from the proof of Theorem 10.19 it can also be worked out that under the Slater constraint qualification, corresponding to $x_\rho \notin C$, there exists $\bar{x} \in C$ such that

$$f(\bar{x}) = f_\rho(\bar{x}) < f_\rho(x_\rho)$$

for every $\rho \geq \rho_0$, where ρ_0 is the same as in the previous theorem. As a matter of fact, because the set C is closed convex, one can always find such an $\bar{x}$ on the boundary of C. As $x_\rho \notin C$ is arbitrarily chosen, then from the above inequality it is obvious that

$$\inf_{x \in C} f(x) \leq f_\rho(x), \ \forall\, x \notin C.$$

Also, for any $x \in C$, $f(x) = f_\rho(x)$, which along with the above condition implies

$$\inf_{x \in C} f_\rho(x) = \inf_{x \in C} f(x) \leq \inf_{x \in \mathbb{R}^n} f_\rho(x).$$

The reverse inequality holds trivially. Therefore,

$$\inf_{x \in C} f(x) = \inf_{x \in \mathbb{R}^n} f_\rho(x). \tag{10.24}$$

This leads to the fact that every ε-solution of (CP) is also an ε-solution of the penalized unconstrained problem. Next we derive the approximate optimality conditions for (CP) using the penalized unconstrained problem.

Theorem 10.20 *Consider the convex programming problem (CP) with C defined by (3.1). Assume that the Slater constraint qualification is satisfied. Let $\varepsilon \geq 0$. Then $\bar{x}$ is an ε-solution of (CP) if and only if there exist $\bar{\varepsilon}_0 \geq 0$, $\bar{\varepsilon}_i \geq 0$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$, such that*

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x}) \quad \textit{and} \quad \sum_{i=0}^{m} \bar{\varepsilon}_i - \varepsilon = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq 0.$$

Proof. As $\bar{x} \in C$ is an ε-solution of (CP), from the above discussion it is also an ε-solution of the penalized unconstrained problem for $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ with $\rho_i \geq \rho_0 > 0$, where ρ_0 is defined in Theorem 10.19. Therefore, by the approximate optimality condition, Theorem 10.4, for the unconstrained penalized problem $(CP)_\rho$,

$$0 \in \partial_\varepsilon f_\rho(\bar{x}).$$

As $dom\ f = dom\ g_i = \mathbb{R}^n$, applying the ε-subdifferential Sum Rule, Theorem 2.115, there exist $\varepsilon_i \geq 0,\ i = 0, 1, \ldots, m$, satisfying $\sum_{i=0}^m \varepsilon_i = \varepsilon$ such that

$$0 \in \partial_{\varepsilon_0} f(\bar{x}) + \sum_{i=1}^m \partial_{\varepsilon_i}(\max\{0, \rho_i g_i(.)\})(\bar{x}).$$

By the ε-subdifferential Max-Function Rule, Remark 2.119, there exist $0 \leq \lambda_i \leq 1$ and $\bar{\varepsilon}_i \geq 0$ satisfying

$$\varepsilon_i = \bar{\varepsilon}_i + \max\{0, \rho_i g_i(\bar{x})\} - \lambda_i \rho_i g_i(\bar{x}) = \bar{\varepsilon}_i - \lambda_i \rho_i g_i(\bar{x}) \tag{10.25}$$

for every $i = 1, 2, \ldots, m$ such that

$$0 \in \partial_{\bar{\varepsilon}_0} f(\bar{x}) + \sum_{i=1}^m \partial_{\bar{\varepsilon}_i}(\bar{\lambda}_i g_i)(\bar{x}),$$

where $\bar{\varepsilon}_0 = \varepsilon_0 \geq 0$ and $\bar{\lambda}_i = \rho_i \lambda_i \geq 0,\ i = 1, 2, \ldots, m$. The condition (10.25) along with $\sum_{i=0}^m \varepsilon_i = \varepsilon$ implies that

$$\sum_{i=0}^m \bar{\varepsilon}_i - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) = \varepsilon,$$

thereby leading to the requisite conditions. The converse can be proved in a similar fashion, as done in Theorem 10.9 with $\varepsilon_s = 0$. $\square$

Note that the conditions obtained in the above theorem are the same as those in Theorem 10.11.

10.6 Ekeland's Variational Principle Approach

In all the earlier sections, we concentrated on the ε-solutions. If $\bar{x}$ is an ε-solution of (CP), then by the Ekeland's variational principle, Theorem 2.113, mentioned in Chapter 2 there exists $\hat{x} \in C$ such that

$$f(\hat{x}) \leq f(x) + \sqrt{\varepsilon}\|x - \hat{x}\|,\ \forall\ x \in C.$$

Any $\hat{x}$ satisfying the above condition is a quasi ε-solution of (CP). Observe that we are emphasizing only one of the conditions of the Ekeland's variational principle and the other two need not be satisfied. In this section we deal with the quasi ε-solution and derive the approximate optimality conditions for this class of approximate solutions for (CP). But before doing so, let us illustrate by an example that a quasi ε-solution may or may not be an ε-solution. Consider the problem

$$\inf \ \frac{1}{x} \quad \text{subject to} \quad x > 0.$$

Note that the infimum of the problem is zero, which is not attained. For $\varepsilon = \dfrac{1}{4}$, it is easy to note that $x_\varepsilon = 4$ is an ε-solution. Now $\bar{x} > 0$ is a quasi ε-solution if

$$\frac{1}{\bar{x}} \leq \frac{1}{x} + \frac{1}{2}|x - \bar{x}|, \ \forall \, x > 0.$$

Observe that $\bar{x} = 4.5$ is a quasi ε-solution that is also an ε-solution satisfying all the conditions of the Ekeland's variational principle, while $\bar{x} = 3.5$ is a quasi ε-solution that is not ε-solution. Also, it does not satisfy the condition $\dfrac{1}{\bar{x}} \leq \dfrac{1}{x_\varepsilon}$. These are not the only quasi ε-solutions. Even points that satisfy only the unique minimizer condition of the variational principle, like $\bar{x} = 3$, are also the quasi ε-solution to the above problem.

Now we move on to discuss the approximate optimality conditions for the quasi ε-solutions.

Theorem 10.21 *Consider the convex programming problem (CP) with C given by (3.1). Let $\varepsilon \geq 0$ be given. Assume that the Slater constraint qualification holds. Then $\bar{x}$ is a quasi ε-solution of (CP) if and only if there exist $\lambda_i \geq 0, \ i = 1, 2, \ldots, m$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + \sqrt{\varepsilon}\mathbb{B} \quad \text{and} \quad \lambda_i g_i(\bar{x}) = 0, \ i = 1, 2, \ldots, m.$$

Proof. A quasi ε-solution $\bar{x}$ of (CP) can be considered a minimizer of the convex programming problem

$$\min \ f(x) + \sqrt{\varepsilon}\|x - \bar{x}\| \quad \text{subject to} \quad x \in C,$$

where $C = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, 2, \ldots, m\}$. By the KKT optimality condition, Theorem 3.7, $\bar{x}$ is a minimizer of the above problem if and only if there exist $\lambda_i \geq 0, \ i = 1, 2, \ldots, m$, such that

$$0 \in \partial(f + \sqrt{\varepsilon}\|. - \bar{x}\|)(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}).$$

As *dom* $f =$ *dom* $\|. - \bar{x}\| = \mathbb{R}^n$, invoking the Sum Rule, Theorem 2.91, along with the fact that $\partial\|.-\bar{x}\| = \mathbb{B}$, the above inclusion becomes

$$0 \in \partial f(\bar{x}) + \sqrt{\varepsilon}\mathbb{B} + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}),$$

along with $\lambda_i g_i(\bar{x}) = 0,\ i = 1, 2, \ldots, m$, thereby yielding the requisite conditions.

Conversely, by the approximate optimality condition, there exist $\xi_0 \in \partial f(\bar{x})$, $\xi_i \in \partial g_i(\bar{x}),\ i = 1, 2, \ldots, m$, and $b \in \mathbb{B}$ such that

$$0 = \xi_0 + \sum_{i=1}^{m} \lambda_i \xi_i + \sqrt{\varepsilon} b. \tag{10.26}$$

By Definition 2.77 of the subdifferential,

$$\begin{aligned} f(x) - f(\bar{x}) &\geq \langle \xi_0, x - \bar{x} \rangle,\ \forall\ x \in \mathbb{R}^n, &(10.27)\\ g_i(x) - g_i(x) &\geq \langle \xi_i, x - \bar{x} \rangle,\ \forall\ x \in \mathbb{R}^n,\ i = 1, 2, \ldots, m, &(10.28)\end{aligned}$$

and by the Cauchy–Schwartz inequality, Proposition 1.1,

$$\|b\|\ \|x - \bar{x}\| \geq \langle b, x - \bar{x} \rangle,\ \forall\ x \in \mathbb{R}^n. \tag{10.29}$$

Combining the inequalities (10.27), (10.28), and (10.29) along with (10.26) implies

$$f(x) - f(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(x) - \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) + \sqrt{\varepsilon}\|b\|\ \|x - \bar{x}\| \geq 0,\ \forall\ x \in \mathbb{R}^n.$$

For any x feasible to (CP), $g_i(x) \leq 0,\ i = 1, 2, \ldots, m$, which along with the complementary slackness condition and the fact that $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, reduces the above inequality to

$$f(x) - f(\bar{x}) + \sqrt{\varepsilon}\|b\|\ \|x - \bar{x}\| \geq 0,\ \forall\ x \in C.$$

As $b \in \mathbb{B}$, $\|b\| \leq 1$, thereby leading to

$$f(x) - f(\bar{x}) + \sqrt{\varepsilon}\|x - \bar{x}\| \geq 0,\ \forall\ x \in C$$

and thus, establishing the requisite result. □

Observe that the above theorem provides a necessary as well as sufficient characterization to the quasi ε-solution. Here the approximate optimality condition is in terms of $\mathbb{B}$ and the subdifferentials, unlike the earlier results of this chapter where the approximate optimality conditions were expressed in terms of the ε-subdifferentials. Also, here we obtain the standard complementary slackness condition instead of the ε-complementary slackness or relaxed ε-complementary slackness conditions. Results similar to the ε-saddle point can also be worked out for quasi ε-saddle points. For more details, one can look into Dutta [37].

10.7 Modified ε-KKT Conditions

In all discussions regarding the KKT optimality conditions in the earlier chapters, it was observed that under some constraint qualification, the optimality conditions are established at the point of minimizer, that is, the KKT optimality conditions are nothing but point conditions. Due to this very reason, the KKT conditions have not been widely incorporated in the optimization algorithm design but only used as stopping criteria. However, if one could find the direction of the minima using the deviations from the KKT conditions, it could be useful from an algorithmic point of view. Work has recently been done in this respect by Dutta, Deb, Arora, and Tulshyan [39]. They introduced a new notion of *modified ε-KKT point* and used it to study the convergence of the sequences of modified ε-KKT points to the minima of the convex programming problem (CP). Below we define this new concept, which is again motivated the Ekeland's variational principle.

Definition 10.22 A feasible point $\bar{x}$ of (CP) is said to be a *modified ε-KKT point* for a given $\varepsilon > 0$ if there exists $\tilde{x} \in \mathbb{R}^n$ satisfying $\|\tilde{x} - \bar{x}\| \leq \sqrt{\varepsilon}$ and there exist $\tilde{\xi}_0 \in \partial f(\tilde{x})$, $\tilde{\xi}_i \in \partial g_i(\tilde{x})$ and $\lambda_i \geq 0,\ i = 1, 2, \ldots, m$, such that

$$\|\tilde{\xi}_0 + \sum_{i=1}^{m} \lambda_i \tilde{\xi}_i\| \leq \sqrt{\varepsilon} \quad \text{and} \quad \varepsilon + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \geq 0.$$

Observe that in the ε-KKT condition, the subdifferentials are calculated at some $\tilde{x} \in \mathbb{B}_{\sqrt{\varepsilon}}(\bar{x})$, whereas the relaxed ε-complementary slackness condition is satisfied at $\bar{x}$ itself. Before moving on to the satability part, we try to relate the already discussed ε-solution with the modified ε-KKT point.

Theorem 10.23 *Consider the convex programming problem (CP) with C given by (3.1). Assume that the Slater constraint qualification holds and let $\bar{x}$ be an ε-solution of (CP). Then $\bar{x}$ is a modified ε-KKT point.*

Proof. Because $\bar{x}$ is an ε-solution of (CP), by Theorem 10.13, there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, such that $\bar{x}$ is also an ε-saddle point along with

$$\varepsilon + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \geq 0. \tag{10.30}$$

As $\bar{x}$ is an ε-saddle point,

$$L(\bar{x}, \lambda) \leq L(x, \lambda) + \varepsilon,\ \forall\, x \in \mathbb{R}^n,$$

which implies $\bar{x}$ is an ε-solution of $L(., \lambda)$ over $\mathbb{R}^n$. Applying Ekeland's variational principle, Theorem 2.113, for $\sqrt{\varepsilon}$, there exists $\tilde{x} \in \mathbb{R}^n$ satisfying $\|\tilde{x} - \bar{x}\| \leq \sqrt{\varepsilon}$ such that $\tilde{x}$ is a minimizer of the problem

$$\min\ L(x, \lambda) + \sqrt{\varepsilon}\|x - \tilde{x}\| \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

By the unconstrained optimality condition, Theorem 2.89,

$$0 \quad \in \quad \partial(L(.,\lambda) + \sqrt{\varepsilon}\|. - \hat{x}\|)(\tilde{x}).$$

As the functions $dom\ f = dom\ g_i = \mathbb{R}^n$, $i = 1, 2, \ldots, m$, applying the Sum Rule, Theorem 2.91,

$$0 \in \partial f(\tilde{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\tilde{x}) + \sqrt{\varepsilon}\mathbb{B},$$

which implies there exist $\tilde{\xi}_0 \in \partial f(\tilde{x})$, $\tilde{\xi}_i \in \partial g_i(\tilde{x})$, $i = 1, 2, \ldots, m$, and $b \in \mathbb{B}$ such that

$$0 = \tilde{\xi}_0 + \sum_{i=1}^{m} \lambda_i \tilde{\xi}_i + \sqrt{\varepsilon} b,$$

thereby leading to

$$\|\tilde{\xi}_0 + \sum_{i=1}^{m} \lambda_i \tilde{\xi}_i\| \leq \sqrt{\varepsilon},$$

which along with the condition (10.30) implies that $\bar{x}$ is a modified ε-KKT point as desired. $\square$

In the case of the exact penalization approach, from Theorem 10.16 we have that every convergent sequence of ε-solutions of a sequence of penalized problems converges to an ε-solution of (CP). Now is it possible to establish such a result by studying the sequence of modified ε-KKT points and the answer is yes, as shown in the following theorem.

Theorem 10.24 *Consider the convex programming problem* (CP) *with* C *given by (3.1). Assume that the Slater constraint qualification holds and let* $\{\varepsilon_k\} \subset \mathbb{R}_+$ *such that* $\varepsilon_k \downarrow 0$ *as* $k \to +\infty$*. For every* k*, let* x_k *be a modified* ε_k*-KKT point of* (CP) *such that* $x_k \to \bar{x}$ *as* $k \to +\infty$*. Then* $\bar{x}$ *is a point of minimizer of* (CP)*.*

Proof. As for every k, x_k is a modified ε_k-KKT point of (CP), there exists $\tilde{x}_k \in \mathbb{R}^n$ satisfying $\|\tilde{x}_k - x_k\| \leq \sqrt{\varepsilon}$ and there exist $\xi_0^k \in \partial f(\tilde{x}_k)$, $\xi_i^k \in \partial g_i(\tilde{x}_k)$, and $\lambda_i^k \geq 0$, $i = 1, 2, \ldots, m$, such that

$$\|\xi_0^k + \sum_{i=1}^{m} \lambda_i^k \xi_i^k\| \leq \sqrt{\varepsilon_k} \quad \text{and} \quad \varepsilon_k + \sum_{i=1}^{m} \lambda_i^k g_i(x_k) \geq 0. \tag{10.31}$$

We claim that $\{\lambda^k\} \subset \mathbb{R}_+^m$ is a bounded sequence. Suppose that $\{\lambda^k\}$ is an unbounded sequence. Define a bounded sequence $\gamma^k = \dfrac{\lambda^k}{\|\lambda^k\|}$ with $\|\gamma^k\| = 1$.

Because $\{\gamma^k\}$ is a bounded sequence, by the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, assume that $\gamma^k \to \gamma$ with $\|\gamma\| = 1$. Observe that

$$\begin{aligned} \|\tilde{x}_k - \bar{x}\| &\leq \|\tilde{x}_k - x_k\| + \|x_k - \bar{x}\| \\ &\leq \sqrt{\varepsilon_k} + \|x_k - \bar{x}\|. \end{aligned}$$

By the given hypothesis, as $k \to +\infty$, $\varepsilon_k \downarrow 0$ and $x_k \to \bar{x}$, which implies $\tilde{x}_k \to \bar{x}$.

Now dividing both the conditions of (10.31) throughout by $\|\lambda^k\|$ yields

$$\|\frac{1}{\|\lambda^k\|}\xi_0^k + \sum_{i=1}^m \gamma_i^k \xi_i^k\| \leq \frac{\sqrt{\varepsilon_k}}{\|\lambda^k\|} \quad \text{and} \quad \sum_{i=1}^m \gamma_i^k g_i(x_k) \geq -\frac{\varepsilon_k}{\|\lambda^k\|}.$$

By Proposition 2.83, f and $g_i,\ i = 1, 2, \ldots, m$, have compact subdifferentials. Thus, $\{\xi_0^k\}$ and $\{\xi_i^k\},\ i = 1, 2, \ldots, m$, are bounded sequences and hence by the Bolzano–Weierstrass Theorem, Proposition 1.3, have a convergent subsequence. Without loss of generality, let $\xi_0^k \to \xi_0$ and $\xi_i^k \to \xi_i,\ i = 1, 2, \ldots, m$. By the Closed Graph Theorem, Theorem 2.84, $\xi_0 \in \partial f(\bar{x})$ and $\xi_i \in \partial g_i(\bar{x})$, $i = 1, 2, \ldots, m$. Therefore, as $k \to +\infty$,

$$\frac{1}{\|\lambda^k\|}\xi_0^k \to 0, \quad \frac{\sqrt{\varepsilon_k}}{\|\lambda^k\|} \to 0 \quad \text{and} \quad \frac{\varepsilon_k}{\|\lambda^k\|} \to 0,$$

which implies $\| \sum_{i=1}^m \gamma_i \xi_i \| \leq 0$, that is, $\sum_{i=1}^m \gamma_i \xi_i = 0$ and $\sum_{i=1}^m \gamma_i g_i(\bar{x}) \geq 0$. By Definition 2.77 of the subdifferential,

$$g_i(x) - g_i(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle,\ \forall\ x \in \mathbb{R}^n,\ i = 1, 2, \ldots, m,$$

which yields

$$\sum_{i=1}^m \gamma_i g_i(x) \geq \sum_{i=1}^m \gamma_i g_i(\bar{x}) \geq 0,\ \forall\ x \in \mathbb{R}^n,$$

thereby contradicting the existence of a point $\hat{x}$ satisfying $g_i(\hat{x}) < 0$, $i = 1, 2, \ldots, m$ by the Slater constraint qualification. Therefore, the sequence $\{\lambda^k\}$ is a bounded sequence and hence by the Bolzano–Weierstrass Theorem, Proposition 1.3, has a convergent subsequence. Without loss of generality, let $\lambda_i^k \to \lambda_i,\ i = 1, 2, \ldots, m$. Taking the limit as $k \to +\infty$ in (10.31) yields

$$\|\xi_0 + \sum_{i=1}^m \lambda_i \xi_i\| \leq 0 \quad \text{and} \quad \sum_{i=1}^m \lambda_i g_i(\bar{x}) \geq 0. \tag{10.32}$$

The norm condition in (10.32) implies that

$$0 = \xi_0 + \sum_{i=1}^m \lambda_i \xi_i,$$

thereby leading to the optimality condition

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}).$$

As $\{x_k\}$ is a modified ε_k-KKT point, it is a feasible point of (CP), that is, $g_i(x_k) \leq 0,\ i = 1, 2, \ldots, m$, which implies $g_i(\bar{x}) \leq 0,\ i = 1, 2, \ldots, m$, as $k \to +\infty$. This along with the condition in (10.32) leads to the complementary slackness condition

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) = 0.$$

Hence, $\bar{x}$ satisfies the standard KKT optimality conditions. As (CP) is a convex programming problem, by the sufficient optimality conditions, $\bar{x}$ is a point of minimizer of (CP), thereby establishing the desired result. $\square$

Theorems 10.23 and 10.24 can be combined together and stated as follows.

Theorem 10.25 *Consider the convex programming problem (CP) with C given by (3.1). Assume that the Slater constraint qualification holds. Let $\{x_k\}$ be a sequence of the ε_k-solution of (CP) such that $x_k \to \bar{x}$ and $\varepsilon_k \downarrow 0$ as $k \to +\infty$. Then $\bar{x}$ is a point of minimizer of (CP).*

10.8 Duality-Based Approach to ε-Optimality

In this chapter, in all the results on approximate optimality conditions, we have assumed the Slater constraint qualification. But what if neither the Slater nor any other constraint qualification is satisfied. Work has been done in this respect by Yokoyama [113] using the exact penalization approach. In this work, he replaced the assumption of Slater constraint qualification by relating the penalty parameter with the ε-maximum solution of the dual problem associated with (CP). The results were obtained relating the ε-solutions of the given problem (CP), its dual problem, and the penalized problem. Here we will discuss some of his results in comparison to the ones derived in Section 10.5. For that purpose, we associate the dual problem

$$\sup\ w(\lambda) \quad \text{subject to} \quad \lambda \in \mathbb{R}^m, \tag{DP}$$

where $w(\lambda) = \inf\limits_{x \in \mathbb{R}^n} L(x, \lambda)$ and $L(x, \lambda)$ is the Lagrange function given by

$$L(x, \lambda) = \begin{cases} f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), & \lambda_i \geq 0,\ i = 1, 2, \ldots, m, \\ -\infty, & \text{otherwise.} \end{cases}$$

Denote the duality gap by $\theta = \inf_{x \in C} f(x) - \sup_{\lambda \in \mathbb{R}^m} w(\lambda)$. Next we present the theorem relating the ε-solution of $(CP)_\rho$ with the almost ε-solution of (CP) under the assumption of the ε-maximum solution of (DP). Recall the penalized problem

$$\min \ f_\rho(x) \quad \text{subject to} \quad x \in \mathbb{R}^n, \qquad (CP)_\rho$$

where $f_\rho(x) = f(x) + \sum_{i=1}^m \rho_i \max\{0, g_i(x)\}$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$ with $\rho_i > 0,\ i = 1, 2, \ldots, m$.

Theorem 10.26 *Consider the convex programming problem (CP) with C given by (3.1) and its associated dual problem (DP). Then for ρ satisfying*

$$\rho \geq 3 + \max_{i=1,\ldots,m} \bar{\lambda}_i + \frac{\theta}{\varepsilon},$$

where $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m)$ is an ε-maximum solution of (DP), every $\bar{x}$ that is an ε-solution of $(CP)_\rho$ is also an almost ε-solution of (CP).

Proof. Consider an ε-solution $\hat{x} \in C$ of (CP), that is,

$$f(\hat{x}) \leq \inf_{x \in C} f(x) + \varepsilon.$$

As $\bar{x}$ is an ε-solution of $(CP)_\rho$, in particular,

$$f_\rho(\bar{x}) \leq f_\rho(\hat{x}) + \varepsilon = f(\hat{x}) + \varepsilon,$$

which implies that

$$f(\bar{x}) + \rho \sum_{i=1}^m \max\{0, g_i(\bar{x})\} \leq \inf_{x \in C} f(x) + 2\varepsilon. \qquad (10.33)$$

By the definition of duality gap θ,

$$\inf_{x \in C} f(x) = \sup_{\lambda \in \mathbb{R}^m} w(\lambda) + \theta. \qquad (10.34)$$

For an ε-maximum solution $\bar{\lambda}$ of the dual problem (DP),

$$\sup_{\lambda \in \mathbb{R}^m} w(\lambda) \leq w(\bar{\lambda}) + \varepsilon \leq f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) + \varepsilon. \qquad (10.35)$$

Therefore, using the conditions (10.34) and (10.35), (10.33) becomes

$$f(\bar{x}) + \rho \sum_{i=1}^m \max\{0, g_i(\bar{x})\} \leq f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) + 3\varepsilon + \theta,$$

that is,

$$\rho \sum_{i=1}^m \max\{0, g_i(\bar{x})\} \leq \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) + 3\varepsilon + \theta.$$

Define the index set $I^{>} = \{i \in \{1, 2, \ldots, m\} : g_i(\bar{x}) > 0\}$. Thus

$$\begin{aligned} \rho \sum_{i \in I^{>}} g_i(\bar{x}) = \rho \sum_{i=1}^{m} \max\{0, g_i(\bar{x})\} &\leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) + 3\varepsilon + \theta \\ &\leq \sum_{i \in I^{>}} \bar{\lambda}_i g_i(\bar{x}) + 3\varepsilon + \theta, \end{aligned}$$

which implies

$$(\rho - \max_{i=1,\ldots,m} \bar{\lambda}_i) \sum_{i \in I^{>}} g_i(\bar{x}) \leq \sum_{i \in I^{>}} (\rho - \bar{\lambda}_i) g_i(\bar{x}) \leq 3\varepsilon + \theta.$$

From the above condition and the given hypothesis on ρ,

$$\sum_{i \in I^{>}} g_i(\bar{x}) \leq \frac{3\varepsilon + \theta}{(\rho - \max_{i=1,\ldots,m} \bar{\lambda}_i)} \leq \varepsilon,$$

thereby implying that $\bar{x} \in C_\varepsilon = \{x \in \mathbb{R}^n : g_i(x) \leq \varepsilon,\ i = 1, 2, \ldots, m\}$.

Also, $f(\bar{x}) \leq f_\rho(\bar{x})$. As $\bar{x}$ is an ε-solution of $(CP)_\rho$,

$$f_\rho(\bar{x}) \leq f_\rho(x) + \varepsilon,\ \forall\, x \in \mathbb{R}^n,$$

which along with the fact that $f(x) = f_\rho(x)$ for every $x \in C$ leads to

$$f(\bar{x}) \leq f(x) + \varepsilon,\ \forall\, x \in C,$$

thus implying that $\bar{x}$ is an almost ε-solution of (CP). □

This result is the same as Theorem 10.17 except for the bound on the penalty parameter. Recall from Theorem 10.17 that $\rho \geq \dfrac{\alpha + \varepsilon}{\varepsilon}$ where $\alpha = \inf_{x \in C} f(x) - \inf_{x \in \mathbb{R}^n} f(x)$. Also in that result the Slater constraint qualification was not assumed. Observe that both the results are similar but the parameter bounds are different. Under the Slater constraint qualification, it is known that strong duality holds and thus the duality gap $\theta = 0$ and the dual problem (DP) is solvable. Consequently, under the Slater constraint qualification, the bound on the parameter now becomes

$$\rho \geq 3 + \max_{i=1,\ldots,m} \bar{\lambda}_i,$$

where $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m)$ is a maximizer of (DP). Here we were discussing the existence of an almost ε-solution of (CP), given an ε-solution of $(CP)_\rho$.

From the discussion in Section 10.5, it is seen that under the Slater constraint qualification and for $\rho \geq \rho_0$ with ρ_0 given in Theorem 10.19,

$$\inf_{x \in C} f(x) = \inf_{x \in \mathbb{R}^n} f_\rho(x),$$

thereby implying that every $\bar{x}$ that is an ε-solution of (CP) is also an ε-solution of $(CP)_\rho$. So in absence of any constraint qualification, Yokoyama [113] obtained that $\bar{x}$ is an $(2\varepsilon + \theta)$-solution of $(CP)_\rho$ presented below.

Theorem 10.27 *Consider the convex programming problem* (CP) *with* C *given by (3.1) and its associated dual problem* (DP)*. Then for* ρ *satisfying*

$$\rho \geq \max_{i=1,\ldots,m} \bar{\lambda}_i,$$

where $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m)$ *is an* ε*-maximum solution of* (DP)*, every* $\bar{x}$ *that is an* ε*-solution of* (CP) *is also a* $(2\varepsilon + \theta)$*-solution of* $(CP)_\rho$*.*

Proof. As $\bar{x}$ is an ε-solution of (CP), $\bar{x} \in C$, which implies

$$f_\rho(\bar{x}) = f(\bar{x}) \leq \inf_{x \in C} f(x) + \varepsilon.$$

As $\bar{\lambda}$ is an ε-maximum solution of (DP), working along the lines of Theorem 10.26, the above condition becomes

$$f_\rho(\bar{x}) \leq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + 2\varepsilon + \theta, \ \forall\, x \in \mathbb{R}^n.$$

Using the hypothesis on ρ, the above inequality leads to

$$f_\rho(\bar{x}) \leq f(x) + \rho \sum_{i=1}^{m} \max\{0, g_i(x)\} + 2\varepsilon + \theta = f_\rho(x) + 2\varepsilon + \theta, \ \forall\, x \in \mathbb{R}^n,$$

thereby implying that $\bar{x}$ is a $(2\varepsilon + \theta)$-solution of $(CP)_\rho$. $\square$

It was mentioned by Yokoyama [113] that in the presence of the Slater constraint qualification and with $\bar{\lambda}$ as some optimal Lagrange multiplier, every $\bar{x}$ that is an ε-solution of (CP) is also an ε-solution of $(CP)_\rho$. In his work, Yokoyama also derived the necessary approximate optimality conditions as established in this chapter in the absence of any constraint qualification. The sufficiency could be established only under the assumption of the Slater constraint qualification.

Chapter 11

Convex Semi-Infinite Optimization

11.1 Introduction

In all the preceding chapters we considered the convex programming problem (CP) with the feasible set C of the form (3.1), that is,

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, 2, \ldots, m\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \ldots, m$, are convex functions. Observe that the problem involved only a finite number of constraints. Now in situations where the number of constraints involved is infinite, the problem extends to the class of semi-infinite programming problems. Such problems come into existence in many physical and social sciences models where it is necessary to consider the constraints on the state or the control of the system during a period of time. For examples from real-life scenarios where semi-infinite programming problem are involved, readers may refer to Hettich and Kortanek [57] and references therein. We consider the following *convex semi-infinite programming problem*,

$$\inf\ f(x) \quad \text{subject to} \quad g(x, i) \leq 0,\ i \in I \qquad (SIP)$$

where $f,\ g(., i) : \mathbb{R}^n \to \mathbb{R},\ i \in I$ are convex functions with infinite index set $I \subset \mathbb{R}^m$. The term "*semi-infinite programming*" is derived from the fact that the decision variable x is finite while the index set I is infinite. But before moving on with the derivation of KKT optimality conditions for (SIP), we present some notations that will be used in subsequent sections.

Denote the feasible set of (SIP) by C_I, that is,

$$C_I = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i \in I\}.$$

Let $\mathbb{R}^I$ be the *product space* of $\lambda = (\lambda_i \in \mathbb{R} : i \in I)$ and

$$\mathbb{R}^{[I]} = \{\lambda \in \mathbb{R}^I : \lambda_i \neq 0 \text{ for finitely many } i \in I\},$$

while the *positive cone* in $\mathbb{R}^{[I]}$, $\mathbb{R}^{[I]}_+$, is defined as

$$\mathbb{R}^{[I]}_+ = \{\lambda \in \mathbb{R}^{[I]} : \lambda_i \geq 0,\ \forall\ i \in I\}.$$

For a given $z \in \mathbb{R}^I$ and $\lambda \in \mathbb{R}^{[I]}$, define the *supporting set* of λ as *supp* $\lambda = \{i \in I : \lambda_i \neq 0\}$,

$$\langle \lambda, z \rangle = \sum_{i \in I} \lambda_i z_i = \sum_{i \in supp\ \lambda} \lambda_i z_i.$$

With these notations, we now move on to study the various approaches to obtain the KKT optimality conditions for (SIP).

11.2 Sup-Function Approach

A possible approach to solve (SIP) is to associate a problem with a finite number of constraints, that is, the reduced form of (SIP)

$$\inf \ f(x) \quad \text{subject to} \quad g(x, i) \leq 0, \ i \in \tilde{I}, \qquad (\widetilde{SIP})$$

where $\tilde{I} \subset I$ is finite and f and $g(., i)$, $i \in \tilde{I}$ are as in (SIP) such that the optimal value of (SIP) and the reduced problem $(\widetilde{SIP})$ coincide. Then $(\widetilde{SIP})$ is said to be the *equivalent reduced problem* of (SIP). One way to reduce (SIP) to an equivalent $(\widetilde{SIP})$ is to replace the infinite inequality constraints by a single constraint,

$$\tilde{g}(x) = \sup_{i \in I} g(x, i),$$

where $\tilde{g} : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a convex function by Proposition 2.53 (iii). Therefore, the reduced problem is

$$\inf \ f(x) \quad \text{subject to} \quad \tilde{g}(x) \leq 0. \qquad (\widetilde{SIP}_{sup})$$

Such a formulation was studied by Pshenichnyi [96], where $g(., i)$ for every $i \in I$, were taken to be convex differentiable functions. Observe that $(\widetilde{SIP}_{sup})$ is of the form (CP) studied in Chapter 3. It was seen that under the Slater constraint qualification, the standard KKT optimality conditions for (CP) can be obtained. Therefore, to apply Theorem 3.7, $(\widetilde{SIP}_{sup})$ should satisfy the Slater constraint qualification. But this problem is equivalent to (SIP), for which we introduce the following Slater constraint qualification for (SIP).

Definition 11.1 The *Slater constraint qualification* for (SIP) is

(i) $I \subset \mathbb{R}^m$ is a compact set,

(ii) $g(x, i)$ is a continuous function of $(x, i) \in \mathbb{R}^n \times I$,

(iii) There exists $\hat{x} \in \mathbb{R}^n$ such that $g(\hat{x}, i) < 0$ for every $i \in I$.

Observe that in the Slater constraint qualification for (CP), only condition (iii) is considered. Here the additional conditions (i) and (ii) ensure that the supremum is attained over I, which holds trivially in the finite index set scenario. We now present the KKT optimality condition for (SIP).

Theorem 11.2 *Consider the convex semi-infinite programming problem (SIP). Assume that the Slater constraint qualification for (SIP) holds. Then $\bar{x} \in \mathbb{R}^n$ is a point of minimizer of (SIP) if and only if there exists $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$ such that*

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x}, i),$$

where $I(\bar{x}) = \{i \in I : g(\bar{x}, i) = 0\}$ denotes the active index set and the subdifferential $\partial g(\bar{x}, i)$ is with respect to x.

Proof. As already observed, (SIP) is equivalent to $(\widetilde{SIP}_{sup})$ and thus $\bar{x}$ is also a point of minimizer of $(\widetilde{SIP}_{sup})$. As the Slater constraint qualification for (SIP) holds, then by conditions (i) and (ii) the supremum is attained over I. Therefore, by condition (iii) of the Slater constraint qualification for (SIP), there exists $\hat{x} \in \mathbb{R}^n$ such that

$$\tilde{g}(\hat{x}) = \sup_{i \in I} g(\hat{x}, i) < 0,$$

which implies that $(\widetilde{SIP}_{sup})$ satisfies the Slater constraint qualification. Invoking Theorem 3.7, there exists $\lambda' \geq 0$ such that

$$0 \in \partial f(\bar{x}) + \lambda' \partial \tilde{g}(\bar{x}) \quad \text{and} \quad \lambda' \tilde{g}(\bar{x}) = 0. \tag{11.1}$$

Now we consider two cases depending on $\tilde{g}(\bar{x})$.

(i) $\tilde{g}(\bar{x}) < 0$: By the complementary slackness condition $\lambda' = 0$. Also, because $\tilde{g}(\bar{x}) < 0$, $g(\bar{x}, i) < 0$ for every $i \in I$, which implies the active index set $I(\bar{x})$ is empty. Thus the optimality condition (11.1) reduces to

$$0 \in \partial f(\bar{x}),$$

and the KKT optimality condition for (SIP) holds with $\lambda = 0 \in \mathbb{R}_+^{[I]}$.

(ii) $\tilde{g}(\bar{x}) = 0$: By the complementary slackness condition, $\lambda \geq 0$. Define the supremum set as

$$\hat{I}(\bar{x}) = \{i \in I : g(\bar{x}, i) = \tilde{g}(\bar{x})\} = \{i \in I : g(\bar{x}, i) = 0\},$$

which implies that $\hat{I}(\bar{x}) = I(\bar{x})$. By the conditions (i) and (ii) of the Slater constraint qualification for (SIP), $\hat{I}(\bar{x})$ and hence $I(\bar{x})$ is nonempty. By the Valadier formula, Theorem 2.97, the optimality condition becomes

$$0 \in \partial f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial g(\bar{x}, i),$$

where $\lambda_i = \lambda' \bar{\lambda}_i \geq 0,\ i \in I(\bar{x})$ with $\bar{\lambda} \in \mathbb{R}_+^{[I(\bar{x})]}$ satisfying $\sum_{i \in supp\ \bar{\lambda}} \bar{\lambda}_i = 1$. As $\lambda' \geq 0$, $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$ and thus the preceding optimality condition can be expressed as

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x}, i).$$

Thus, the KKT optimality condition is obtained for (SIP).

Conversely, suppose that the optimality condition holds, which implies that there exist $\xi \in \partial f(\bar{x})$ and $\xi_i \in \partial g(\bar{x}, i)$ such that

$$0 = \xi + \sum_{i \in supp\ \lambda} \lambda_i \xi_i, \tag{11.2}$$

where $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$. By Definition 2.77 of the subdifferential, for every $x \in \mathbb{R}^n$,

$$\begin{array}{rcl} f(x) & \geq & f(\bar{x}) + \langle \xi, x - \bar{x} \rangle, \\ g(x, i) & \geq & g(\bar{x}, i) + \langle \xi_i, x - \bar{x} \rangle,\ i \in supp\ \lambda, \end{array}$$

which along with the condition (11.2) implies that

$$f(x) + \sum_{i \in supp\ \lambda} \lambda_i g(x, i) \geq f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i g(\bar{x}, i),\ \forall\ x \in \mathbb{R}^n.$$

The above inequality along with the fact that $g(\bar{x}, i) = 0,\ i \in I(\bar{x})$ leads to

$$f(x) + \sum_{i \in supp\ \lambda} \lambda_i g(x, i) \geq f(\bar{x}),\ \forall\ x \in \mathbb{R}^n.$$

In particular, for $x \in C_I$, that is, $g(x, i) \leq 0,\ i \in I$, the above condition reduces to

$$f(x) \geq f(\bar{x}),\ \forall\ x \in C_I,$$

thereby implying that $\bar{x}$ is the minimizer of (SIP). □

11.3 Reduction Approach

As already mentioned in the preceding section, the reduction approach is one possible method to establish the KKT optimality condition for (SIP). The sup-function approach was one such reduction technique. Another way to formulate an equivalent $(\widetilde{SIP})$ is to use the approach by Ben-Tal, Rosinger, and Ben-Israel [9] to derive a Helly-type Theorem for open convex sets using the result by Klee [72]. But this approach was a bit difficult to follow. So Borwein [16] provided a self-contained proof of the reduction approach involving quasiconvex functions. Here we present the same under the assumptions that $g(., i)$ is convex for every $i \in I$ and $g(.,.)$ is jointly continuous as a function of $(x, i) \in \mathbb{R}^n \times I$. In the proof, one only needs $g(x, i)$ to be jointly usc as a function of $(x, i) \in \mathbb{R}^n \times I$ along with the convexity assumption.

Proposition 11.3 *Consider open and closed convex sets $U \subset \mathbb{R}^n$ and $C \subset \mathbb{R}^n$, respectively. The following are equivalent when I is compact.*

(i) There exists $x \in C$ and $\varepsilon > 0$ such that

$$x + \varepsilon\mathbb{B} \subset U, \quad g(y, i) < 0, \ \forall\, y \in x + \varepsilon\mathbb{B}, \ \forall\, i \in I.$$

(ii) (a) For every set of $n+1$ points $\{i_0, i_1, \ldots, i_n\} \subset I$, there exists $x \in C$ such that

$$g(x, i_0) < 0, \ g(x, i_1) < 0, \ \ldots \ , \ g(x, i_n) < 0.$$

(b) For every set of n points $\{i_1, i_2, \ldots, i_n\} \subset I$, there exists $x \in C$ such that

$$x \in U, \ g(x, i_1) < 0, \ g(x, i_2) < 0, \ \ldots \ , \ g(x, i_n) < 0.$$

Proof. It is obvious that (i) implies (ii)(b). Also, in particular, taking $y = x \in x + \varepsilon\mathbb{B}$ in (i) yields (ii)(a). Therefore, to establish the result, we show that (ii) implies (i).

Suppose that both (ii)(a) and (b) are satisfied. We first prove that (ii)(a) implies (i) with $U = \mathbb{R}^n$. For any $r \in \mathbb{N}$ and any $i \in I$, define the set

$$C^r(i) = \{x \in C \cap r \ cl \ \mathbb{B} : g(y, i) < 0, \ \forall\, y \in x + \frac{1}{r}\mathbb{B}\}.$$

Observe that $C^r(i) \subset r \ cl \ \mathbb{B}$ and hence is bounded.

We claim that $C^r(i)$ is convex. Consider $x_1, x_2 \in C^r(i)$, which implies that $x_j \in C \cap r \ cl \ \mathbb{B}, \ j = 1, 2$. Because C and $cl \ \mathbb{B}$ are convex sets, $C \cap r \ cl \ \mathbb{B}$ is also convex. Thus,

$$(1-\lambda)x_1 + \lambda x_2 \in C \cap r \ cl \ \mathbb{B}, \ \forall\, \lambda \in [0, 1].$$

For any $y_j \in x_j + \frac{1}{r}\mathbb{B}, \ j = 1, 2,$

$$\begin{aligned} y = (1-\lambda)y_1 + \lambda y_2 \quad &\in \quad (1-\lambda)(x_1 + \frac{1}{r}\mathbb{B}) + \lambda(x_2 + \frac{1}{r}\mathbb{B}) \\ &\subset \quad (1-\lambda)x_1 + \lambda x_2 + \frac{1}{r}\mathbb{B}. \end{aligned}$$

As $x_1, x_2 \in C^r(i)$, for $j = 1, 2$,

$$g(y_j, i) < 0, \ \forall \ y_j \in x_j + \frac{1}{r}\mathbb{B}.$$

By the convexity of $g(., i)$, for any $\lambda \in [0, 1]$,

$$g(y, i) \leq (1 - \lambda)g(y_1, i) + \lambda g(y_2, i) < 0.$$

Because the above conditions hold for arbitrary $y_j \in x_j + \frac{1}{r}\mathbb{B}$, $j = 1, 2$,

$$g(y, i) < 0, \ \forall \ y \in (1 - \lambda)x_1 + \lambda x_2 + \frac{1}{r}\mathbb{B}.$$

Therefore, from the definition of $C^r(i)$, it is obvious that

$$(1 - \lambda)x_1 + \lambda x_2 \in C^r(i), \ \forall \ \lambda \in [0, 1].$$

Because $x_1, x_2 \in C^r(i)$ are arbitrary, $C^r(i)$ is a convex set.

Next we prove that $C^r(i)$ is closed. Suppose that $\bar{x} \in cl \ C^r(i)$, which implies there exists a sequence $\{x_k\} \subset C^r(i)$ with $x_k \to \bar{x}$. Because $x_k \in C^r(i)$, $x_k \in C \cap r \ cl \ \mathbb{B}$ such that

$$g(y, i) < 0, \ \forall \ y \in x_k + \frac{1}{r}\mathbb{B}. \tag{11.3}$$

Because C and $cl \ \mathbb{B}$ are closed sets, $C \cap r \ cl \ \mathbb{B}$ is also closed and thus, $\bar{x} \in C \cap r \ cl \ \mathbb{B}$. Now if $\bar{x} \notin C^r(i)$, there exists some $\bar{y} \in \bar{x} + \frac{1}{r}\mathbb{B}$ such that $g(\bar{y}, i) \geq 0$. As $x_k \to \bar{x}$, for sufficiently large k, $\bar{y} \in x_k + \frac{1}{r}\mathbb{B}$ with $g(\bar{y}, i) \geq 0$, which is a contradiction to condition (11.3). Thus $C^r(i)$ is a closed set.

Finally, we claim that for some $\bar{r} \in \mathbb{N}$ and every set of $n + 1$ points $\{i_0, i_1, \ldots, i_n\} \subset I$,

$$\bigcap_{j=0}^{n} C^{\bar{r}}(i_j) \neq \emptyset.$$

On the contrary, suppose that for every $r \in \mathbb{N}$, there exist $n + 1$ points $\{i_0^r, i_1^r, \ldots, i_n^r\} \subset I$ such that

$$\bigcap_{j=0}^{n} C^r(i_j^r) = \emptyset. \tag{11.4}$$

Define the sequence $s_r = (i_0^r, i_1^r, \ldots, i_n^r) \in I^{n+1}$. As I is a compact set, I^{n+1} is also compact and thus $\{s_r\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, assume that $s_r \to \bar{s}$, where $\bar{s} = (\bar{i}_0, \bar{i}_1, \ldots, \bar{i}_n) \in I^{n+1}$. As (ii)(a) is satisfied, there exists $\bar{x} \in C$ such that

$$g(\bar{x}, \bar{i}_0) < 0, \ g(\bar{x}, \bar{i}_1) < 0, \ \ldots \ , \ g(\bar{x}, \bar{i}_n) < 0.$$

Because $g(.,.)$ is jointly continuous on $(x,i) \in \mathbb{R}^n \times I$, hence jointly usc on $(x,i) \in \mathbb{R}^n \times I$. Therefore, by the above condition there exist $\varepsilon > 0$ and a neighborhood of $\bar{i}_j$, $\mathcal{N}(\bar{i}_j)$, $j = 0, 1, \ldots, n$, such that

$$g(y, i_j) < 0,\ \forall\ y \in \bar{x} + \varepsilon\mathbb{B},\ \forall\ i_j \in \mathcal{N}(\bar{i}_j),\ j = 0, 1, \ldots, n. \tag{11.5}$$

As $i_j^r \to \bar{i}_j$, one may choose $\bar{r} \in \mathbb{N}$ sufficiently large such that

$$\|\bar{x}\| \leq \bar{r}, \quad \varepsilon > \frac{1}{\bar{r}} \quad \text{and} \quad i_j^{\bar{r}} \in \mathcal{N}(\bar{i}_j),\ j = 0, 1, \ldots, n. \tag{11.6}$$

Combining (11.5) and (11.6), $\bar{x} \in C \cap \bar{r}\ cl\ \mathbb{B}$ such that

$$g(y, i_j^{\bar{r}}) < 0,\ \forall\ y \in \bar{x} + \frac{1}{\bar{r}}\mathbb{B}.$$

Therefore, $\bar{x} \in C^{\bar{r}}(i_j^{\bar{r}})$ for every $j = 0, 1, \ldots, n$, which contradicts our assumption (11.4). Thus, for some $\bar{r} \in \mathbb{N}$ and every set of $n+1$ points $\{i_0, i_1, \ldots, i_n\} \subset I$,

$$\bigcap_{j=0}^{n} C^{\bar{r}}(i_j) \neq \emptyset.$$

As $C^{\bar{r}}(i_j)$, $j = 0, 1, \ldots, n$, are nonempty compact convex sets, invoking Helly's Theorem, Proposition 2.28,

$$\bigcap_{i \in I} C^{\bar{r}}(i) \neq \emptyset.$$

From the above condition, there exists $\tilde{x} \in C^r(i)$ for every $i \in I$, which implies $\tilde{x} \in C$ such that

$$g(y, i) < 0,\ \forall\ y \in \tilde{x} + \frac{1}{r}\mathbb{B},\ i \in I.$$

Taking $U = \mathbb{R}^n$ and defining $\varepsilon = \dfrac{1}{r}$ for $r \in \mathbb{N}$, the above condition yields (i).

To complete the proof, we have to finally show that (ii)(b) also implies (i). This can be done by expressing (ii(b) in the form of (ii)(a). Consider a point $i' \notin I$ and define $I' = \{i'\} \cup I$, which is again a compact set. Also define the function g' on $\mathbb{R}^n \times I'$ as

$$g'(x, i') = \begin{cases} -\delta, & x \in U, \\ +\infty, & x \notin U, \end{cases} \qquad \text{and} \qquad g'(x, i) = g(x, i),\ i \in I,$$

where $\delta > 0$. Observe that $g'(., i),\ i \in I'$ satisfies the convexity assumption and is jointly usc on $\mathbb{R}^n \times I'$. Therefore, (ii)(b) is equivalent to the existence of $x \in C$ for every n points $\{i_1, i_2, \ldots, i_n\} \subset I$,

$$g'(x, i') < 0,\ g'(x, i_1) < 0,\ \ldots\ ,\ g'(x, i_n) < 0. \tag{11.7}$$

As (ii)(a) is also satisfied, for every $n+1$ points $\{i_0, i_1, \ldots, i_n\} \subset I$ there exists $x \in C$ such that

$$g'(x, i_0) < 0, \; g'(x, i_1) < 0, \; \ldots \;, \; g'(x, i_n) < 0. \tag{11.8}$$

Combining the conditions (11.7) and (11.8), (ii)(b) implies that for every $n+1$ points $\{i_0, i_1, \ldots, i_n\} \subset I'$ there exists $x \in C$ such that

$$g'(x, i_0) < 0, \; g'(x, i_1) < 0, \; \ldots \;, \; g'(x, i_n) < 0,$$

which is of the form (ii)(a). As we have already seen that (ii)(a) implies (i) with $U = \mathbb{R}^n$, there exists $x \in C$ and $\varepsilon > 0$ such that

$$g'(x, i) < 0, \; \forall\, y \in x + \varepsilon\mathbb{B}, \; \forall\, i \in I',$$

which by the definition of the function g' implies that

$$y \in U, \; g(x, i) < 0, \; \forall\, y \in x + \varepsilon\mathbb{B}, \; \forall\, i \in I,$$

that is,

$$x + \varepsilon\mathbb{B} \subset U, \; g(x, i) < 0, \; \forall\, y \in x + \varepsilon\mathbb{B}, \; \forall\, i \in I.$$

Thus, (ii)(b) implies (i) and hence establishes the result. $\square$

Using the above proposition, Borwein [16] obtained the equivalent reduced form of (SIP) under the relaxed Slater constraint qualification. The convex semi-infinite programming (SIP) is said to satisfy the *relaxed Slater constraint qualification* for (SIP) if given any $n+1$ points $\{i_0, i_1, \ldots, i_n\} \subset I$, there exists $\hat{x} \in \mathbb{R}^n$ such that

$$g(\hat{x}, i_0) < 0, \; g(\hat{x}, i_1) < 0, \; \ldots \;, \; g(\hat{x}, i_n) < 0.$$

Observe that the Slater constraint qualification for (SIP) also implies the relaxed Slater constraint qualification for (SIP). Now we present the KKT optimality condition for (SIP) by reducing it to the equivalent $(\widetilde{SIP})$.

Theorem 11.4 *Consider the convex semi-infinite programming problem* (SIP)*. Suppose that the relaxed Slater constraint qualification for* (SIP) *holds. Then* $\bar{x}$ *is a point of minimizer of* (SIP) *if and only if there exist* n *points* $\{i_1, i_2, \ldots, i_n\} \subset I$, $\lambda_{i_j} \geq 0, \; j = 1, 2, \ldots, n$, *such that*

$$0 \in \partial f(\bar{x}) + \sum_{j=1}^{n} \lambda_{i_j} \partial g(\bar{x}, i_j).$$

Proof. Define an open set

$$U = \{x \in \mathbb{R}^n : f(x) < f(\bar{x})\}.$$

Consider $x_1, x_2 \in U$. By the convexity of f,

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2) < f(\bar{x}), \ \forall \ \lambda \in [0,1],$$

which implies $(1-\lambda)x_1 + \lambda x_2 \in U$. Because $x_1, x_2 \in U$ were arbitrary, U is a convex set. As $\bar{x}$ is a point of minimizer of (SIP), there does not exist any $x \in \mathbb{R}^n$ and $\varepsilon > 0$ such that

$$x + \varepsilon\mathbb{B} \subset U, \quad g(y,i) < 0, \ \forall \ y \in x + \varepsilon\mathbb{B}, \ \forall \ i \in I,$$

which implies that (i) of Proposition 11.3 does not hold. Therefore either (ii)(a) or (ii)(b) is not satisfied. As the relaxed Slater constraint qualification for (SIP), which is the same as (ii)(a), holds, (ii)(b) cannot be satisfied. Thus, there exist n points $\{i_1, i_2, \ldots, i_n\} \subset I$ such that

$$f(x) < f(\bar{x}), \quad g(x, i_j) < 0, \ j = 1, 2, \ldots, n, \tag{11.9}$$

has no solution. We claim that $\bar{x}$ is a point of minimizer of the reduced problem

$$\inf \ f(x) \quad \text{subject to} \quad g(x, i_j) \leq 0, \ j = 1, 2, \ldots, n. \tag{$\widetilde{SIP}$}$$

Consider a feasible point $\tilde{x}$ of $(\widetilde{SIP})$, that is,

$$g(x, i_j) \leq 0, \ j = 1, 2, \ldots, n. \tag{11.10}$$

Also, by the relaxed Slater constraint qualification for (SIP), corresponding to the $n+1$ points $\{i_0, i_1, i_2, \ldots, i_n\} \subset I$ with $\{i_1, i_2, \ldots, i_n\}$ as in $(\widetilde{SIP})$, there exists $\hat{x}$ such that

$$g(\hat{x}, i_j) < 0, \ j = 0, 1, 2, \ldots, n. \tag{11.11}$$

By the convexity of $g(., i_j)$, $j = 1, 2, \ldots, n$, along with the conditions (11.10) and (11.11),

$$g((1-\lambda)\tilde{x} + \lambda\hat{x}, i_j) \leq (1-\lambda)g(\tilde{x}, i_j) + \lambda g(\hat{x}, i_j) < 0, \ \forall \ \lambda \in (0,1).$$

Because the system (11.9) has no solution,

$$f((1-\lambda)\tilde{x} + \lambda\hat{x}) \geq f(\bar{x}), \ \forall \ \lambda \in (0,1). \tag{11.12}$$

As $dom\ f = \mathbb{R}^n$, by Theorem 2.69, f is continuous on $\mathbb{R}^n$. Thus, taking the limit as $\lambda \to 0$, the inequality (11.12) leads to

$$f(\tilde{x}) \geq f(\bar{x}).$$

Because $\tilde{x}$ is an arbitrary feasible point of $(\widetilde{SIP})$, $\bar{x}$ is a point of minimizer of $(\widetilde{SIP})$. Observe that by (11.11), the Slater constraint qualification is satisfied by the reduced problem. Therefore, invoking Theorem 3.7, there exist $\lambda_{i_j} \geq 0$, $j = 1, 2, \ldots, n$, such that

$$0 \in \partial f(\bar{x}) + \sum_{j=1}^{n} \lambda_{i_j} \partial g(\bar{x}, i_j) \quad \text{and} \quad \lambda_{i_j} g(\bar{x}, i_j) = 0, \; j = 1, 2, \ldots, n. \tag{11.13}$$

We claim that $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$. From the complementary slackness condition in the optimality condition (11.13), if $i_j \notin I(\bar{x})$, $\lambda_{i_j} = 0$, whereas for $i_j \in I(\bar{x})$, $\lambda_{i_j} \geq 0$. For $i \notin \{i_1, i_2, \ldots, i_n\}$ but $i \in I(\bar{x})$, define $\lambda_i = 0$. Therefore, $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$ such that

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x}, i),$$

thereby yielding the KKT optimality condition for (SIP). The converse can be worked out along the lines of Theorem 11.2. □

11.4 Lagrangian Regular Point

In both the preceding sections on reduction techniques to establish the KKT optimality condition for (SIP), the index set I was taken to be compact. But then what about the scenarios where the index set I need not be compact. To look into such situations, López and Vercher [75] introduced the concept of Lagrangian regular point, which we present next. Before we define this concept, we introduce the following notations.

For $\bar{x} \in C_I$ having nonempty $I(\bar{x})$, define

$$\bar{S}(\bar{x}) = \{\partial g(\bar{x}, i) \in \mathbb{R}^n : i \in I(\bar{x})\} = \bigcup_{i \in I(\bar{x})} \partial g(\bar{x}, i)$$

and

$$\widehat{S}(\bar{x}) = cone\ co\ \bar{S}(\bar{x}) = \{ \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x}, i) \in \mathbb{R}^n : \lambda \in \mathbb{R}_+^{[I(\bar{x})]} \}.$$

For any $i \in I(\bar{x})$, consider $\xi_i \in \partial g(\bar{x}, i) \subset \bar{S}(\bar{x})$. By Definition 2.77 of the subdifferential,

$$g(x, i) - g(\bar{x}, i) \geq \langle \xi_i, x - \bar{x} \rangle, \; \forall\ x \in \mathbb{R}^n.$$

In particular, for $\{x_k\} \subset C_I$, that is, $g(x_k, i) \leq 0$ along with the fact that $g(\bar{x}, i) = 0, \; i \in I(\bar{x})$, the above inequality reduces to

$$\langle \xi_i, x_k - \bar{x} \rangle \leq 0, \; \forall\ k \in \mathbb{N}.$$

For any $\{\alpha_k\} \subset \mathbb{R}_+$,

$$\langle \xi_i, \alpha_k(x_k - \bar{x}) \rangle \leq 0, \ \forall\, k \in \mathbb{N}.$$

Taking the limit as $k \to +\infty$ in the above inequality,

$$\langle \xi_i, z \rangle = \lim_{k \to \infty} \langle \xi_i, \alpha_k(x_k - \bar{x}) \rangle \leq 0,$$

where $z \in cl\ cone\ (C_I - \bar{x})$. By Theorem 2.35, $z \in T_{C_I}(\bar{x})$. Because $i \in I(\bar{x})$ and $\xi_i \in \bar{S}(\bar{x})$ were arbitrary,

$$\langle \xi, z \rangle \leq 0, \ \forall\, \xi \in \bar{S}(\bar{x}),$$

which implies $z \in (\bar{S}(\bar{x}))^\circ$. Because $z \in T_{C_I}(\bar{x})$ is arbitrary,

$$T_{C_I}(\bar{x}) \subset (\bar{S}(\bar{x}))^\circ, \tag{11.14}$$

which by Propositions 2.31(iii) and 2.37 implies that $cl\ \widehat{S}(\bar{x}) \subset N_{C_I}(\bar{x})$. As (SIP) is equivalent to the unconstrained problem,

$$\inf\ (f + \delta_{C_I})(x) \quad \text{subject to} \quad x \in \mathbb{R}^n, \tag{SIP_u}$$

therefore, if $\bar{x}$ is a point of minimizer of (SIP), it is also a minimizer of (SIP_u). By Theorem 3.1, the following optimality condition

$$0 \in \partial f(\bar{x}) + N_{C_I}(\bar{x}) \tag{11.15}$$

holds. So rather than $cl\ \widehat{S}(\bar{x}) \subset N_{C_I}(\bar{x})$, one would prefer the reverse relation so that the above condition may be explicitly expressed in terms of the subdifferential of the constraints. Thus, we move on with the notion of Lagrangian regular point studied in López and Vercher [75].

Definition 11.5 $\bar{x} \in C_I$ is said to be a *Lagrangian regular point* if

(i) $I(\bar{x})$ is empty: $T_{C_I}(\bar{x}) = \mathbb{R}^n$.

(ii) $I(\bar{x})$ is nonempty: $(\bar{S}(\bar{x}))^\circ \subset T_{C_I}(\bar{x})$ and $\widehat{S}(\bar{x})$ is closed.

Recall the equivalent Abadie constraint qualification for (CP) studied in Chapter 3, that is, $S(\bar{x}) \subset T_C(\bar{x})$, where

$$S(\bar{x}) = \{v \in \mathbb{R}^n : g_i'(\bar{x}, v) \leq 0, \ \forall\, i \in I(\bar{x})\}.$$

By Proposition 3.9,

$$(S(\bar{x}))^\circ = cl\ \widehat{S}(\bar{x})$$

which by Proposition 2.31(ii) and (iii) along with the fact that $S(\bar{x})$ is a closed convex cone implies that

$$(\widehat{S}(\bar{x}))^\circ = (\bar{S}(\bar{x}))^\circ = S(\bar{x})$$

where

$$\bar{S}(\bar{x}) = \{\partial g_i(\bar{x}) : i = 1, 2, \ldots, m\} = \bigcup_{i=1}^{m} \partial g_i(\bar{x}).$$

Therefore, the Abadie constraint qualification is equivalent to

$$(\bar{S}(\bar{x}))^\circ \subset T_C(\bar{x}).$$

Moreover, in the derivation of the standard KKT optimality condition for (CP), Theorem 3.10, under the Abadie constraint qualification, we further assumed that $\widehat{S}(\bar{x})$ was closed. A careful look at the Lagrangian regular point when $I(\bar{x})$ is nonempty shows that it is an extension of the Abadie constraint qualification to (SIP) along with the closedness condition. Next we derive the KKT optimality condition for (SIP) under the Lagrangian regularity. The result is due to López and Vercher [75].

Theorem 11.6 *Consider the convex semi-infinite programming problem* (SIP). *Assume that* $\bar{x} \in C_I$ *is a Lagrangian regular point. Then* $\bar{x}$ *is a point of minimizer of* (SIP) *if and only if there exists* $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$ *such that*

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x}, i).$$

Proof. Suppose that $\bar{x}$ is a point of minimizer of (SIP), which by the condition (11.15) implies

$$0 \in \partial f(\bar{x}) + N_{C_I}(\bar{x}).$$

Therefore, there exists $\xi \in \partial f(\bar{x})$ such that

$$-\xi \in N_{C_I}(\bar{x}).$$

Depending on the emptiness and nonemptiness of $I(\bar{x})$, we consider the following two cases.

(i) $I(\bar{x})$ is empty: As $\bar{x}$ is a Lagrangian regular point, $T_{C_I}(\bar{x}) = \mathbb{R}^n$, which by Proposition 2.37 implies that

$$N_{C_I}(\bar{x}) = (T_{C_I}(\bar{x}))^\circ = \{0\}.$$

Therefore, the optimality condition reduces to

$$0 \in \partial f(\bar{x}).$$

(ii) $I(\bar{x})$ is nonempty: As $\bar{x}$ is a Lagrangian regular point, $(\bar{S}(\bar{x}))^\circ \subset T_{C_I}(\bar{x})$, which by Proposition 2.31(i) and (iii) yields that

$$\begin{aligned} N_{C_I}(\bar{x}) &\subset (\bar{S}(\bar{x}))^{\circ\circ} \\ &= cl\ cone\ co\ \bar{S}(\bar{x}) = cl\ \widehat{S}(\bar{x}). \end{aligned} \tag{11.16}$$

Also, by relation (11.14), $T_{C_I}(\bar{x}) \subset (\bar{S}(\bar{x}))^\circ$, which implies that

$$cl\ \widehat{S}(\bar{x}) \subset N_{C_I}(\bar{x}) \tag{11.17}$$

is always true. Combining the conditions (11.16) and (11.17),

$$N_{C_I}(\bar{x}) = cl\ \widehat{S}(\bar{x}).$$

Again, by Definition 11.5 of the Lagrangian regular point, $\widehat{S}(\bar{x})$ is closed and hence, the KKT optimality condition becomes

$$0 \in \partial f(\bar{x}) + \widehat{S}(\bar{x}),$$

which implies that there exists $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$ such that

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x}, i),$$

as desired. The converse can be worked out as in Theorem 11.2, thereby establishing the requisite result. $\square$

In Goberna and López [50], they consider the *feasible direction cone* to $F \subset \mathbb{R}^n$ at $\bar{x} \in F$, $D_F(\bar{x})$, defined as

$$D_F(\bar{x}) = \{d \in \mathbb{R}^n : \text{there exists } \lambda > 0 \text{ such that } \bar{x} + \lambda d \in F\}.$$

It is easy to observe that

$$D_F(\bar{x}) \subset cone\ (F - \bar{x}). \tag{11.18}$$

In case F is a convex set, by Definition 2.46 of convex set, for every $x \in F$,

$$\bar{x} + \lambda(x - \bar{x}) \in F, \ \forall\ \lambda \in (0, 1),$$

which implies $x - \bar{x} \in D_F(\bar{x})$. Because $x \in F$ was arbitrary, $F - \bar{x} \subset D_F(\bar{x})$. As $D_F(\bar{x})$ is a cone,

$$cone\ (F - \bar{x}) \subset D_F(\bar{x}). \tag{11.19}$$

Combining the conditions (11.18) and (11.19),

$$D_F(\bar{x}) = cone\ (F - \bar{x})$$

and hence, the tangent cone to F at $\bar{x}$ is related to the feasible direction set as

$$T_F(\bar{x}) = cl\ D_F(\bar{x}).$$

For the convex semi-infinite programming problem (SIP), the feasible set is C_I. In particular, taking $F = C_I$ in the above condition yields

$$T_{C_I}(\bar{x}) = cl\ D_{C_I}(\bar{x}).$$

From (11.14) we have that $T_{C_I}(\bar{x}) \subset (\bar{S}(\bar{x}))^\circ$; thus the above condition yields

$$D_{C_I}(\bar{x}) \subset T_{C_I}(\bar{x}) \subset (\bar{S}(\bar{x}))^\circ. \tag{11.20}$$

In Definition 11.5 of the Lagrangian regular point, for $\bar{x} \in C_I$ with nonempty $I(\bar{x})$,

$$(\bar{S}(\bar{x}))^\circ \subset T_{C_I}(\bar{x}) = cl\ D_{C_I}(\bar{x}).$$

Combining the above condition with (11.20), which along with Proposition 2.31 and the fact that $\widehat{S}(\bar{x}) = cone\ co\ \bar{S}(\bar{x})$ implies that

$$cl\ \widehat{S}(\bar{x}) = (\bar{S}(\bar{x}))^{\circ\circ} = (D_{C_I}(\bar{x}))^\circ.$$

By the closedness condition of $\widehat{S}(\bar{x})$ at the Lagrangian regular point $\bar{x}$, the preceding condition reduces to

$$\widehat{S}(\bar{x}) = (D_{C_I}(\bar{x}))^\circ.$$

The above qualification condition is referred to as the *convex locally Farkas-Minkowski problem* in Goberna and López [50]. For $\bar{x} \in ri\ C_I$, $T_{C_I}(\bar{x}) = \mathbb{R}^n$, which by Proposition 2.37 implies that

$$N_{C_I}(\bar{x}) = (T_{C_I}(\bar{x}))^\circ = \{0\}.$$

As $T_{C_I}(\bar{x}) \subset (\bar{S}(\bar{x}))^\circ$ always holds, by Proposition 2.31,

$$\{0\} \subset (\bar{S}(\bar{x}))^{\circ\circ} \subset (T_{C_I}(\bar{x}))^\circ = \{0\}$$

which implies

$$\widehat{S}(\bar{x}) = cl\ cone\ co\ \bar{S}(\bar{x}) = (\bar{S}(\bar{x}))^{\circ\circ} = \{0\}.$$

Thus, $\widehat{S}(\bar{x}) = N_{C_I}(\bar{x})$ for every $\bar{x} \in ri\ C_I$. Therefore one needs to impose the Lagrangian regular point condition to boundary points only. This fact was mentioned in Goberna and López [50] and was proved in Fajardo and López [44].

Recall that in Chapter 3, under the Slater constraint qualification, Proposition 3.3 leads to $N_C(\bar{x}) = \widehat{S}(\bar{x})$, which by Propositions 2.31 and 2.37 is equivalent to

$$T_C(\bar{x}) = (\widehat{S}(\bar{x}))^\circ = (cl\ \widehat{S}(\bar{x}))^\circ = S(\bar{x}).$$

Also, under the Slater constraint qualification, by Lemma 3.5, $\widehat{S}(\bar{x})$ is closed. Hence the Slater constraint qualification leads to the Abadie constraint qualification along with the closedness criteria. A similar result also holds for (SIP). But before that we present *Gordan's Theorem of Alternative* which plays an important role in establishing the result.

Proposition 11.7 *(Gordan's Theorem of Alternative) Consider $x_i \in \mathbb{R}^n$ for $i \in I$, where I is an arbitrary index set. If $co\{x_i : i \in I\}$ is a closed set, then the equivalence holds between the negation of system (I) and system (II), where*

$$(I) \quad \{x \in \mathbb{R}^n : \langle x_i, x\rangle < 0, \ i \in I\} \neq \emptyset,$$

$$(II) \quad 0 \in co\ \{x_i : i \in I\}.$$

Proof. If $x_i = 0$ for some $i \in I$, then the result holds trivially as system (I) is not satisfied while system (II) holds. So without loss of generality, assume that $x_i \neq 0$ for every $i \in I$.

Suppose (I) does not hold. Let $0 \notin co\ \{x_i : i \in I\}$. As by hypothesis $co\ \{x_i : i \in I\}$ is closed, by the Strict Separation Theorem, Theorem 2.26(iii), there exists $a \in \mathbb{R}^n$ with $a \neq 0$ such that

$$\langle a, x\rangle < 0, \ \forall\ x \in co\ \{x_i : i \in I\}.$$

In particular, for $x_i \in co\ \{x_i : i \in I\}$,

$$\langle a, x_i\rangle < 0, \ \forall\ i \in I,$$

which implies system (I) holds, a contradiction to our supposition. Thus $0 \in co\ \{x_i : i \in I\}$, that is, system (II) holds.

Suppose that system (II) holds, which implies that there exists $\lambda \in \mathbb{R}^{[I]}_+$ with $\sum_{i \in supp\ \lambda} \lambda_i = 1$ such that

$$0 = \sum_{i \in supp\ \lambda} \lambda_i x_i.$$

Let $\bar{x} \in \{x \in \mathbb{R}^n : \langle x_i, x\rangle < 0, \ i \in I\}$. Therefore,

$$0 = \langle 0, \bar{x}\rangle = \sum_{i \in supp\ \lambda} \lambda_i \langle x_i, \bar{x}\rangle < 0,$$

which is a contradiction. Thus, system (I) does not hold, thereby completing the proof. $\square$

The hypothesis that $co\ \{x_i : i \in I\}$ is a closed set is required as shown in the example from López and Vercher [75]. Consider $x_i = (\cos i, \sin i)$ and $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Observe that $(0, 0) \notin co\ \{x_i : i \in I\}$ as

$$0 \in co\ \{cos\ i : i \in I\} \quad \text{and} \quad 0 \in co\ \{sin\ i : i \in I\}$$

cannot hold simultaneously because $0 \in co\ \{cos\ i : i \in I\}$ is possible only if $i = -\frac{\pi}{2}$ at which $sin\ i = -1$. Thus system (II) is not satisfied. Also, there does not exist any $x = (x_c, x_s)$ such that

$$x_c cos\ i + x_s sin\ i < 0, \ \forall\ i \in I. \tag{11.21}$$

On the contrary, suppose that such an x exists. In particular, taking $i = -\dfrac{\pi}{2}$ and $i = 0$ yields that $x_s > 0$ and $x_c < 0$, respectively. But as the limit $i \to \dfrac{\pi}{2}$,

$$x_c cos\ i + x_s sin\ i \to x_s,$$

that is, for some $i \in I$, $x_c cos\ i + x_s sin\ i > 0$, which is a contradiction to (11.21). Hence, system (I) is also not satisfied. Note that $co\ \{x_i : i \in I\}$ is not closed because taking the limit as $i \to \dfrac{\pi}{2}$,

$$x_i = (cos\ i, sin\ i) \to (0, 1)$$

and $(0, 1) \notin co\ \{x_i : i \in I\}$.

Now we present the result from López and Vercher [75] showing that the Slater constraint qualification for (SIP) implies that every feasible point $x \in C_I$ is a Lagrangian regular point.

Proposition 11.8 *Consider the convex semi-infinite programming problem (SIP). If the Slater constraint qualification for (SIP) holds, then every $\bar{x} \in C_I$ is a Lagrangian regular point.*

Proof. Suppose that $\bar{x} \in C_I$, that is,

$$g(\bar{x}, i) \leq 0, \ \ i \in I.$$

Define $g(x) = \sup_{i \in I} g(x, i)$.

(i) $I(\bar{x})$ is empty: By conditions (i) and (ii) of the Slater constraint qualification for (SIP), $g(\bar{x}) < 0$ which by Proposition 2.67 implies that $\bar{x} \in ri\ C_I$. Therefore,

$$T_{C_I}(\bar{x}) = cl\ cone\ (C_I - \bar{x}) = \mathbb{R}^n.$$

(ii) $I(\bar{x})$ is nonempty: We claim that $I(\bar{x})$ is compact. By condition (i) of the Slater constraint qualification for (SIP), I is compact. Because $I(\bar{x}) \subset I$, $I(\bar{x})$ is bounded. Now consider $\{i_k\} \subset I(\bar{x})$ such that $i_k \to i$. By the compactness of I and the fact that $I(\bar{x}) \subset I$, $i \in I$. As $i_k \in I(\bar{x})$,

$$g(\bar{x}, i_k) = 0, \ \ \forall\ k \in \mathbb{N},$$

which by condition (ii) of the Slater constraint qualification for (SIP), that is, the continuity of $g(x, i)$ with respect to (x, i) in $\mathbb{R}^n \times I$, implies that as the limit $k \to +\infty$, $g(\bar{x}, i) = 0$ and thus $i \in I(\bar{x})$. Therefore, $I(\bar{x})$ is closed, which along with the boundedness implies that $I(\bar{x})$ is compact.

Next we will show that $\bar{S}(\bar{x}) = \bigcup_{i \in I(\bar{x})} \partial g(\bar{x}, i)$ is compact. Suppose that $\{\xi_k\} \subset \bar{S}(\bar{x})$ with $\xi_k \to \xi$. As $\xi_k \in \bar{S}(\bar{x})$, there exists $i_k \in I(\bar{x})$ such that $\xi_k \in \partial g(\bar{x}, i_k)$, that is, by Definition 2.77 of the subdifferential,

$$g(x, i_k) - g(\bar{x}, i_k) \geq \langle \xi_k, x - \bar{x} \rangle, \ \ \forall\ x \in \mathbb{R}^n.$$

As $I(\bar{x})$ is compact, without loss of generality, assume that $i_k \to i \in I(\bar{x})$. Taking the limit as $k \to +\infty$ in the above inequality along with condition (ii) of the Slater constraint qualification for (SIP) yields

$$g(x,i) - g(\bar{x},i) \geq \langle \xi, x - \bar{x} \rangle, \ \forall \ x \in \mathbb{R}^n,$$

that is, $\xi \in \partial g(\bar{x},i)$ with $i \in I(\bar{x})$. Therefore $\xi \in \bar{S}(\bar{x})$, thereby implying that $\bar{S}(\bar{x})$ is closed. As $dom\ g(.,i) = \mathbb{R}^n, \ i \in I(\bar{x})$, by Proposition 2.83, $\partial g(\bar{x},i)$ is compact for $i \in I(\bar{x})$, which implies for every $\xi_i \in \partial g(\bar{x},i)$ there exists $M_i > 0$ such that

$$\|\xi_i\| \leq M_i, \ \forall \ i \in I(\bar{x}).$$

Because $I(\bar{x})$ is compact, the supremum of M_i over $I(\bar{x})$ is attained, that is,

$$\sup_{i \in I(\bar{x})} M_i = M < +\infty.$$

Therefore, for every $i \in I(\bar{x})$,

$$\|\xi\| \leq M, \ \forall \ \xi \in \partial g(\bar{x},i),$$

which implies that $\bar{S}(\bar{x})$ is bounded. Thus $\bar{S}(\bar{x})$ is compact.

As $I(\bar{x})$ is nonempty, $g(\bar{x}) = 0$. By condition (iii) of the Slater constraint qualification for (SIP), there exists $\hat{x}$ such that $g(\hat{x},i) < 0, \ i \in I$, which along with condition (i) yields that

$$g(\hat{x}) < 0 = g(\bar{x}).$$

Thus, $\bar{x}$ is not a point of minimizer of g and hence $0 \notin \partial g(\bar{x})$. By the Valadier formula, Theorem 2.97,

$$0 \notin co \bigcup_{i \in I(\bar{x})} \partial g(\bar{x},i) = co\ \bar{S}(\bar{x}).$$

Because $\bar{S}(\bar{x})$ is compact, by Theorem 2.9, $co\ \bar{S}(\bar{x})$ is also compact. By Proposition 3.4, *cone co* $\bar{S}(\bar{x})$ and hence $\widehat{S}(\bar{x})$ is closed.

Finally to establish that $\bar{x}$ is a Lagrangian regular point, we will prove that $(\bar{S}(\bar{x}))^\circ \subset T_{C_I}(\bar{x})$. Define the set

$$S'(\bar{x}) = \{x \in \mathbb{R}^n : \langle \xi, x \rangle < 0, \ \forall \ \xi \in \bar{S}(\bar{x})\}.$$

As $co\ \bar{S}(\bar{x})$ is closed with $0 \notin co\ \bar{S}(\bar{x})$, by the Gordan's Theorem of Alternative, Proposition 11.7, $S'(\bar{x})$ is nonempty. Therefore, by Proposition 2.67, $ri\ (\bar{S}(\bar{x}))^\circ$ is nonempty. Consider $z \in ri\ (\bar{S}(\bar{x}))^\circ$, which implies

$$\langle \xi, z \rangle < 0, \ \forall \ \xi \in \bar{S}(\bar{x}),$$

which leads to

$$\langle \xi, z \rangle < 0, \ \forall \ \xi \in co \ \bar{S}(\bar{x}).$$

Again, by the Valadier formula, Theorem 2.97,

$$\partial g(\bar{x}) = co \bigcup_{i \in I(\bar{x})} \partial g(., i) = co \ \bar{S}(\bar{x}).$$

Thus,

$$\langle \xi, z \rangle < 0, \ \forall \ \xi \in \partial g(\bar{x}).$$

By the compactness of I, the supremum is attained by $g(., i)$ over I. As $dom \ g(., i) = \mathbb{R}^n$, $dom \ g = \mathbb{R}^n$. Therefore, by Theorem 2.79,

$$g'(\bar{x}, z) = \max_{\xi \in \partial g(\bar{x})} \langle \xi, z \rangle.$$

Also, because $dom \ g = \mathbb{R}^n$, by Proposition 2.83, $\partial g(\bar{x})$ is compact and thus $g'(\bar{x}, z) < 0$, which implies that there exists $\bar{\lambda} > 0$ such that

$$g(\bar{x} + \lambda z) < 0, \ \forall \ \lambda \in (0, \bar{\lambda}),$$

which implies that for every $\lambda \in (0, \bar{\lambda})$,

$$g(\bar{x} + \lambda z, i) < 0, \ \forall \ i \in I.$$

Hence, $\bar{x} + \lambda z \in C_I$ for every $\lambda \in (0, \bar{\lambda})$, which yields

$$z \in \frac{1}{\lambda}(C_I - \bar{x}) \subset cl \ cone \ (C_I - \bar{x}) = T_{C_I}(\bar{x}).$$

Because $z \in ri \ (\bar{S}(\bar{x}))^\circ$ was arbitrary, $ri \ (\bar{S}(\bar{x}))^\circ \subset T_{C_I}(\bar{x})$, which along with the closedness of the tangent cone, $T_{C_I}(\bar{x})$, leads to

$$(\bar{S}(\bar{x}))^\circ = cl \ (ri \ (\bar{S}(\bar{x}))^\circ) \subset T_{C_I}(\bar{x}).$$

From both cases, we obtain that $\bar{x} \in C_I$ is a Lagrangian regular point. Because $\bar{x}$ was arbitrary, every feasible point is a Lagrangian regular point under the assumption of the Slater constraint qualification for (SIP), thereby establishing the result. $\square$

11.5 Farkas–Minkowski Linearization

In the previous section on the Lagrangian regular point, observe that it is defined at a point and hence is a local qualification condition. We observed that the notion of Lagrangian regular point is also known as the convex locally Farkas–Minkowski problem. In this section, we will discuss about the global qualification condition, namely the Farkas–Minkowski qualification studied in Goberna, López, and Pastor [51] and López and Vercher [75]. Before introducing this qualification condition, let us briefly discuss the concept of Farkas–Minkowski system for a linear semi-infinite system from Goberna and López [50].

Consider a *linear semi-infinite system*

$$\Theta = \{\langle x_i, x\rangle \geq c_i,\ i \in I\} \tag{$LSIS$}$$

The relation $\langle \tilde{x}, x\rangle \geq \tilde{c}$ is a *consequence relation* of the system Θ if every solution of Θ satisfies the relation. A consistent $(LSIS)$ Θ is said to be a *Farkas–Minkowski system*, in short, an *FM system*, if every consequent relation is a consequence of some finite subsystem. Before we state the *Farkas–Minkowski qualification* for convex semi-infinite programming problem (SIP), we present some results on the consequence relation and the FM system from Goberna and López [50].

Proposition 11.9 *$\langle \tilde{x}, x\rangle \geq \tilde{c}$ is a consequence relation of the consistent $(LSIS)$ Θ if and only if*

$$(\tilde{x}, \tilde{c}) \in cl\ cone\ co\ \{(x_i, c_i),\ i \in I;\ (0, -1)\}.$$

Proof. Denote by $\mathcal{K} \subset \mathbb{R}^{n+1}$ the convex cone

$$\mathcal{K} = cone\ co\ \{(x_i, c_i),\ i \in I;\ (0, -1)\}.$$

Consider $i' \notin I$ and define $(x_{i'}, c_{i'}) = (0, -1)$ and $I' = \{i'\} \cup I$. Thus

$$\mathcal{K} = cone\ co\ \{(x_i, c_i),\ i \in I'\}.$$

Suppose that $(\tilde{x}, \tilde{c}) \in cl\ \mathcal{K}$, which implies there exist $\{\lambda_k\} \subset \mathbb{R}_+^{[I']}$, $\{s_k\} \in \mathbb{N}$, $\{x_{i_{k_j}}\} \subset \mathbb{R}^n$ and $\{c_{i_{k_j}}\} \subset \mathbb{R}$ satisfying $i_{k_j} \in I'$ for $j = 1, 2, \ldots, s_k$, such that

$$(\tilde{x}, \tilde{c}) = \lim_{k\to\infty} \sum_{j=1}^{s_k} \lambda_{k_j}(x_{i_{k_j}}, c_{i_{k_j}}). \tag{11.22}$$

As $\mathcal{K} \subset \mathbb{R}^{n+1}$, by the Carathéodory Theorem, Theorem 2.8, $1 \leq s_k \leq n+2$. For any $k \in \mathbb{N}$ with $s_k < n+2$, define $\lambda_{k_j} = 0$ and any arbitrary $(x_{i_{k_j}}, c_{i_{k_j}})$ with $i_{k_j} \in I'$ for $j = s_k+1, s_k+2, \ldots, n+2$. Therefore, condition (11.22) becomes

$$(\tilde{x}, \tilde{c}) = \lim_{k\to\infty} \sum_{j=1}^{n+2} \lambda_{k_j}(x_{i_{k_j}}, c_{i_{k_j}}). \tag{11.23}$$

If $\bar{x}$ is a solution of $(LSIS)$ Θ,

$$\langle x_{i_{k_j}}, \bar{x}\rangle \geq c_{i_{k_j}},\ j = 1, 2, \ldots, n+2,$$

which along with the fact $\lambda_{k_j} \geq 0,\ j = 1, 2, \ldots, n+2$, leads to

$$\sum_{j=1}^{n+2} \lambda_{k_j} \langle x_{i_{k_j}}, \bar{x} \rangle \geq \sum_{j=1}^{n+2} \lambda_{k_j} c_{i_{k_j}}.$$

Taking the limit as $k \to +\infty$ in the above condition along with (11.23) yields

$$\langle \tilde{x}, \bar{x} \rangle \geq \tilde{c}.$$

Because $\bar{x}$ was arbitrary, $\langle \tilde{x}, x \rangle \geq \tilde{c}$ is a consequence relation of $(LSIS)$ Θ.

Conversely, suppose that $\langle \tilde{x}, x \rangle \geq \tilde{c}$ is a consequence relation of $(LSIS)$ Θ. We claim that $(\tilde{x}, \tilde{c}) \in cl\ \mathcal{K}$. On the contrary, suppose that $(\tilde{x}, \tilde{c}) \notin cl\ \mathcal{K}$. By the Strict Separation Theorem, Theorem 2.26 (iii), there exist $(\gamma, \gamma_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ with $(\gamma, \gamma_{n+1}) \neq (0, 0)$ such that

$$\langle \gamma, x \rangle + \gamma_{n+1} c > \tilde{\alpha} = \langle \gamma, \tilde{x} \rangle + \gamma_{n+1} \tilde{c},\ \forall\ (x, c) \in cl\ \mathcal{K}. \tag{11.24}$$

As $0 \in \mathcal{K} \subset cl\ \mathcal{K}$, the above condition implies that $\tilde{\alpha} < 0$. We claim that

$$\langle \gamma, x \rangle + \gamma_{n+1} c \geq 0,\ \forall\ (x, c) \in cl\ \mathcal{K}.$$

On the contrary, suppose that there exists $(\bar{x}, \bar{c}) \in cl\ \mathcal{K}$ such that

$$\tilde{\alpha} < \langle \gamma, \bar{x} \rangle + \gamma_{n+1} \bar{c} < 0.$$

Because $cl\ \mathcal{K}$ is a cone, $\lambda(\bar{x}, \bar{c}) \in cl\ \mathcal{K}$ for $\lambda > 0$. Therefore, the above inequality along with the condition (11.24) implies that

$$\tilde{\alpha} < \langle \gamma, \lambda \bar{x} \rangle + \gamma_{n+1} \lambda \bar{c} < 0. \tag{11.25}$$

Taking the limit as $\lambda \to +\infty$,

$$\langle \gamma, \lambda \bar{x} \rangle + \gamma_{n+1} \lambda \bar{c} \to -\infty,$$

thereby contradicting the relation (11.25). Thus,

$$\langle \gamma, x \rangle + \gamma_{n+1} c \geq 0 > \tilde{\alpha},\ \forall\ (x, c) \in cl\ \mathcal{K}. \tag{11.26}$$

As $(0, -1) \in \mathcal{K} \subset cl\ \mathcal{K}$, for $\lambda > 0$, $(0, -\lambda) \in cl\ \mathcal{K}$. Therefore, (11.26) leads to

$$-\lambda \gamma_{n+1} \geq 0,\ \forall\ \lambda > 0,$$

which implies that $\gamma_{n+1} \leq 0$. We now consider the following two cases.

(i) $\gamma_{n+1} = 0$: The condition (11.26) reduces to

$$\langle \gamma, x \rangle \geq 0 > \langle \gamma, \tilde{x} \rangle,\ \forall\ (x, c) \in cl\ \mathcal{K}.$$

In particular, for $(x_i, c_i) \in cl\ \mathcal{K},\ i \in I'$,

$$\langle \gamma, x_i \rangle \geq 0 > \langle \gamma, \tilde{x} \rangle,\ \forall\ i \in I'. \tag{11.27}$$

As $(LSIS)$ Θ is consistent, there exists $\bar{x} \in \mathbb{R}^n$ such that

$$\langle \bar{x}, x_i \rangle \geq c_i, \ \forall \ i \in I. \tag{11.28}$$

Therefore, from the inequalities (11.27) and (11.28), for any $\lambda > 0$,

$$\langle \bar{x} + \lambda\gamma, x_i \rangle \geq c_i, \ \forall \ i \in I,$$

which implies $\bar{x} + \lambda\gamma$ is a solution of $(LSIS)$ Θ. By our supposition, $\langle \tilde{x}, x \rangle \geq \tilde{c}$ is a consequence relation of Θ, which implies that

$$\langle \tilde{x}, \bar{x} + \lambda\gamma \rangle \geq \tilde{c}. \tag{11.29}$$

By the condition (11.27), as the limit $\lambda \to +\infty$,

$$\langle \bar{x} + \lambda\gamma, \tilde{x} \rangle \to -\infty,$$

thereby contradicting the inequality (11.29).

(ii) $\gamma_{n+1} < 0$: As $\gamma_{n+1} \neq 0$, dividing (11.26) throughout by $-\gamma_{n+1}$ and setting $\bar{x} = \dfrac{-\gamma}{\gamma_{n+1}}$,

$$\langle \bar{x}, x \rangle - c \geq 0 > \langle \bar{x}, \tilde{x} \rangle - \tilde{c}, \ \forall \ (x, c) \in cl\ \mathcal{K}.$$

The above condition holds in particular for $(x_i, c_i) \in \mathcal{K} \subset cl\ \mathcal{K}, \ i \in I$. Thus,

$$\langle \bar{x}, x_i \rangle - c_i \geq 0 > \langle \bar{x}, \tilde{x} \rangle - \tilde{c}, \ \forall \ i \in I,$$

that is,

$$\langle \bar{x}, x_i \rangle \geq c_i, \ i \in I \quad \text{and} \quad \langle \bar{x}, \tilde{x} \rangle < \tilde{c}.$$

Therefore, $\bar{x}$ is a solution of $(LSIS)$ Θ but does not satisfy the consequence relation $\langle \tilde{x}, x \rangle \geq \tilde{c}$, which is again a contradiction.

Hence, our assumption was wrong and $(\tilde{x}, \tilde{c}) \in cl\ \mathcal{K}$, thereby completing the proof. □

Next we present the characterization of the FM system Θ from Goberna, López, and Pastor [51].

Proposition 11.10 *$(LSIS)$ Θ is an FM system if and only if*

$$(\tilde{x}, \tilde{c}) \in cone\ co\ \{(x_i, c_i), \ i \in I; \ (0, -1)\}.$$

Proof. Suppose that

$$(\tilde{x}, \tilde{c}) \in cone\ co\ \{(x_i, c_i), \ i \in I; \ (0, -1)\}.$$

As in the proof of Proposition 11.9, consider $i' \notin I$. Define $(x_{i'}, c_{i'}) = (0, -1)$ and $I' = \{i'\} \cup I$. Therefore,

$$(\tilde{x}, \tilde{c}) \in cone\ co\ \{(x_i, c_i),\ i \in I'\} \subset \mathbb{R}^{n+1},$$

which by the Carathéodory Theorem, Theorem 2.8, implies that there exist $\lambda_j \geq 0$ and $i_j \in I',\ j = 1, \ldots, s$, with $1 \leq s \leq n+2$ such that

$$(\tilde{x}, \tilde{c}) = \sum_{j=1}^{s} \lambda_j (x_{i_j}, c_{i_j}).$$

Invoking Proposition 11.9 to the finite system, $\langle \tilde{x}, x \rangle \geq \tilde{c}$ is a consequence relation of the finite system

$$\langle x_{i_j}, x \rangle \geq c_{i_j},\ j = 1, 2, \ldots, s.$$

Therefore, $(LSIS)$ Θ is an FM system.

Conversely, suppose that Θ is an FM system, which implies that a consequence relation $\langle \tilde{x}, x \rangle \geq \tilde{c}$ of the infinite system

$$\langle x_i, x \rangle \geq c_i,\ i \in I$$

can be expressed as a consequence of finite subsystem, that is, $\langle \tilde{x}, x \rangle \geq \tilde{c}$ is a consequence relation of a finite subsystem

$$\langle x_i, x \rangle \geq c_i,\ i = 1, 2, \ldots, s.$$

Applying Proposition 11.9 to the above finite system,

$$\begin{aligned} (\tilde{x}, \tilde{c}) &\in cl\ cone\ co\ \{(x_i, c_i),\ i = 1, 2, \ldots, s; (0, -1)\} \\ &= cone\ co\ \{(x_i, c_i),\ i = 1, 2, \ldots, s; (0, -1)\} \end{aligned}$$

which implies that

$$(\tilde{x}, \tilde{c}) \in cone\ co\ \{(x_i, c_i),\ i \in I;\ (0, -1)\},$$

thereby establishing the desired result. $\square$

Now we introduce the Farkas–Minkowski qualification for (SIP) from Goberna, López, and Pastor [51].

Definition 11.11 The convex semi-infinite programming problem (SIP) is said to satisfy the *Farkas–Minkowski (FM) qualification* if $(LSIS)$

$$\Theta = \{g(y, i) + \langle \xi, x - y \rangle \leq 0 : (y, i) \in \mathbb{R}^n \times I,\ \xi \in \partial g(y, i)\}$$

is an FM system.

Using the FM qualification, we present the KKT optimality condition for (SIP) from Goberna, López, and Pastor [51].

Theorem 11.12 *Consider the convex semi-infinite programming problem (SIP). Assume that the FM qualification holds. Then $\bar{x} \in \mathbb{R}^n$ is a point of minimizer of (SIP) if and only if there exists $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$ such that*

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g_i(\bar{x}).$$

Proof. Define the solution set of $(LSIS)$ Θ by $\widetilde{C}$, that is,

$$\widetilde{C} = \{x \in \mathbb{R}^n : g(y,i) + \langle \xi, x - y \rangle \leq 0,\ \forall\ (y,i) \in \mathbb{R}^n \times I,\ \forall\ \xi \in \partial g(y,i)\}.$$

Note that as $dom\ g(.,i) = \mathbb{R}^n,\ i \in I$, by Proposition 2.83, $\partial g(y,i)$ is nonempty for every $y \in \mathbb{R}^n$ and hence $\widetilde{C}$ is defined. We claim that $C_I = \widetilde{C}$.

Suppose that $\tilde{x} \in C_I$, which implies that $g(\tilde{x},i) \leq 0,\ i \in I$. For any $y \in \mathbb{R}^n$ and $\xi_i \in \partial g(y,i)$ for $i \in I$, by Definition 2.77 of the subdifferential,

$$g(y,i) + \langle \xi_i, x - y \rangle \leq g(x,i),\ \forall\ x \in \mathbb{R}^n.$$

In particular, for $x = \tilde{x}$, the above inequality becomes

$$g(y,i) + \langle \xi_i, \tilde{x} - y \rangle \leq g(\tilde{x},i) \leq 0,\ \forall\ i \in I,$$

which leads to $\tilde{x} \in \widetilde{C}$. Because $\tilde{x} \in C_I$ was arbitrary, $C_I \subset \widetilde{C}$.

Conversely, suppose that $\tilde{x} \in \widetilde{C}$, which implies that for every $y \in \mathbb{R}^n$ and $i \in I$,

$$g(y,i) + \langle \xi, \tilde{x} - y \rangle \leq 0,\ \forall\ \xi \in \partial g(y,i).$$

In particular, taking $y = \tilde{x}$, the above condition reduces to

$$g(\tilde{x},i) \leq 0,\ i \in I,$$

thereby implying that $\tilde{x} \in C_I$. Because $\tilde{x} \in \widetilde{C}$ was arbitrary, $\widetilde{C} \subset C_I$. Hence $C_I = \widetilde{C}$ and thus the FM system Θ is a linearization of C_I.

As $\bar{x}$ is a point of minimizer of (SIP), it is also the point of minimizer of the equivalent problem

$$\inf\ f(x) \quad \text{subject to} \quad x \in C_I.$$

Because $dom\ f = \mathbb{R}^n$, by Theorem 3.1,

$$0 \in \partial f(\bar{x}) + N_{C_I}(\bar{x}),$$

which implies that there exists $\xi \in \partial f(\bar{x})$ such that $-\xi \in N_{C_I}(\bar{x})$. By Definition 2.36 of the normal cone,

$$\langle \xi, x - \bar{x} \rangle \geq 0,\ \forall\ x \in C_I,$$

and thus it is a consequence relation of $(LSIS)$ Θ. As Θ is an FM system, by Proposition 11.10, there exist $\lambda_j \geq 0,\ \xi_j \in \partial g(\bar{x}, i_j),\ i_j \in I,\ j = 1, 2, \ldots, s$, and $\mu \geq 0$ such that

$$(\xi, \langle \xi, \bar{x} \rangle) = \sum_{j=1}^{s} \lambda_j(-\xi_j, g(y_j, i_j) - \langle \xi_j, y_j \rangle) + (0, -\mu).$$

Without loss of generality, assume that $\lambda_j > 0,\ j = 1, 2, \ldots, s$. Now multiplying the above condition throughout by $(-x, 1)$ leads to

$$\langle \xi, \bar{x} - x \rangle = \sum_{j=1}^{s} \lambda_j(g(y_j, i_j) + \langle \xi_j, x - y_j \rangle) - \mu.$$

As $\mu \geq 0$, the above relation leads to

$$\langle \xi, \bar{x} - x \rangle \leq \sum_{j=1}^{s} \lambda_j(g(y_j, i_j) + \langle \xi_j, x - y_j \rangle). \tag{11.30}$$

As $\xi_j \in \partial g(\bar{x}, i_j),\ j = 1, 2, \ldots, s$,

$$g(y_j, i_j) + \langle \xi_j, x - y_j \rangle \leq g(x, i_j),\ \forall\, x \in \mathbb{R}^n. \tag{11.31}$$

Also, because $\xi \in \partial f(\bar{x})$,

$$f(\bar{x}) - f(x) \leq \langle \xi, \bar{x} - x \rangle,\ \forall\, x \in \mathbb{R}^n. \tag{11.32}$$

Combining the conditions (11.30), (11.31) and (11.32), yields that

$$f(\bar{x}) \leq f(x) + \sum_{j=1}^{s} \lambda_j g(x, i_j),\ \forall\, x \in \mathbb{R}^n. \tag{11.33}$$

In particular, taking $x = \bar{x}$ in the above inequality, which along with the feasibility of $\bar{x}$ leads to

$$0 \leq \sum_{j=1}^{s} \lambda_j g(\bar{x}, i_j) \leq 0,$$

that is, $\sum_{j=1}^{s} \lambda_j g(\bar{x}, i_j) = 0$. Thus,

$$\lambda_j g(\bar{x}, i_j) = 0,\ \forall\, j = 1, 2, \ldots, s.$$

By our supposition, $\lambda_j > 0,\ j = 1, 2, \ldots, s$ which implies that $g(\bar{x}, i_j) = 0$, that is, $i_j \in I(\bar{x}),\ j = 1, 2, \ldots, s$. Define $\lambda_i = 0$ for $i \in I(\bar{x})$ and $i \notin \{i_1, i_2, \ldots, i_s\}$.

Therefore, from the inequality (11.33), $\bar{x}$ is the minimizer of the unconstrained problem

$$\inf \; f(x) + \sum_{i \in supp\ \lambda} \lambda_i g(x,i) \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

where $\lambda \in \mathbb{R}_+^{[I(\bar{x})]}$. By the optimality condition for the unconstrained problem, Theorem 2.89,

$$0 \in \partial(f + \sum_{i \in supp\ \lambda} \lambda_i g(.,i))(\bar{x}).$$

As $dom\ f = dom\ g(.,i) = \mathbb{R}^n$ for $i \in I(\bar{x})$, by the Sum Rule, Theorem 2.91,

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x},i),$$

thereby establishing the KKT optimality conditions for (SIP). The converse can be worked out along the lines of Theorem 11.2. $\square$

Another notion that implies $(LSIS)$ Θ is an FM system is that of the canonically closed system. Below we define this concept and a result relating a canonically closed system and an FM system from Goberna, López, and Pastor [51].

Definition 11.13 $(LSIS)$ $\Theta = \{\langle x_i, x\rangle \geq c_i,\ i \in I\}$ is *canonically closed* if the following conditions hold:

(i) There exists $\hat{x} \in \mathbb{R}^n$ such that $\langle x_i, \hat{x}\rangle > c_i,\ i \in I$.

(ii) The set $\{(x_i, c_i),\ i \in I\}$ is compact.

The following result provides different conditions under which Θ is an FM system, part of the proof is due to Hestenes [55].

Proposition 11.14 *If the consistent* $(LSIS)$ Θ *satisfies one of the following conditions, then it is an FM system:*

(i) cone co $\{(x_i, c_i),\ i \in I;\ (0,-1)\})$ *is closed.*

(ii) cone co $\{(x_i, c_i),\ i \in I\}$ *is closed.*

(iii) $(LSIS)$ Θ *is canonically closed.*

Proof. (i) Suppose that *cone co* $\{(x_i, c_i),\ i \in I;\ (0,-1)\}$ is closed. Then by Proposition 11.9, $\langle \tilde{x}, x\rangle \geq \tilde{c}$ is the consequence relation of $(LSIS)$ Θ if and only if

$$(\tilde{x}, \tilde{c}) \in cone\ co\ \{(x_i, c_i),\ i \in I;\ (0,-1)\},$$

which by Proposition 11.10 is equivalent to Θ being an FM system.

(ii) Define

$$\mathcal{K} = cone\ co\ \{(x_i, c_i),\ i \in I;\ (0, -1)\} \quad \text{and} \quad \widetilde{\mathcal{K}} = cone\ co\ \{(x_i, c_i),\ i \in I\}.$$

It is easy to observe that

$$\mathcal{K} = \widetilde{\mathcal{K}} + cone\ (0, -1). \tag{11.34}$$

Suppose that $\widetilde{\mathcal{K}}$ is closed. We claim that $\mathcal{K}$ is closed, which by (i) will then imply that Θ is an FM system. Consider a bounded sequence $\{(\tilde{x}_k, \tilde{c}_k)\} \subset \mathcal{K}$ such that $(\tilde{x}_k, \tilde{c}_k) \to (\tilde{x}, \tilde{c})$. Note that $(\tilde{x}_k, \tilde{c}_k)$ for $k \in \mathbb{N}$ can be expressed as

$$(\tilde{x}_k, \tilde{c}_k) = (x_k, c_k) + \lambda_k(0, -1),\ \forall\ k \in \mathbb{N}, \tag{11.35}$$

where $\{(x_k, c_k)\} \subset \widetilde{\mathcal{K}}$ and $\{\lambda_k\} \subset \mathbb{R}_+$. Assume that $\{\lambda_k\}$ is an unbounded sequence, which implies $\lambda_k \to +\infty$. From the condition (11.35)

$$\frac{1}{\lambda_k}(\tilde{x}_k, \tilde{c}_k) = \frac{1}{\lambda_k}(x_k, c_k) + (0, -1),\ \forall\ k \in \mathbb{N}.$$

As $(\tilde{x}_k, \tilde{c}_k) \to (\tilde{x}, \tilde{c})$, taking the limit as $k \to +\infty$ in the above condition implies that

$$\frac{1}{\lambda_k}(x_k, c_k) \to (0, 1),$$

that is, $(0, 1) \in cl\ \widetilde{\mathcal{K}} \subset cl\ \mathcal{K}$. By Proposition 11.9,

$$0 = \langle 0, x \rangle \geq 1$$

is a consequent relation of $(LSIS)$ Θ, which is impossible. Thus, $\{\lambda_k\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, assume that $\lambda_k \to \lambda$. By the condition (11.35) and boundedness of the sequences $\{(\tilde{x}_k, \tilde{c}_k)\}$ and $\{\lambda_k\}$, $\{(x_k, c_k)\}$ is also a bounded sequence. Without loss of generality, by the Bolzano–Weierstrass Theorem, let $(x_k, c_k) \to (x, c)$. As $\widetilde{\mathcal{K}}$ is closed, $(x, c) \in \widetilde{\mathcal{K}}$. Therefore, taking the limit as $k \to +\infty$, (11.35) along with (11.34) yields that

$$(\tilde{x}, \tilde{c}) = (x, c) + \lambda(0, -1) \in \mathcal{K},$$

and hence $\mathcal{K}$ is closed, which by (i) implies that $(LSIS)$ Θ is an FM system.

(iii) Suppose that Θ is a canonically closed system. Therefore, the set $\{(x_i, c_i),\ i \in I\}$ is compact. We claim that

$$\widetilde{\mathcal{K}} = cone\ co\ \{(x_i, c_i),\ i \in I\} \subset \mathbb{R}^{n+1}$$

is closed. On the contrary, assume that it is not closed, which implies there exists a convergent sequence $\{(\tilde{x}_k, \tilde{c}_k)\} \subset \widetilde{\mathcal{K}}$ such that $(\tilde{x}_k, \tilde{c}_k) \to (\tilde{x}, \tilde{c})$ with $(\tilde{x}, \tilde{c}) \in cl\, \widetilde{\mathcal{K}}$ but $(\tilde{x}, \tilde{c}) \notin \widetilde{\mathcal{K}}$. Because $(\tilde{x}_k, \tilde{c}_k) \in \widetilde{\mathcal{K}}$, there exist $\lambda_{i_{k_j}} \geq 0$, $i_{k_j} \in I$ for $j = 1, 2, \ldots, s_k$, $\{s_k\} \subset \mathbb{N}$ such that

$$(\tilde{x}_k, \tilde{c}_k) = \sum_{j=1}^{s_k} \lambda_{i_{k_j}} (x_{i_{k_j}}, c_{i_{k_j}}).$$

By the Carathéodory Theorem, Theorem 2.8, $1 \leq s_k \leq n+2$. For $s_k < n+2$, choose any $i_{k_j} \in I$ with $\lambda_{i_{k_j}} = 0$, $j = s_k+1, s_k+2, \ldots, n+2$. Therefore the above condition can be rewritten as

$$(\tilde{x}_k, \tilde{c}_k) = \sum_{j=1}^{n+2} \lambda_{i_{k_j}} (x_{i_{k_j}}, c_{i_{k_j}}). \tag{11.36}$$

As $\{(x_i, c_i),\ i \in I\}$ is compact, by the Bolzano–Weierstrass Theorem, $\{(x_{i_{k_j}}, c_{i_{k_j}})\}$ has a convergent subsequence. Without loss of generality, assume that $(x_{i_{k_j}}, c_{i_{k_j}}) \to (x_{i_j}, c_{i_j}) \in \{(x_i, c_i),\ i \in I\}$. By assumption, $(\tilde{x}, \tilde{c}) \notin \widetilde{\mathcal{K}}$, which implies $\{\lambda_{i_{k_j}}\}$ is unbounded. Otherwise, there exists some convergent subsequence of $\{\lambda_{i_{k_j}}\}$. Without relabeling, assume that $\lambda_{i_{k_j}} \to \lambda_{i_j}$. Therefore, taking the limit as $k \to +\infty$ in (11.36) leads to

$$(\tilde{x}, \tilde{c}) = \sum_{j=1}^{n+2} \lambda_{i_j} (x_{i_j}, c_{i_j}) \in \widetilde{\mathcal{K}},$$

which is a contradiction of our assumption.

Denote $\lambda_k = \sum_{j=1}^{n+2} \lambda_{i_{k_j}}$. Observe that the sequence $\{\dfrac{\lambda_{i_{k_j}}}{\lambda_k}\} \subset \mathbb{R}_+$ is a bounded sequence and hence by the Bolzano–Weierstrass Theorem has a convergent subsequence. Without loss of generality, assume that $\dfrac{\lambda_{i_{k_j}}}{\lambda_k} \to \lambda_{i_j} \geq 0$, $j = 1, 2, \ldots, n+2$, with $\sum_{j=1}^{n+2} \lambda_{i_j} = 1$. Dividing the condition (11.36) throughout by λ_k and taking the limit as $k \to +\infty$, which along with the fact that $\lambda_k \to +\infty$ yields

$$\sum_{j=1}^{n+2} \lambda_j (x_{i_j}, c_{i_j}) = 0 \quad \text{with} \quad \sum_{j=1}^{n+2} \lambda_{i_j} = 1. \tag{11.37}$$

As $(LSIS)$ Θ is canonically closed, there exists $\hat{x} \in \mathbb{R}^n$ such that $\langle x_i, \hat{x} \rangle > c_i$, $i \in I$, that is,

$$\langle (x_i, c_i), (\hat{x}, -1) \rangle = \langle x_i, \hat{x} \rangle - c_i > 0,\ \forall\, i \in I. \tag{11.38}$$

Combining the relations (11.37) and (11.38) along with the fact that $\lambda_{i_j} \geq 0$, $j = 1, 2, \ldots, n+2$, not all simultaneously zero, implies that

$$0 = \sum_{j=1}^{n+2} \lambda_{i_j} \langle (x_{i_j}, c_{i_j}), (\hat{x}, -1) \rangle = \sum_{j=1}^{n+2} \lambda_{i_j} (\langle x_{i_j}, \hat{x} \rangle - c_{i_j}) > 0,$$

which is impossible. Thus our assumption was wrong and hence $\widetilde{\mathcal{K}}$ is closed, which by (ii) yields that Θ is an FM system. $\square$

As seen in Section 11.4, the Slater constraint qualification for (SIP) implies that every feasible point is a Lagrangian regular point; we will now present the relation between the Slater constraint qualification for (SIP) and the FM qualification. For this we will need the following result from Goberna, López, and Pastor [51].

Proposition 11.15 *Consider a closed convex set $F \subset \mathbb{R}^n$ and let F_b denote the boundary points of F. Also consider $(LSIS)$*

$$\Theta = \{\langle x_i, x\rangle \leq c_i,\ i \in I\}$$

such that

(i) every point of F is a solution of the system Θ,

(ii) there exists $\hat{x} \in F$ such that $\langle x_i, x\rangle < c_i,\ i \in I$,

(iii) given any $y \in F_b$, there exists some $i \in I$ such that $\langle x_i, y\rangle = c_i$.

Then F is the solution set of Θ, that is,

$$F = \{x \in \mathbb{R}^n : \langle x_i, x\rangle \leq c_i,\ i \in I\}.$$

Proof. Observe that by condition (i), $F \subset \{x \in \mathbb{R}^n : \langle x_i, x\rangle \leq c_i,\ i \in I\}$. Conversely, suppose that there exists

$$z \in \{x \in \mathbb{R}^n : \langle x_i, x\rangle \leq c_i,\ i \in I\}$$

and $z \notin F$. By (ii), there exists $\hat{x} \in F$ such that $\langle x_i, x\rangle < c_i$ for every $i \in I$. As F is a closed convex set, the line segment joining $\hat{x}$ and z meets the boundary F_b at only one point, say $y \in (\hat{x}, z)$. Therefore, there exists $\lambda \in (0,1)$ such that $y = (1-\lambda)\hat{x} + \lambda z \in F_b$. By condition (iii), there exists $\bar{i} \in I$ such that

$$\langle x_{\bar{i}}, y\rangle = c_{\bar{i}}. \tag{11.39}$$

By the conditions on $\hat{x}$ and z,

$$\langle x_{\bar{i}}, \hat{x}\rangle < c_{\bar{i}} \quad \text{and} \quad \langle x_{\bar{i}}, z\rangle \leq c_{\bar{i}},$$

respectively. Thus

$$\langle x_{\bar{i}}, y\rangle = (1-\lambda)\langle x_{\bar{i}}, \hat{x}\rangle + \lambda\langle x_{\bar{i}}, z\rangle < c_{\bar{i}},$$

which is a contradiction to (11.39). Hence, $F \supset \{x \in \mathbb{R}^n : \langle x_i, x\rangle \leq c_i,\ i \in I\}$, thereby establishing the result. $\square$

Now we are in a position to present the implication that the Slater constraint qualification for (SIP) leads to the FM qualification from López and Vercher [75].

Proposition 11.16 *Consider the convex semi-infinite programming problem* (SIP) *with bounded feasible set* C_I*. If the Slater constraint qualification for* (SIP) *holds, then the FM qualification condition is also satisfied.*

Proof. Define $g(x) = \sup_{i \in I} g(x,i)$ and $C_I^b = \{x \in C_I : g(x) = 0\}$. Consider the $(LSIS)$

$$\widetilde{\Theta} = \{\langle \xi, x \rangle \leq \langle \xi, y \rangle,\ y \in C_I^b,\ \xi \in \partial g(y)\}.$$

We claim that $\widetilde{\Theta}$ is a linear representation of C_I. For any $\xi \in \partial g(y)$, by Definition 2.77 of the subdifferential,

$$\langle \xi, x - y \rangle \leq g(x) - g(y),\ \forall\, x \in \mathbb{R}^n. \tag{11.40}$$

As the Slater constraint qualification for (SIP) holds, by condition (i) and (ii), the supremum $g(x)$ is attained. Therefore, in particular, for $y \in C_I^b$ and $x \in C_I$, that is, $g(y) = 0$ and $g(x) = \sup_{i \in I} g(x,i) \leq 0$, respectively, the above inequality reduces to

$$\langle \xi, x \rangle \leq \langle \xi, y \rangle,\ \forall\, \xi \in \partial g(y).$$

Because $x \in C_I$ was arbitrary, every point of C_I is a solution of $\widetilde{\Theta}$.

By condition (iii) of the Slater constraint qualification for (SIP), there exists $\hat{x} \in \mathbb{R}^n$ such that

$$g(\hat{x}, i) < 0,\ \forall\, i \in I,$$

that is, $\hat{x} \in C_I$. By the conditions (i) and (ii) of the Slater constraint qualification for (SIP), $g(\hat{x}) < 0$. Therefore, in particular, taking $y \in C_I^b$ and $x = \hat{x}$, the condition (11.40) becomes

$$\langle \xi, \hat{x} - y \rangle \leq g(\hat{x}) < 0,\ \forall\, \xi \in \partial g(y) \tag{11.41}$$

for every $y \in C_I^b$. Also, in particular, taking $y = \bar{y} \in C_I^b$ and $x = \bar{y}$ in the inequality (11.40), the relation holds with equality.

From the above discussion, it is obvious that the conditions of Proposition 11.15 are satisfied and thus, C_I coincides with the solution set of $(LSIS)$ $\widetilde{\Theta}$, that is,

$$C_I = \{x \in \mathbb{R}^n : \langle \xi, x \rangle \leq \langle \xi, y \rangle,\ \forall\, y \in C_I^b,\ \forall\, \xi \in \partial g(y)\}. \tag{11.42}$$

We now show that $\widetilde{\Theta}$ is canonically closed and hence is an FM system.

By the condition (11.41),

$$\langle \xi, \hat{x} \rangle < \langle \xi, y \rangle,\ \forall\, y \in C_I^b,\ \forall\, \xi \in \partial g(y)$$

and thus, the condition of (ii) of Definition 11.13 is satisfied. Therefore, for $\widetilde{\Theta}$ to be a canonically closed system, we need to show that the set

$$\widetilde{\mathcal{K}} = \{(\xi, \langle \xi, y \rangle),\ y \in C_I^b,\ \xi \in \partial g(y)\}$$

is compact.

As C_I is bounded and $C_I^b \subset C_I$, therefore C_I^b is bounded. Also by condition (i) and (ii) of the Slater constraint qualification for (SIP), the supremum is attained over I. Therefore, as $dom\ g(.,i) = \mathbb{R}^n,\ i \in I$, $dom\ g = \mathbb{R}^n$, which by Theorem 2.69 is continuous over $\mathbb{R}^n$. Thus, for a sequence $\{y_k\} \subset C_I^b$ with $y_k \to \bar{y}$, $g(y_k) \to g(\bar{y})$. Also, as $g(y_k) = 0$ for every $k \in \mathbb{N}$, $g(\bar{y}) = 0$, which implies $\bar{y} \in C_I^b$. Hence, C_I^b is closed, thereby yielding the compactness of C_I^b. By Proposition 2.85,

$$\partial g(C_I^b) = \{\xi \in \mathbb{R}^n : \xi \in \partial g(y),\ y \in C_I^b\} = \bigcup_{y \in C_I^b} \partial g(y)$$

is compact.

Now consider a convergent sequence $\{(\xi_k, \langle \xi_k, y_k \rangle)\} \subset \widetilde{\mathcal{K}}$ where $\{y_k\} \subset C_I^b$ and $\xi_k \in \partial g(y_k) \subset \partial g(C_I^b)$. Suppose that $(\xi_k, \langle \xi_k, y_k \rangle) \to (\tilde{\xi}, \tilde{\gamma})$, that is $\xi_k \to \tilde{\xi}$ and $\langle \xi_k, y_k \rangle \to \tilde{\gamma}$. As $\xi_k \to \tilde{\xi}$, which by the compactness of $\partial g(C_I^b)$ implies that $\tilde{\xi} \in \partial g(C_I^b)$. Because $\{y_k\} \subset C_I^b$, $\{y_k\}$ is a bounded sequence. By the Bolzano–Weierstrass Theorem, Proposition 1.3, it has a convergent subsequence. Without loss of generality, assume that $y_k \to \tilde{y}$, which by compactness of C_I^b leads to $\tilde{y} \in C_I^b$. As $\langle \xi_k, y_k \rangle \to \gamma$, which by the convergence of $\{\xi_k\}$ and $\{y_k\}$ implies that $\tilde{\gamma} = \langle \tilde{\xi}, \tilde{y} \rangle$. Because $\xi_k \in \partial g(y_k)$ with $\xi_k \to \tilde{\xi}$ and $y_k \to \tilde{y}$, by the Closed Graph Theorem, Theorem 2.84, $\tilde{\xi} \in \partial g(\tilde{y})$. Thus $(\tilde{\xi}, \langle \tilde{\xi}, \tilde{y} \rangle) \in \widetilde{\mathcal{K}}$, thereby yielding the closedness of $\widetilde{\mathcal{K}}$.

By the compactness of C_I^b and $\partial g(C_I^b)$, $\|y\| \leq M_1$ for every $y \in C_I^b$ and $\|\xi\| \leq M_2$ for every $\xi \in \partial g(C_I^b)$, respectively. Therefore, for any $(\xi, \langle \xi, y \rangle) \in \widetilde{\mathcal{K}}$ along with the Cauchy–Schwarz inequality, Proposition 1.1,

$$\|\xi\|^2 + |\langle \xi, y \rangle| \leq \|\xi\|^2 + \|\xi\|\ \|y\| \leq M_1(M_1 + M_2)$$

and hence $\widetilde{\mathcal{K}}$ is bounded. Therefore, $\widetilde{\mathcal{K}}$ is compact, thus implying that the system $\widetilde{\Theta}$ is canonically closed, which by Proposition 11.14 yields that $\widetilde{\Theta}$ is an FM system.

Next we claim that $(LSIS)$

$$\Theta = \{\langle \xi, x - y \rangle \leq -g(y,i),\ (y,i) \in \mathbb{R}^n \times I,\ \xi \in \partial g(y,i)\}$$

is equivalent to $\widetilde{\Theta}$, that is, both Θ and $\widetilde{\Theta}$ have the same solution set. To establish this claim, we will prove that C_I is the solution set of Θ.

For any $(y,i) \in \mathbb{R}^n \times I$ and $\xi_i \in \partial g(y,i)$, by Definition 2.77 of the subdifferential,

$$\langle \xi_i, x - y \rangle \leq g(x,i) - g(y,i),\ \forall\ x \in \mathbb{R}^n. \tag{11.43}$$

In particular, taking $x \in C_I$, that is,

$$g(x,i) \leq 0,\ \forall\ i \in I,$$

the inequality (11.43) reduces to

$$\langle \xi_i, x - y\rangle \leq -g(y,i),\ \forall\ (y,i) \in \mathbb{R}^n \times I,\ \forall\ \xi_i \in \partial g(y,i),$$

which implies x is a solution of $(LSIS)$ Θ. Because $x \in C_I$ was arbitrary, every point of C_I is a solution of Θ.

By condition (iii) of the Slater constraint qualification, there exists $\hat{x} \in \mathbb{R}^n$ such that

$$g(\hat{x}, i) < 0,\ \forall\ i \in I.$$

In particular, taking $x = \hat{x}$ in (11.43) yields that for every $(y,i) \in \mathbb{R}^n \times I$,

$$\langle \xi_i, \hat{x} - y\rangle \leq g(\hat{x}, i) - g(y,i) < -g(y,i),\ \forall\ \xi_i \in \partial g(y,i).$$

Also, taking $y = \tilde{y} \in C_I^b$, where

$$C_I^b = \{y \in \mathbb{R}^n : \text{there exists some } i \in I \text{ such that } g(y,i) = 0\},$$

along with $x = \tilde{y}$ and $\tilde{i} \in I(\tilde{y})$ in the condition (11.43) leads to

$$\langle \tilde{\xi}_i, \tilde{y} - \tilde{y}\rangle = 0 = -g(\tilde{y}, \tilde{i}),\ \forall\ \xi_i \in \partial g(\tilde{y}, \tilde{i}).$$

As the conditions of Proposition 11.15 are satisfied,

$$C_I = \{x \in \mathbb{R}^n : \langle \xi, x - y\rangle \leq -g(y,i),\ \forall\ (y,i) \in \mathbb{R}^n \times I,\ \forall\ \xi \in \partial g(y,i)\}, \tag{11.44}$$

that is, C_I is a solution set of $(LSIS)$ Θ.

From the conditions (11.42) and (11.44), both $\widetilde{\Theta}$ and Θ are equivalent $(LSIS)$. Because $\widetilde{\Theta}$ is an FM system, Θ is also an FM system, which along with Definition 11.11 yields that (SIP) satisfies the FM qualification condition, thereby establishing the requisite result. □

11.6 Noncompact Scenario: An Alternate Approach

In this section we discuss the recent epigraphical approach, or more precisely the sequential approach studied in Chapter 7 as a tool to establish the KKT optimality conditions for (SIP). This approach has been studied for convex programming problems with infinite constraints by Jeyakumar [66, 67] and Goberna, Jeyakumar, and López [49]. Here we will present the KKT optimality conditions for (SIP) from the work of Dinh, Mordukhovich, and Nghia [33] under the following relaxed closed cone constraint qualification for (SIP), that is,

$$cone\ co \bigcup_{i \in I} epi\ g(.,i)^* \quad \text{is closed.}$$

But before establishing the optimality conditions for (SIP) as a consequence of the optimality conditions expressed in terms of the epigraph of the conjugate functions, we present a result from Jeyakumar [67].

Proposition 11.17 *Consider an lsc proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and define*

$$F = \{x \in \mathbb{R}^n : \phi(x) \leq 0\}.$$

If F is nonempty, then epi $\delta_F^* =$ cl cone co epi ϕ^*.

Proof. Suppose that F is nonempty. From the definition of the indicator function to the set F, δ_F,

$$\begin{aligned} \phi(x) \leq 0 &= \delta_F(x), \quad \text{for } x \in F, \\ \phi(x) \leq +\infty &= \delta_F(x), \quad \text{for } x \notin F. \end{aligned}$$

Therefore, $\phi(x) \leq \delta_F(x)$ for every $x \in \mathbb{R}^n$. By Proposition 2.103,

$$\delta_F^*(\xi) \leq \phi^*(\xi), \ \forall\ \xi \in \mathbb{R}^n. \tag{11.45}$$

We claim that *cl cone co epi* $\phi^* \subset$ *epi* δ_F^*. By Definition 2.101 of the conjugate function, δ_F^* is the same as the support function to the set F, that is, $\delta_F^* = \sigma_F$. By Proposition 2.102, δ_F^* is lsc, hence by Theorem 1.9, *epi* δ_F^* is closed. Also, as σ_F is a sublinear function, by Theorem 2.59 *epi* σ is a convex cone. So it is sufficient to establish that *epi* $\phi^* \subset$ *epi* δ_F^*. Consider any $(\xi, \alpha) \in$ *epi* ϕ^*, which by condition (11.45) implies that

$$\delta_F^*(\xi) \leq \phi^*(\xi) \leq \alpha.$$

Therefore, $(\xi, \alpha) \in$ *epi* δ_F^*. As $(\xi, \alpha) \in$ *epi* ϕ^* was arbitrary, *epi* $\phi^* \subset$ *epi* δ_F^*. Because *epi* δ_F^* is a closed convex cone,

$$cl\ cone\ co\ epi\ \phi^* \subset epi\ \delta_F^*. \tag{11.46}$$

To complete the proof, we will prove the converse inclusion, that is, *epi* $\delta_F^* \subset$ *cl cone co epi* ϕ^*. Suppose that $(\xi, \alpha) \notin$ *cl cone co epi* ϕ^*. As $\delta_F^* = \sigma_F$ is a sublinear function with $\delta_F^*(0) = 0$. Therefore, $(0, -1) \notin$ *epi* δ_F^*, which by the relation (11.46) implies that $(0, -1) \notin$ *cl cone co epi* ϕ^*. Define the convex set

$$\widetilde{F} = \{(1-\lambda)(\xi, \alpha) + \lambda(0, -1) \in \mathbb{R}^n \times \mathbb{R} : \lambda \in [0, 1]\}.$$

We claim that

$$\widetilde{F} \cap cl\ cone\ co\ epi\ \phi^* = \emptyset.$$

On the contrary, suppose that there exists $\tilde{\lambda} \in (0, 1)$ such that

$$(1-\tilde{\lambda})(\xi, \alpha) + \tilde{\lambda}(0, -1) \in cl\ cone\ co\ epi\ \phi^*. \tag{11.47}$$

We claim that $\{0\} \times \mathbb{R}_+ \subset cl\ cone\ co\ epi\ \phi^*$. To establish this fact, it is sufficient to show that $(0,1) \in cl\ cone\ co\ epi\ \phi^*$. On the contrary, suppose that

$$(0,1) \notin cl\ cone\ co\ epi\ \phi^*.$$

Then by the Strict Separation Theorem, Theorem 2.26 (iii), there exist $(a,\gamma) \in \mathbb{R}^n \times \mathbb{R}$ with $(a,\gamma) \neq (0,0)$ such that

$$\langle a, \xi \rangle + \gamma\alpha > \gamma, \ \forall\ (\xi,\alpha) \in cl\ cone\ co\ epi\ \phi^*. \tag{11.48}$$

As $(0,0) \in cl\ cone\ co\ epi\ \phi^*$, $\gamma < 0$. We will show that

$$\langle a, \xi \rangle + \gamma\alpha \geq 0 > \gamma, \ \forall\ (\xi,\alpha) \in cl\ cone\ co\ epi\ \phi^*.$$

On the contrary, suppose that $(\xi,\alpha) \in cl\ cone\ co\ epi\ \phi^*$ such that

$$0 > \langle a, \xi \rangle + \gamma\alpha > \gamma. \tag{11.49}$$

For any $\lambda > 0$, $\lambda(\xi,\alpha) \in cl\ cone\ co\ epi\ \phi^*$, which by the conditions (11.48) and (11.49) implies that

$$0 > \lambda(\langle a, \xi \rangle + \gamma\alpha) > \gamma.$$

Taking the limit as $\lambda \to +\infty$ in the above inequality,

$$\lambda(\langle a, \xi \rangle + \gamma\alpha) \to -\infty,$$

which is a contradiction. Therefore,

$$\langle a, \xi \rangle + \gamma\alpha \geq 0 > \gamma, \ \forall\ (\xi,\alpha) \in cl\ cone\ co\ epi\ \phi^*. \tag{11.50}$$

Consider any $\xi \in dom\ \phi^*$ and $\varepsilon > 0$. Thus, $(\xi, \phi^*(\xi) + \varepsilon) \in cl\ cone\ co\ epi\ \phi^*$. Therefore, from the condition (11.50),

$$\langle a, \xi \rangle + \gamma(\phi^*(\xi) + \varepsilon) \geq 0,$$

which implies

$$\frac{1}{\varepsilon}(\langle a, \xi \rangle + \gamma\phi^*(\xi)) + \gamma \geq 0.$$

Taking the limit as $\varepsilon \to +\infty$ in the above inequality, which along with (11.50) yields that $0 > \gamma \geq 0$, which is a contradiction. Thus, $(0,1) \in cl\ cone\ co\ epi\ \phi^*$ and hence

$$\{0\} \times \mathbb{R}_+ = cone\ (0,1) \subset cl\ cone\ co\ epi\ \phi^*. \tag{11.51}$$

From the relations (11.47) and (11.51),

$$(1-\tilde{\lambda})(\xi,\alpha) = (1-\tilde{\lambda})(\xi,\alpha) + \tilde{\lambda}(0,-1) + (0,\tilde{\lambda}) \in cl\ cone\ co\ epi\ \phi^*,$$

which implies

$$(\xi, \alpha) = \frac{1}{(1-\tilde{\lambda})}\{(1-\tilde{\lambda})(\xi, \alpha)\} \in cl\ cone\ co\ epi\ \phi^*,$$

thereby contradicting our assumption. Thus

$$\widetilde{F} \cap cl\ cone\ co\ epi\ \phi^* = \emptyset.$$

As $\widetilde{F}$ is a compact convex set and *cl cone co epi* ϕ^* is a closed convex cone, by the Strict Separation Theorem, Theorem 2.26 (iii), there exists $(a, \gamma) \in \mathbb{R}^n \times \mathbb{R}$ with $(a, \gamma) \neq (0, 0)$ such that

$$\langle a, z \rangle + \gamma\beta > \langle a, \tilde{z} \rangle + \gamma\tilde{\beta} \tag{11.52}$$

for every $(z, \beta) \in cl\ cone\ co\ \phi^*$ and $(\tilde{z}, \tilde{\beta}) \in \widetilde{F}$. As $(0, 0) \in cl\ cone\ co\ epi\ \phi^*$,

$$0 > \langle a, z \rangle + \gamma\beta,\ \forall\ (z, \beta) \in \widetilde{F}.$$

Also, as $(0, -1),\ (\xi, \alpha) \in \widetilde{F}$, from condition (11.52),

$$\gamma > 0 \quad \text{and} \quad \langle a, \xi \rangle + \gamma\alpha < 0. \tag{11.53}$$

Repeating the discussion as before, we can show that

$$\langle a, z \rangle + \gamma\beta \geq 0 > \langle a, \tilde{z} \rangle + \gamma\tilde{\beta}$$

for every $(z, \beta) \in cl\ cone\ co\ \phi^*$ and $(\tilde{z}, \tilde{\beta}) \in \widetilde{F}$. For any $\xi \in dom\ \phi^*$, $(\xi, \phi^*(\xi)) \in cl\ cone\ co\ epi\ \phi^*$, which by the above inequality implies that

$$\langle a, \xi \rangle + \gamma\phi^*(\xi) \geq 0,\ \forall\ \xi \in dom\ \phi^*. \tag{11.54}$$

Because ϕ is lsc, by Theorem 2.105, $\phi = \phi^{**}$. Therefore, by the conditions (11.53) and (11.54),

$$\phi(\frac{-a}{\gamma}) = \phi^{**}(\frac{-a}{\gamma}) = \sup_{\xi \in \mathbb{R}^n} \{\langle \xi, \frac{-a}{\gamma} \rangle - \phi^*(\xi)\} \leq 0,$$

which implies that $\dfrac{-a}{\gamma} \in F$. Again using the condition (11.53),

$$\delta^*_F(\frac{-a}{\gamma}) = \sigma_F(\frac{-a}{\gamma}) \geq \langle \xi, \frac{-a}{\gamma} \rangle > \alpha,$$

which implies $(\xi, \alpha) \notin epi\ \delta^*_F$, thereby establishing the desired result. $\square$

Now we move on to derive the optimality conditions in epigraphical form. Similar results have been studied in the form of generalized Farkas' Lemma in Dinh, Goberna, and López [31] and Dinh, Goberna, López, and Son [32].

Theorem 11.18 *Consider the convex semi-infinite programming problem* (SIP)*. Then* $\bar{x}$ *is a point of minimizer of* (SIP) *if and only if*

$$(0, -f(\bar{x})) \in epi\ f^* + cl\ cone\ co \bigcup_{i \in I} epi\ g^*(., i). \tag{11.55}$$

Proof. Suppose that $\bar{x}$ is a point of minimizer of (SIP) and hence of the following unconstrained problem

$$\inf\ f(x) + \delta_{C_I}(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

Therefore, by Theorem 2.89,

$$0 \in \partial(f + \delta_{C_I})(\bar{x}),$$

which by Theorem 2.108 and the fact that $\bar{x} \in C_I$ implies that

$$f(\bar{x}) + (f + \delta_{C_I})^*(0) = \langle 0, \bar{x} \rangle = 0.$$

Therefore, the above condition leads to

$$(0, -f(\bar{x})) \in epi\ (f + \delta_{C_I})^*.$$

As $dom\ f = \mathbb{R}^n$, by Theorem 2.69, f is continuous over $\mathbb{R}^n$. Thus, by Proposition 2.124,

$$(0, -f(\bar{x})) \in epi\ f^* + epi\ \delta^*_{C_I}. \tag{11.56}$$

Define the supremum function $g(x) = \sup_{i \in I} g(x, i)$, which implies that

$$C_I = \{x \in \mathbb{R}^n : g(x) \leq 0\}.$$

Because $\bar{x} \in C_I$, C_I is nonempty. Invoking Proposition 11.17, the condition (11.56) yields

$$(0, -f(\bar{x})) \in epi\ f^* + cl\ cone\ co\ epi\ g^*.$$

Applying Theorem 2.123 to the above relation leads to

$$(0, -f(\bar{x})) \in epi\ f^* + cl\ cone\ co \bigcup_{i \in I} epi\ g^*(., i),$$

thereby leading to the desired condition.

Conversely, suppose that the epigraphcal condition (11.55) is satisfied, which implies that there exist $\xi \in dom\ f^*$, $\alpha \geq 0$, $\lambda_i^k \geq 0$, $\xi_i^k \in dom\ g^*(., i)$ and $\alpha_i^k \geq 0$ for $i \in I$ such that

$$(0, -f(\bar{x})) = (\xi, f^*(\xi) + \alpha) + \lim_{k \to \infty} \sum_{i \in I} \lambda_i^k (\xi_i^k, g^*(\xi_i^k, i) + \alpha_i^k).$$

As *cone co* $\bigcup_{i\in I} epi\ g^*(.,i) \subset \mathbb{R}^{n+1}$, by the Carathéodory Theorem, Theorem 2.8, any element in the convex cone can be expressed as a sum of $n+2$ elements from $\bigcup_{i\in I} epi\ g^*(.,i)$. Therefore, the above condition becomes

$$(0,-f(\bar{x})) = (\xi, f^*(\xi)+\alpha) + \lim_{k\to\infty}\sum_{j=1}^{n+2}\lambda^k_{i_j}(\xi^k_{i_j}, g^*(\xi^k_{i_j}, i_j)+\alpha^k_{i_j}),$$

where $i_j \in I,\ j=1,2,\ldots,n+2$. Componentwise comparison leads to

$$0 = \xi + \lim_{k\to\infty}\sum_{j=1}^{n+2}\lambda^k_{i_j}\xi^k_{i_j}, \tag{11.57}$$

$$-f(\bar{x}) = f^*(\xi)+\alpha + \lim_{k\to\infty}\sum_{j=1}^{n+2}\lambda^k_{i_j}(g^*(\xi^k_{i_j}, i_j)+\alpha^k_{i_j}). \tag{11.58}$$

By Definition 2.101 of the conjugate function, condition (11.58) yields

$$\begin{aligned} f(\bar{x})-f(x) &\leq -\langle\xi,x\rangle - \alpha - \lim_{k\to\infty}\sum_{i=1}^{n+2}\lambda^k_{i_j}(g^*(\xi^k_{i_j}, i_j)+\alpha^k_{i_j}) \\ &\leq -\langle\xi,x\rangle - \alpha - \lim_{k\to\infty}\sum_{i=1}^{n+2}\lambda^k_{i_j}(\langle\xi^k_{i_j},x\rangle - g(\xi^k_{i_j}, i_j)+\alpha^k_{i_j}),\ \forall\ x\in\mathbb{R}^n. \end{aligned}$$

Using condition (11.57), for every $x \in C_I$, the above inequality leads to

$$f(\bar{x})-f(x) \leq -\alpha - \lim_{k\to\infty}\sum_{i=1}^{n+2}\lambda^k_{i_j}\alpha^k_{i_j},$$

which by the nonnegativity of α and $\alpha^k_{i_j},\ j=1,2,\ldots,n+2$, yields

$$f(\bar{x}) \leq f(x),\ \forall\ x\in C_I.$$

Thus, $\bar{x}$ is a point of minimizer of (SIP), thereby establishing the result. □

We end this chapter by presenting the KKT optimality condition for (SIP) from Dinh, Mordukhovich, and Nghia [33]. But before that we define the set of *active constraint multipliers* as

$$\mathcal{A}(\bar{x}) = \{\lambda \in \mathbb{R}^{[I]}_+ : \lambda_i g(\bar{x},i) = 0,\ \forall\ i \in supp\ \lambda\}.$$

Theorem 11.19 *Consider the convex semi-infinite programming problem (SIP). Assume that the closed cone constraint qualification holds. Then $\bar{x}$ is a point of minimizer of (SIP) if and only if there exists $\lambda \in \mathcal{A}(\bar{x})$ such that*

$$0 \in \partial f(\bar{x}) + \sum_{i\in supp\ \lambda}\lambda_i\partial g(\bar{x},i).$$

Proof. By Theorem 11.18, $\bar{x}$ is a point of minimizer of (SIP) if and only if condition (11.55) is satisfied. As the closed cone constraint qualification is satisfied, (11.55) reduces to

$$(0, -f(\bar{x})) \in epi\ f^* + cone\ co \bigcup_{i \in I} epi\ g^*(., i).$$

By Theorem 2.122, there exist $\xi \in \partial_\varepsilon f(\bar{x}),\ \varepsilon \geq 0,\ \lambda_i \geq 0,\ \xi_i \in \partial_{\varepsilon_i} g(\bar{x}, i)$ and $\varepsilon_i \geq 0,\ i \in I$ such that

$$(0, -f(\bar{x})) = (\xi, \langle \xi, \bar{x} \rangle - f(\bar{x}) + \varepsilon) + \sum_{i \in I} \lambda_i (\xi_i, \langle \xi_i, \bar{x} \rangle - g(\bar{x}, i) + \varepsilon_i).$$

Componentwise comparison leads to

$$0 = \xi + \sum_{i \in I} \lambda_i \xi_i, \tag{11.59}$$

$$-f(\bar{x}) = (\langle \xi, \bar{x} \rangle - f(\bar{x}) + \varepsilon) + \sum_{i \in I} \lambda_i (\langle \xi_i, \bar{x} \rangle - g(\bar{x}, i) + \varepsilon_i). \tag{11.60}$$

Using the condition (11.59), (11.60) reduces to

$$0 = \varepsilon + \sum_{i \in I} \lambda_i (-g(\bar{x}, i) + \varepsilon_i).$$

The above condition along with the fact that $\bar{x} \in C_I$, that is, $g(\bar{x}, i) \leq 0,\ i \in I$ and the nonnegativity of $\varepsilon,\ \varepsilon_i$ and $\lambda_i,\ i \in I$, implies that

$$\varepsilon = 0, \quad \lambda_i \varepsilon_i = 0 \quad \text{and} \quad \lambda_i g(\bar{x}, i) = 0,\ i \in I.$$

Thus, for $i \in supp\ \lambda$, $\varepsilon_i = 0$ and $\lambda \in \mathcal{A}(\bar{x})$. Therefore, $\xi \in \partial f(\bar{x})$ and $\xi_i \in \partial g(\bar{x}, i),\ i \in supp\ \lambda$ satisfying

$$0 = \xi + \sum_{i \in supp\ \lambda} \lambda_i \xi_i.$$

Therefore, for $\lambda \in \mathcal{A}(\bar{x})$,

$$0 \in \partial f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i \partial g(\bar{x}, i), \tag{11.61}$$

thereby yielding the KKT optimality condition for (SIP).

Conversely, suppose that (11.61) holds, which implies that there exist $\xi \in \partial f(\bar{x})$ and $\xi_i \in \partial g(\bar{x}, i),\ i \in supp\ \lambda$ such that

$$0 = \xi + \sum_{i \in supp\ \lambda} \xi_i. \tag{11.62}$$

By Definition 2.77 of the subdifferential, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} f(x) &\geq f(\bar{x}) + \langle \xi, x - \bar{x} \rangle, \\ g(x,i) &\geq g(\bar{x},i) + \langle \xi, x - \bar{x} \rangle, \ i \in supp\ \lambda, \end{aligned}$$

which along with the condition (11.62) implies that

$$f(x) + \sum_{i \in supp\ \lambda} \lambda_i g(x,i) \geq f(\bar{x}) + \sum_{i \in supp\ \lambda} \lambda_i g(\bar{x},i), \ \forall\ x \in \mathbb{R}^n.$$

As $\lambda \in \mathcal{A}(\bar{x})$, $\lambda_i g(\bar{x},i) = 0$ for $i \in supp\ \lambda$, which for every $x \in C_I$ reduces the above inequality to

$$f(x) \geq f(x) + \sum_{i \in supp\ \lambda} \lambda_i g(x,i) \geq f(\bar{x}), \ \forall\ x \in C_I.$$

Therefore, $\bar{x}$ is a point of minimizer of (SIP), hence completing the proof. $\square$

Chapter 12

Convexity in Nonconvex Optimization

12.1 Introduction

This is the final chapter of the book. What we want to discuss here is essentially outside the preview of convex optimization. Yet as we will see, convexity will play a fundamental role in the issues discussed. We will discuss here two major areas in nonconvex optimization, namely maximization of a convex function and minimization of a *d.c. function*. The acronym *d.c.* stands for difference convex function, that is, functions expressed as the difference of two convex functions. Thus, more precisely, we would look into the following problems:

$$\max f(x) \quad \text{subject to} \quad x \in C \tag{$P1$}$$

and

$$\min f(x) - g(x) \quad \text{subject to} \quad x \in C, \tag{$P2$}$$

where $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $C \subset \mathbb{R}^n$ is a convex set. A large class of nonconvex optimization problems actually come into this setting. Note that $(P1)$ can be posed as

$$\min \ -f(x) \quad \text{subject to} \quad x \in C$$

and thus as

$$\min \ \phi(x) - f(x) \quad \text{subject to} \quad x \in C,$$

where ϕ is the zero function. Thus the problem $(P1)$ can also be viewed as a special case of $(P2)$, though we will consider them separately for a better understanding.

12.2 Maximization of a Convex Function

The problem of maximizing a convex function over a convex set is a complete paradigm shift from that of minimization of a convex function over a convex set. The problem of maximization of a convex function is, in fact, a hard nonconvex minimization problem. One of the early results in this direction appears in the classic text of Rockafellar [97] and we will mention a few of them here in order to motivate the reader. The first point that the reader should note is that local maxima of a convex function need not be global maxima. We leave it to the reader to create some examples that bring out this fact. The following result is given in Rockafellar [97]. We will not provide any proof. See Rockafellar [97] for the proof.

Theorem 12.1 *Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a convex set $C \subset \mathbb{R}^n$. If f attains its supremum relative to C at some point in ri C, then f is constant on C.*

The above theorem says that if f is a nonconstant convex function and if it attains its supremum on C, then it must be attained at the boundary. Of course the more interesting question is when does the convex function actually attains its maximum? In this respect, one has the following interesting result from [97] where the set C is assumed to be polyhedral.

Theorem 12.2 *Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a convex set $C \subset \mathbb{R}^n$ that is polyhedral. Suppose that there are no half lines in C on which f is unbounded above. Then f attains its supremum over C.*

For more general results, see [97]. One of the earliest papers dealing exclusively with the optimality conditions of maximizing a convex function over a convex set is due to Strekalovskii [104]. Though Strekalovskii [104] frames his problem in a general setting, his results are essentially useful for the convex case and the main results in his paper are given only for the convex case.

Observe that if $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $\bar{x} \in C$ is the point where f attains a global maximum, then for every $x \in C$,

$$0 \geq f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall\, \xi \in \partial f(\bar{x}),$$

which implies

$$\langle \xi, x - \bar{x} \rangle \leq 0, \ \forall\, \xi \in \partial f(\bar{x}), \ \forall\, x \in C.$$

Thus the necessary condition is $\partial f(\bar{x}) \subset N_C(\bar{x})$. The reader should try to find a necessary condition when $\bar{x}$ is a local maximum. Can we find a sufficient condition for a global maximum? Strekalovskii [104] attempts to answer this question by developing a set of necessary and sufficient conditions.

Theorem 12.3 *Consider the problem of maximizing a convex function f over a closed convex set C. Assume that $\bar{x} \in C$ is a point such that*

$$-\infty < \inf_{x \in \mathbb{R}^n} f(x) < f(\bar{x}) < +\infty$$

and the set

$$\bar{C} = \{x \in \mathbb{R}^n : f(x) \leq f(\bar{x})\}$$

is compact having a nonempty interior, that is, int $\bar{C} \neq \emptyset$. *Then* $\bar{x} \in C$ *is a global maximum of* f *on* C *if and only if*

(a) *for every* $x^* \in \partial f(\bar{x})$, $\langle x^*, x - \bar{x}\rangle \leq 0, \ \forall\, x \in C$ *or,*

(b) *for every* $y^* \in S(f, \bar{x})$, $\langle y^*, x - \bar{x}\rangle \leq 1, \ \forall\, x \in C$ *where*

$$S(f,\bar{x}) = \{y^* \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^n,\ y \neq \bar{x},\ f(y) = f(\bar{x}) \text{ and } \exists\, \alpha > 0,\ \alpha y^* \in \partial f(y) \text{ satisfying } \langle y^*, y - \bar{x}\rangle = 1\}.$$

Proof. We will only prove (a) and leave (b) to the readers. If $\bar{x}$ is a global maximum, then our discussion preceding the theorem shows that (a) holds, that the condition in (a) is necessary. Now we will look into the reverse, that is, whether (a) is sufficient for a global maximum or not. Observe that under the given hypothesis, for every $x^* \in \partial f(\bar{x})$,

$$\langle x^*, x - \bar{x}\rangle \leq 0, \ \forall\, x \in C.$$

As *dom* $f = \mathbb{R}^n$, by Theorem 2.69, f is a continuous convex function, thus the set $\bar{C}$ is closed and convex. Also, from the above inequality,

$$\textit{cone}\ \partial f(\bar{x}) \subset N_C(\bar{x}).$$

Further, as $\bar{C}$ has a nonempty interior, there exists $\hat{x}$ such that $f(\hat{x}) < f(\bar{x})$. Hence

$$N_{\bar{C}}(\bar{x}) = \{\lambda\xi : \lambda \geq 0,\ \xi \in \partial f(\bar{x})\}.$$

Thus, $N_{\bar{C}}(\bar{x}) = \textit{cone}\ \partial f(\bar{x})$. This shows that $N_{\bar{C}}(\bar{x}) \subset N_C(\bar{x})$, which implies that $C \subset \bar{C}$. Hence $\bar{x}$ is the point where the maximum is achieved as $\bar{x}$ is already given to be an element of C. $\square$

It is important to note that without the additional conditions, $\partial f(\bar{x}) \subset N_C(\bar{x})$ does not render a global maximum. Here we put forward an example from Dutta [38]. Consider $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \max\{x^2, x\}.$$

Now suppose that we want to maximize f over $C = [-1, 0]$. Consider $\bar{x} = 0$. Thus $N_C(\bar{x}) = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Observe that $\partial f(0) = [0, 1]$. Therefore, $\partial f(0) \subset N_C(0)$. However, $\bar{x} = 0$ is a global minimizer of f over C and not a global maximizer.

Strekalovskii refined the above result slightly to provide the following result. This appeared in [105].

Theorem 12.4 *Consider a closed convex set $C \subset \mathbb{R}^n$ and let $\bar{x} \in C$. Assume that*

$$-\infty \leq \inf_{x \in \mathbb{R}^n} f(x) < f(\bar{x}),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Then $\bar{x} \in C$ is a global maximum for $(P1)$ if and only if

$$\partial f(x) \subset N_C(x), \ \forall\, x \in \mathbb{R}^n \text{ satisfying } f(x) = f(\bar{x}).$$

Readers are requested to have a look at the difference between Strekalovskii's result in Theorem 12.3 and this result. Though the above result is elegant, it suffers from a drawback, that is, one needs to calculate $N_C(x)$ for every $x \in \mathbb{R}^n$ satisfying $f(x) = f(\bar{x})$. Now if $x \notin C$, then traditionally we define $N_C(x) = \emptyset$. However, for a convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\partial f(x) \neq \emptyset$ for every $x \in \mathbb{R}^n$. This drawback was overcome by Hiriart-Urruty and Ledyaev [61]. We now present their result but with a different approach to the proof.

Theorem 12.5 *Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a closed convex set $C \subset \mathbb{R}$. Let $\bar{x} \in C$ be such that*

$$-\infty \leq \inf_{x \in C} f(x) < f(\bar{x}).$$

Then $\bar{x} \in C$ is a maximizer for $(P1)$ if and only if

$$\partial f(x) \subset N_C(x), \ \forall\, x \in C \text{ satisfying } f(x) = f(\bar{x}).$$

Proof. If $\bar{x} \in C$ is the global maximizer of the function f over C, then we have already seen that $\partial f(\bar{x}) \subset N_C(\bar{x})$. It is simple to see that if $f(x) = f(\bar{x})$, $\partial f(x) \subset N_C(x)$. We leave this very simple proof to the reader.

Conversely, assume on the contrary that $\bar{x} \in C$ is not a global maximizer of $(P1)$. Therefore, there exists $\hat{x} \in C$ such that $f(\hat{x}) > f(\bar{x})$. Consider the following level set

$$S(\bar{x}) = \{x \in C : f(x) \leq f(\bar{x})\},$$

which is a closed convex set. It is clear that $\hat{x} \notin S(\bar{x})$. Thus, the following projection problem:

$$\min \ \frac{1}{2}\|x - \hat{x}\|^2 \quad \text{subject to } f(x) \leq f(\bar{x}), \ x \in C$$

has a unique solution. Let $\tilde{x} \in C$ be that unique solution. Now using the Fritz John optimality conditions for a convex optimization problem, Theorem 5.1, there exist $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$ with $(\lambda_0, \lambda_1) \neq (0, 0)$ such that

(i) $0 \in \lambda_0(\tilde{x} - \hat{x}) + \lambda_1 \partial f(\tilde{x}) + N_C(\tilde{x})$,

(ii) $\lambda_1(f(\tilde{x}) - f(\bar{x})) = 0$.

Assume that $\lambda_0 = 0$, which implies $\lambda_1 > 0$. Thus the above conditions reduce to

$$0 \in \lambda_1 \partial f(\tilde{x}) + N_C(\tilde{x}) \text{ and } f(\tilde{x}) = f(\bar{x}).$$

The condition $0 \in \lambda_1 \partial f(\tilde{x}) + N_C(\tilde{x})$ leads to the expression

$$0 \in \partial f(\tilde{x}) + N_C(\tilde{x}).$$

This is obtained by dividing both sides by λ_1 and noting that $N_C(\tilde{x})$ is a cone. As f is convex, invoking Theorem 3.1, $\tilde{x} \in C$ is a point of minimizer of f over C, that is,

$$f(\tilde{x}) = \inf_{x \in C} f(x).$$

The condition $f(\tilde{x}) = f(\bar{x})$ along with the given hypothesis yields that

$$-\infty \leq \inf_{x \in C} f(x) < f(\tilde{x}),$$

thereby contradicting the fact that $\tilde{x}$ is the point of minimizer of f over C. Hence $\lambda_0 > 0$. Now assume that $\lambda_1 = 0$. Therefore, the facts that $\lambda_0 > 0$ and $N_C(\tilde{x})$ is a cone yield that

$$0 \in (\tilde{x} - \hat{x}) + N_C(\tilde{x}),$$

that is,

$$\hat{x} - \tilde{x} \in N_C(\tilde{x}).$$

Because $\hat{x} \in C$,

$$0 \geq \langle \hat{x} - \tilde{x}, \hat{x} - \tilde{x} \rangle = \|\hat{x} - \tilde{x}\|^2,$$

implying that $\tilde{x} = \hat{x}$. This is indeed a contradiction. Hence $\lambda_1 > 0$. Thus there exist $\xi \in \partial f(\tilde{x})$ and $\eta \in N_C(\tilde{x})$ such that

$$0 = \lambda_0(\tilde{x} - \hat{x}) + \lambda_1 \xi + \eta. \tag{12.1}$$

As $f(\tilde{x}) = f(\bar{x})$, by the given hypothesis, $\partial f(\tilde{x}) \subset N_C(\tilde{x})$, which implies

$$-\langle \lambda_1 \xi, \hat{x} - \tilde{x} \rangle \geq 0. \tag{12.2}$$

Further, it is simple to see that

$$-\langle \eta, \hat{x} - \tilde{x} \rangle + \lambda_0 \|\hat{x} - \tilde{x}\|^2 > 0. \tag{12.3}$$

The conditions (12.1), (12.2), and (12.3) lead to a contradiction, thereby establishing the result. $\square$

12.3 Minimization of d.c. Functions

In this section we will concentrate on deriving the optimality condition for local and global minimization of a very important class of nonconvex problems. These problems are the ones where the objective function can be expressed as the difference of two convex functions. Such functions are referred to as *difference convex functions* or *d.c. functions.* Thus we will concentrate on the problem

$$\min f(x) - g(x) \quad \text{subject to} \quad x \in C \tag{$P2$}$$

where $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $C \subset \mathbb{R}^n$ is a convex set. Note that $f(x) - g(x)$ need not be a convex function unless g is a linear or affine function. So in general it is a nonconvex function. We begin by providing a necessary optimality condition for a local optimal point.

Theorem 12.6 *Consider the problem $(P2)$ and let $\bar{x}$ be a local minimizer of $(P2)$ where $C = \mathbb{R}^n$. Then $\partial f(\bar{x}) \cap \partial g(\bar{x}) \neq \emptyset$.*

Proof. Let $\bar{x}$ be a local minimum. As $f - g$ is locally Lipschitz,

$$0 \in \partial^\circ (f - g)(\bar{x}).$$

For details, see Clarke [27] or Chapter 3. Hence, by the Sum Rule of the Clarke subdifferential,

$$0 \in \partial^\circ f(\bar{x}) + \partial^\circ(-g)(\bar{x}).$$

Noting that $\partial^\circ f(\bar{x}) = \partial f(\bar{x})$ and $\partial^\circ(-g)(\bar{x}) = -\partial^\circ g(\bar{x}) = -\partial g(\bar{x})$, the above condition becomes

$$0 \in \partial f(\bar{x}) - \partial g(\bar{x}).$$

This yields that

$$\partial f(\bar{x}) \cap \partial g(\bar{x}) \neq \emptyset.$$

We would again like to stress that for details on the Clarke subdifferential, see Clarke [27]. $\square$

Let us note that the above condition is only necessary and not sufficient. Consider $h(x) = f(x) - g(x)$, where $f(x) = x^2$ and $g(x) = |x|$. At $\bar{x} = 0$, $\partial f(0) = 0$ and $\partial g(0) = [-1, 1]$. Thus,

$$\partial f(0) \cap \partial g(0) = \{0\}.$$

But it is clear that $\bar{x} = 0$ is not a local minimizer of h.

Now let us see what happens if we consider $C \subset \mathbb{R}^n$. In this case, one would have

$$0 \in \partial^\circ (f-g)(\bar{x}) + N_C(\bar{x}),$$

(see Clarke [27] for more details). Hence,

$$0 \in \partial f(\bar{x}) - \partial g(\bar{x}) + N_C(\bar{x}).$$

Thus there exist $\xi_f \in \partial f(\bar{x})$, $\xi_g \in \partial g(\bar{x})$ and $\eta \in N_C(\bar{x})$ such that

$$\xi_g = \xi_f + \eta.$$

Thus, the optimality condition can now be stated as follows:

If $\bar{x}$ is a local minimum for $(P2)$, there there exists $\xi_g \in \partial g(\bar{x})$ such that

$$\xi_g \in \partial f(\bar{x}) + N_C(\bar{x}).$$

For $C = \mathbb{R}^n$, if $\bar{x}$ is a global minimum for $(P2)$,

$$f(x) - g(x) \geq f(\bar{x}) - g(\bar{x}), \ \forall\ x \in \mathbb{R}^n.$$

Therefore,

$$f(x) - f(\bar{x}) \geq g(x) - g(\bar{x}) \geq \langle \xi_g, x - \bar{x} \rangle, \ \forall\ \xi_g \in \partial g(\bar{x}),$$

thereby implying that

$$\partial g(\bar{x}) \subset \partial f(\bar{x}).$$

Note that this is again a necessary condition and not sufficient. We urge the reader to find an example demonstrating this fact.

We now present interesting and important necessary and sufficient optimality conditions for the global optimization of problem $(P2)$. Here the optimality conditions will be expressed in terms of the ε-subdifferential. We present this result as given in Bomze [15].

Theorem 12.7 *Consider the problem $(P2)$ with $C = \mathbb{R}^n$. Then $\bar{x} \in \mathbb{R}^n$ is a global minimizer of $(P2)$ if and only if*

$$\partial_\varepsilon g(\bar{x}) \subset \partial_\varepsilon f(\bar{x}), \ \forall\ \varepsilon > 0.$$

Proof. As $\bar{x} \in \mathbb{R}^n$ is a global minimizer of $(f-g)$ over $\mathbb{R}^n$,

$$f(x) - f(\bar{x}) \geq g(x) - g(\bar{x}), \ \forall\ x \in \mathbb{R}^n.$$

If $\xi \in \partial_\varepsilon g(\bar{x})$ for any $\varepsilon > 0$,

$$f(x) - f(\bar{x}) \geq g(x) - g(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall\ x \in \mathbb{R}^n,$$

thereby implying that $\xi \in \partial_\varepsilon f(\bar{x})$. Because $\varepsilon > 0$ was arbitrary, this establishes that

$$\partial_\varepsilon g(\bar{x}) \subset \partial_\varepsilon f(\bar{x}), \ \forall\, \varepsilon > 0.$$

Let us now look at the converse. On the contrary, assume that $\bar{x}$ is not a global minimizer of $(P2)$, which implies that there exists $\hat{x} \in \mathbb{R}^n$ such that

$$f(\hat{x}) - g(\hat{x}) < f(\bar{x}) - g(\bar{x}).$$

This yields that

$$f(\bar{x}) - f(\hat{x}) - g(\bar{x}) + g(\hat{x}) > 0.$$

Set $\delta = (1/2)(f(\bar{x}) - f(\hat{x}) - g(\bar{x}) + g(\hat{x}))$. It is simple to see that $\delta > 0$.

Now consider $\hat{\xi} \in \partial g(\hat{x})$, which implies that

$$g(\bar{x}) - g(\hat{x}) - \langle \hat{\xi}, \bar{x} - \hat{x} \rangle \geq 0.$$

Because $\delta > 0$,

$$g(\bar{x}) - g(\hat{x}) - \langle \hat{\xi}, \bar{x} - \hat{x} \rangle + \delta > 0.$$

Set $\varepsilon = g(\bar{x}) - g(\hat{x}) - \langle \hat{\xi}, \bar{x} - \hat{x} \rangle + \delta$. Then for any $x \in \mathbb{R}^n$,

$$\begin{aligned} \langle \hat{\xi}, x - \bar{x} \rangle - \varepsilon &= \langle \hat{\xi}, x - \hat{x} + \hat{x} - \bar{x} \rangle - \varepsilon \\ &= \langle \hat{\xi}, x - \hat{x} \rangle - \delta + g(\hat{x}) - g(\bar{x}). \end{aligned}$$

As $\hat{\xi} \in \partial g(\hat{x})$, it is clear that $\hat{\xi} \in \partial_\delta g(\hat{x})$, which leads to

$$\langle \hat{\xi}, x - \hat{x} \rangle - \delta + g(\hat{x}) \leq g(x).$$

Thus,

$$\langle \hat{\xi}, x - \bar{x} \rangle - \varepsilon \leq g(x) - g(\bar{x}), \ \forall\, x \in \mathbb{R}^n,$$

thereby implying that $\hat{\xi} \in \partial_\varepsilon g(\bar{x})$. By the given hypothesis, $\hat{\xi} \in \partial_\varepsilon f(\bar{x})$. Therefore, in particular for $x = \hat{x}$,

$$f(\hat{x}) - f(\bar{x}) \geq \langle \hat{\xi}, \hat{x} - \bar{x} \rangle - \varepsilon.$$

Now

$$\begin{aligned} 2\delta &= f(\bar{x}) - f(\hat{x}) - (g(\bar{x}) - g(\hat{x})) \\ &\leq \varepsilon - \langle \hat{\xi}, \hat{x} - \bar{x} \rangle - (g(\bar{x}) - g(\hat{x})). \end{aligned}$$

The way in which ε is defined leads to

$$\varepsilon - (g(\bar{x}) - g(\hat{x})) = \delta + \langle \hat{\xi}, \hat{x} - \bar{x} \rangle.$$

Hence,

$$2\delta \leq \delta + \langle \hat{\xi}, \hat{x} - \bar{x} \rangle - \langle \hat{\xi}, \hat{x} - \bar{x} \rangle = \delta < 2\delta,$$

which is a contradiction. Thus, $\bar{x}$ is indeed a global solution for $(P2)$. $\square$

Note that the above result also holds true if we assume $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \to \mathbb{R}$. In that case, one just has to assume that $\bar{x} \in dom\ f$. The reader is encouraged to sketch the proof for such a scenario. However, we present the result here for the sake of convenience.

Theorem 12.8 *Consider the problem $(P2)$ with $C = \mathbb{R}^n$ and a lower semi-continuous convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $dom\ f \neq \emptyset$. Let $\bar{x} \in dom\ f$. Then $\bar{x}$ is a global minimum for $(P2)$ if and only if*

$$\partial_\varepsilon g(\bar{x}) \subset \partial_\varepsilon f(\bar{x}),\ \forall\ \varepsilon > 0.$$

Using the above result, one can deduce an optimality conditions for the case when $C \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. Observe that when $C \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$, the problem $(P2)$ can be equivalently written as

$$\min\ (f + \delta_C)(x) - g(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

Hence, $\bar{x}$ is a global minimum for $(P2)$ if and only if

$$\partial_\varepsilon g(\bar{x}) \subset \partial_\varepsilon (f + \delta_C)(\bar{x}),\ \forall\ \varepsilon > 0.$$

This is done of course by applying Theorem 12.8. Invoking the Sum Rule of ε-subdifferential, Theorem 2.115,

$$\partial_\varepsilon (f + \delta_C)(\bar{x}) = \bigcup_{\substack{\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1} f(\bar{x}) + \partial_{\varepsilon_2} \delta_C(\bar{x})).$$

Hence,

$$\partial_\varepsilon g(\bar{x}) \subset \bigcup_{\substack{\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1} f(\bar{x}) + N_{\varepsilon_2, C}(\bar{x})),\ \forall\ \varepsilon > 0.$$

We just recall that $\partial_{\varepsilon_2} \delta_C(\bar{x}) = N_{\varepsilon_2, C}(\bar{x})$ for any $\varepsilon_2 \geq 0$.

The result Theorem 12.8 can also be used to deduce necessary and sufficient optimality conditions for the problem $(P1)$.

Corollary 12.9 *Consider the problem $(P1)$. Assume that $C \subset \mathbb{R}^n$ is a closed convex set. The $\bar{x} \in C$ is a global maximum for $(P1)$ if and only if*

$$\partial_\varepsilon f(\bar{x}) \subset N_{\varepsilon, C}(\bar{x}),\ \forall\ \varepsilon > 0.$$

Proof. Observe that the problem $(P1)$ can be written as

$$\min \ -f(x) \quad \text{subject to} \quad x \in C.$$

A further equivalent version can be given by

$$\min \ (\delta_C + f)(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.$$

Using Theorem 12.8, the optimality condition is

$$\partial_\varepsilon f(\bar{x}) \subset \partial_\varepsilon \delta_C(\bar{x}), \ \forall\, \varepsilon > 0,$$

that is,

$$\partial_\varepsilon f(\bar{x}) \subset N_{\varepsilon,C}(\bar{x}), \ \forall\, \varepsilon > 0,$$

thereby establishing the result. □

We end our discussion and the book here. However for more details on the use of the above results, see for example Bomze [15], Hiriart-Urruty [60], and the references therein.

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